Reiteration formulae for interpolation methods associated to polygons

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Abstract

We study spaces generated by applying the interpolation methods defined by a polygon Π to an N-tuple of real interpolation spaces with respect to a fixed Banach couple $\{X, Y\}$. We show that if the interior point (α, β) of the polygon does not lie in any diagonal of Π then the interpolation spaces coincide with sums and intersections of real interpolation spaces generated by $\{X, Y\}$. Applications are given to N-tuples formed by Lorentz function spaces and Besov spaces. Moreover, we show that results fail in general if (α, β) is in a diagonal.

Key words:
Interpolation methods associated to polygons; reiteration; Lorentz function spaces; Besov spaces.
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1 Introduction

As it is well-known, abstract interpolation theory has its roots on interpolation theorems for bounded operators between L_p spaces due to M. Riesz and Thorin, and to Marcinkiewicz. It was established in the early 1960s with the work of J.-L. Lions, Peetre, A.P. Calderón, Gagliardo, S.G. Krein and other authors. Since then it has attracted considerable interest in itself and has found many important applications, not only in analysis but also in some other areas of mathematics as one can see, for example, in the books by Butzer and Berens [5], Bergh and Löfström [3], Triebel [24], Bennett and Sharpley [2], Brudnyĭ and Krugljak [4], Connes [14] and Amrein, Boutet de Monvel and Georgescu [1].

The greater part of interpolation theory refers to couples of spaces and operators acting on couples but some problems in functional analysis have also led to study interpolation spaces generated by three or more spaces, and even an infinite family of spaces. The first contribution to this question can be found in the paper by Foiaş and J.-L. Lions [20] and then in the papers by Yoshikawa [26], Favini [17], Sparr [23] and Fernandez [18], among others. The step from two to more than two spaces bears a number of difficulties of combinatorial and geometrical nature, to the effect that basic results in the classical theory for couples are no longer true for finite families (N-tuples) of spaces.

We work here with the interpolation methods for N-tuples introduced by Peetre and the first present author [13], and further developed in [12,8,11,16,10,19] among other papers. These methods are defined by means of a convex polygon Π in the plane $\mathbb{R}^2$ and a point (α, β) in the interior of Π. The spaces of the N-tuple should be thought as sitting on the vertices of Π. Using this picture, one introduces K- and J-functionals with two parameters and then K- and J-spaces by using an (α, β)-weighted L_q-norm (see Section 2 for the proper definitions). In the special case where Π is equal to the simplex, we recover (the first nontrivial case of) spaces studied by Yoshikawa and Sparr, and if Π coincides with the unit square, then we get spaces introduced by Fernandez.

Working with the multidimensional methods, an important question is to de-

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termine the spaces that come up by interpolation of concrete N-tuples. For this task, since many natural N-tuples are formed by real interpolation spaces, it turns out to be very useful to have reiteration results between the real method and the methods associated to polygons. This question has been considered in [9,16,7]. Let $\bar{A} = \{A_1, ..., A_N\}$ be an N-tuple formed by Banach spaces A_j which are of class θ_j with respect to a fixed Banach couple $\{X, Y\}$ (precise definitions are given in Section 2). Reiteration formulae describe the interpolation spaces generated by $\bar{A}$ in terms of sums and intersections of real interpolation spaces generated by $\{X, Y\}$. Results established by Ericsson [16] require extra assumptions on the polygon Π, which are stated by using four auxiliary numbers $\delta, \delta', \rho, \rho'$ and relations between them and the vertices of Π, the θ_j and the point (α, β). Recently, in the limit case $q = \infty$ for the K-method and $q = 1$ for the J-method, Cobos, Fernández-Cabrera and Martín [7] have proved formulae which do not need any auxiliary assumption.

In the present paper we continue the research of [7] by showing that if (α, β) does not lie in any diagonal of Π then the reiteration formulae hold for any $1 \leq q \leq \infty$ without any extra condition on the polygon. Moreover, we show by counterexamples that if (α, β) lies on a diagonal then the known results for (∞, K)- and $(1, J)$-methods fail in general for other values of q.

The approach we follow is different from that in [7] which is based on special features of ℓ_1- and ℓ_∞-norms. Our strategy is to establish several geometrical and algebraic results that allow to apply some ideas developed by Ericsson [16] without requiring any extra condition on the polygon.

The paper is organized as follows. In Section 2 we review some basic results on K- and J-spaces defined by polygons. In Section 3 we establish the reiteration formulae and we write down some concrete cases for Lorentz function spaces and Besov spaces. Finally, Section 4 contains the counterexamples for the case when (α, β) lies on a diagonal.

2 Preliminaries

By a *Banach N-tuple* we mean a family $\bar{A} = \{A_1, ..., A_N\}$ of N Banach spaces A_j which are continuously embedded in a common Hausdorff topological vector space. When $N = 2$ we simply call $\{A_1, A_2\}$ a *Banach couple.*

Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon in the plane $\mathbb{R}^2$, with vertices $P_j = (x_j, y_j)$. Given the Banach N-tuple $\bar{A}$, each space A_j should be thought of as sitting on the vertex P_j. For $t, s > 0$ and $a \in \Sigma(\bar{A}) = A_1 + ... + A_N$ we define the K-functional by

$$K(t,s;a) = K(t,s;a;\bar{A}) = \inf\left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} \ : \ a = \sum_{j=1}^{N} a_j \ , \ a_j \in A_j \right\}.$$

For $a \in \Delta(\bar{A}) = A_1 \cap ... \cap A_N$, the J-functional is given by

$$J(t,s;a) = J(t,s;a;\bar{A}) = \max\{ t^{x_j} s^{y_j} \|a\|_{A_j} \ : \ 1 \le j \le N \}.$$

Given any interior point (α,β) of Π $[(\alpha,\beta) \in \operatorname{Int}\Pi]$ and any $1 \le q \le \infty$, the *K-space* $\bar{A}_{(\alpha,\beta),q;K}$ is defined as the collection of all elements $a \in \Sigma(\bar{A})$ for which the norm

$$\|a\|_{\bar{A}_{(\alpha,\beta),q;K}} = \left(\int_0^\infty \int_0^\infty [t^{-\alpha} s^{-\beta} K(t,s;a)]^q \frac{dt}{t} \frac{ds}{s} \right)^{\frac{1}{q}}$$

is finite (the integral must be replaced by the supremum if $q = \infty$).

The *J-space* $\bar{A}_{(\alpha,\beta),q;J}$ consists of all those $a \in \Sigma(\bar{A})$ which can be represented as

$$a = \int_0^\infty \int_0^\infty v(t,s) \frac{dt}{t} \frac{ds}{s} \qquad \text{(convergence in } \Sigma(\bar{A})) \tag{2.1}$$

with a strongly measurable $\Delta(\bar{A})$-valued function v satisfying

$$\left(\int_0^\infty \int_0^\infty [t^{-\alpha} s^{-\beta} J(t,s;v(t,s))]^q \frac{dt}{t} \frac{ds}{s} \right)^{\frac{1}{q}} < \infty. \tag{2.2}$$

The norm in $\bar{A}_{(\alpha,\beta),q;J}$ is given by

$$\|a\|_{\bar{A}_{(\alpha,\beta),q;J}} = \inf_v \left\{ \left(\int_0^\infty \int_0^\infty [t^{-\alpha} s^{-\beta} J(t,s;v(t,s))]^q \frac{dt}{t} \frac{ds}{s} \right)^{\frac{1}{q}} \right\}$$

where the infimum is taken over all v satisfying (2.1) and (2.2) (see [27] for properties of the Bochner-integral).

These spaces were introduced by Cobos and Peetre [13]. In the special case where Π is equal to the simplex $\{(0,0),(1,0),(0,1)\}$ we get

$$K(t,s;a) = \inf\{\|a_1\|_{A_1} + t\|a_2\|_{A_2} + s\|a_3\|_{A_3} \ : \ a = \sum_{j=1}^{3} a_j \ , \ a_j \in A_j\}$$

and we recover (the first nontrivial case of) spaces investigated by Sparr [23]. When Π is the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$ then

$$K(t,s;a) = \inf\{\|a_1\|_{A_1} + t\|a_2\|_{A_2} + s\|a_3\|_{A_3} + ts\|a_4\|_{A_4} \ : \ a = \sum_{j=1}^{4} a_j \ , \ a_j \in A_j\}$$

and we obtain spaces studied by Fernandez [18].

Given any Banach couple $\{X, Y\}$, the *real interpolation space* $(X, Y)_{\theta,q}$ can be described in a similar way, but by replacing the polygon by the segment $[0, 1]$, the point (α, β) by $\theta \in (0, 1)$ and by imagining that X is sitting on 0 and Y on 1. We denote the relevant functionals by

$$\bar{K}(t, a) = \bar{K}(t, a; X, Y) = \inf\{\|x\|_X + t\|y\|_Y : a = x + y\ ,\ x \in X\ ,\ y \in Y\}$$

and

$$\bar{J}(t, a) = \bar{J}(t, a; X, Y) = \max\{\|a\|_X, t\|a\|_Y\}\,.$$

Note that $\bar{K}(1, \cdot)$ and $\bar{J}(1, \cdot)$ coincide with the canonical norms on $X + Y$ and $X \cap Y$, respectively. For the real method it is well-known that K- and J-spaces coincide with equivalence of norms, i.e.

$$\begin{aligned}
&(X, Y)_{\theta,q} \\
&= \left\{a \in X + Y : \|a\|_{(X,Y)_{\theta,q}} = \left(\int_0^\infty [t^{-\theta}\bar{K}(t, a)]^q \frac{dt}{t}\right)^{1/q} < \infty\right\} \\
&= \left\{a \in X + Y\ :\ a = \int_0^\infty u(t)\,\frac{dt}{t} \text{ with } \left(\int_0^\infty [t^{-\theta}\bar{J}(t, u(t))]^q \frac{dt}{t}\right)^{1/q} < \infty\right\}
\end{aligned}$$

(see [3,24]). However, working with N-tuples ($N \geq 3$), K- and J-spaces do not agree in general. We only have $\bar{A}_{(\alpha,\beta),q;J} \hookrightarrow \bar{A}_{(\alpha,\beta),q;K}$ (see [13]), where $\hookrightarrow$ means continuous inclusion.

The following property of invariance under affine bijections is shown in [12, Remark 4.1]: If R is any affine bijection of $\mathbb{R}^2$ then the K- and the J-space defined by means of Π and (α, β) coincide (with equivalence of norms) with those defined by $R(\Pi) = \overline{RP_1 \cdots RP_N}$ and the point $R(\alpha, \beta)$.

The location of (α, β) in Π will play an important role in our later considerations. Let $\mathcal{P}_{(\alpha,\beta)}$ be the collection of all triples $\{i, k, r\}$ such that (α, β) belongs to the triangle with vertices P_i, P_k, P_r (see Figure 2.1). We allow that (α, β) lies in any of the edges of this triangle.

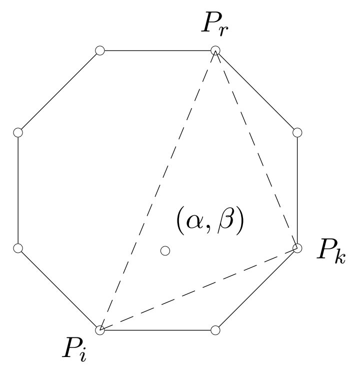

Figure 2.1

Given any $\{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)}$ we write (c_i, c_k, c_r) for the unique barycentric coordinates of (α, β) with respect to P_i, P_k, P_r. So,

$$(\alpha, \beta) = c_i P_i + c_k P_k + c_r P_r \quad , \quad c_i + c_k + c_r = 1 \, .$$

Let $\{X, Y\}$ be a Banach couple and let Z be a Banach space such that $X \cap Y \hookrightarrow Z \hookrightarrow X + Y$. For $0 \leq \theta \leq 1$, we say that Z is of *class* $\mathcal{C}(\theta; X, Y)$ if there is a constant $c > 0$ such that

$$\bar{K}(t, a) \leq c t^{\theta} \|a\|_Z \quad \text{for all} \quad a \in Z \tag{2.3}$$

and

$$\|a\|_Z \leq c t^{-\theta} \bar{J}(t, a) \quad \text{for all} \quad a \in X \cap Y. \tag{2.4}$$

It is clear that X is of class $\mathcal{C}(0; X, Y)$ and Y is of class $\mathcal{C}(1; X, Y)$. When $0 < \theta < 1$, conditions (2.3) and (2.4) can be formulated by the inclusions

$$(X, Y)_{\theta,1} \hookrightarrow Z \hookrightarrow (X, Y)_{\theta,\infty} \quad \text{(see [3,24]).}$$

Let $\bar{A} = \{A_1, ..., A_N\}$ be a Banach N-tuple. For each $\{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)}$ such that (α, β) belongs to the interior of the triangle $\Delta_{ikr} = \overline{P_i P_k P_r}$, we put $\tilde{A}_{ikr} = \{A_i, A_k, A_r\}$ and we write $\tilde{K}$, $\tilde{J}$ for K- and J-functionals defined by means of Δ_{ikr}. Clearly,

$$K(t, s; a; \bar{A}) \leq \tilde{K}(t, s; a; \tilde{A}_{ikr}) \quad , \quad a \in A_i + A_k + A_r$$

and

$$\tilde{J}(t, s; a; \tilde{A}_{ikr}) \leq J(t, s; a; \bar{A}) \quad , \quad a \in \Delta(\bar{A}) \, .$$

Hence,

$$\bar{A}_{(\alpha,\beta),q;J} \hookrightarrow (A_i, A_k, A_r)_{(\alpha,\beta),q;J} \hookrightarrow (A_i, A_k, A_r)_{(\alpha,\beta),q;K} \hookrightarrow \bar{A}_{(\alpha,\beta),q;K} \, . \tag{2.5}$$

Suppose now, in addition, that each A_j is of class $\mathcal{C}(\theta_j; X, Y)$, $0 \leq \theta_j \leq 1$, $1 \leq j \leq N$. Let

$$\theta_{ikr} = c_i\theta_i + c_k\theta_k + c_r\theta_r.$$

It is shown in [16, Corollary 4] or [7, (2.9)] that if $\theta_i, \theta_k, \theta_r$ are not all equal then we have with equivalence of norms

$$(A_i, A_k, A_r)_{(\alpha,\beta),q;J} = (A_i, A_k, A_r)_{(\alpha,\beta),q;K} = (X, Y)_{\theta_{ikr},q} \,. \tag{2.6}$$

We close this section with a remark concerning notation. Subsequently, given two quantities or two real-valued functions f, g we write $f \lesssim g$ whenever there is a constant $c > 0$ such that $f(x) \leq cg(x)$ for all x. If $f \lesssim g$ and $g \lesssim f$ we put $f \sim g$.

3 The reiteration results

In this section we establish the reiteration formulae between the real method and the methods associated to polygons. We start by proving several geometrical and algebraic results that will allow to apply some ideas developed in [16] to polygons in a much more general setting.

Definition 3.1 *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$, let $(\alpha, \beta) \in Int\,\Pi$ and let $\mathcal{P}_{(\alpha,\beta)}$ be the set introduced in Section 2. Let further $\{\theta_1, ..., \theta_N\}$ be a family of N numbers with $0 \leq \theta_j \leq 1$ such that for each $\{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}$, where $(\alpha, \beta) \in Int\,\overline{P_iP_kP_r}$, the values $\theta_i, \theta_k, \theta_r$ are not all equal. If $(\alpha, \beta) \in \overline{P_iP_k}$ we assume that $\theta_i \neq \theta_k$. For $\{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}$ let $\theta_{ikr} = c_i\theta_i + c_k\theta_k + c_r\theta_r$ be the number introduced above and define $0 < \bar{\theta}, \breve{\theta} < 1$ by*

$$\bar{\theta} = \min\{\theta_{ikr} \;:\; \{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}\} \quad , \quad \breve{\theta} = \max\{\theta_{ikr} \;:\; \{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}\}.$$

Definition 3.2 *Under the assumptions of Definition 3.1, two affine functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ are said to be admissible if f satisfies*

$$f(x_j, y_j) \leq \theta_j \quad \textit{for} \quad 1 \leq j \leq N \quad \textit{and} \quad f(\alpha, \beta) = \bar{\theta} \;,$$

and g satisfies

$$g(x_j, y_j) \geq \theta_j \quad \textit{for} \quad 1 \leq j \leq N \quad \textit{and} \quad g(\alpha, \beta) = \breve{\theta} \;.$$

Lemma 3.3 *Under the assumptions of Definition 3.1, there exists a pair of non-constant admissible functions f and g.*

Proof. Let $\hat{P}_j = (x_j, y_j, \theta_j)$ for $1 \leq j \leq N$ and let $\hat{\Pi}$ be the convex hull of $\hat{P}_1, \ldots, \hat{P}_N$. Then $\hat{\Pi}$ is a convex polyhedron in $\mathbb{R}^3$ whose projection onto the first two coordinates is Π. The polyhedron $\hat{\Pi}$ does not necessarily have inner points. The vertical line $l_{(\alpha,\beta)} = \{(\alpha, \beta, z) : z \in \mathbb{R}\}$ intersects $\hat{\Pi}$ in a line segment, that may be degenerated into a single point.

Let $(\alpha, \beta, \bar{z})$ be the lower end-point of $l_{(\alpha,\beta)} \cap \hat{\Pi}$. We pick a face $\bar{F}$ of $\hat{\Pi}$ that contains $(\alpha, \beta, \bar{z})$. (There are two possible choices if $(\alpha, \beta, \bar{z})$ belongs to an edge of $\hat{\Pi}$. If $\hat{\Pi}$ has no inner points, we put $\bar{F} = \hat{\Pi}$.) The plane $\bar{\pi}$ determined by $\bar{F}$ supports $\hat{\Pi}$ in the sense that $\hat{\Pi}$ is contained in the half-space $\bar{\pi}^+ = \{(x, y, z + w) : (x, y, z) \in \bar{\pi},\ w \geq 0\}$. We interpret $\hat{\pi}$ as the graph of an affine functional $f : \mathbb{R}^2 \to \mathbb{R}$ and shall show that f satisfies our claim.

We have $f(x_j, y_j) \leq \theta_j$ for $1 \leq j \leq N$, because

$$(x_j, y_j, \theta_j) = \hat{P}_j \in \hat{\Pi} \subseteq \bar{\pi}^+ = \{(x, y, f(x, y) + w) : x, y \in \mathbb{R},\ w \geq 0\}.$$

The face $\bar{F}$ contains $(\alpha, \beta, \bar{z})$ and has its vertices in $\{\hat{P}_1, \ldots, \hat{P}_N\}$. Hence there exist three vertices $\hat{P}_j, \hat{P}_l, \hat{P}_s$ of $\bar{F}$ such that $(\alpha, \beta, \bar{z})$ belongs to the convex hull $\hat{\Delta}_{jls}$ of $\hat{P}_j, \hat{P}_l, \hat{P}_s$. Then, by projection, $(\alpha, \beta) \in \Delta_{jls}$; that is, $\{j, l, s\} \in \mathcal{P}_{(\alpha,\beta)}$. Hence $f(x_j, y_j) = \theta_j$, $f(x_l, y_l) = \theta_l$, and $f(x_s, y_s) = \theta_s$ do not all agree. This shows that f is not constant.

Let (c_j, c_l, c_s) be the barycentric coordinates of (α, β) with respect to the vertices of the triangle Δ_{jls} . Then

$$\begin{aligned} f(\alpha, \beta) &= f(c_j P_j + c_l P_l + c_s P_s) = c_j f(P_j) + c_l f(P_l) + c_s f(P_s) \\ &= c_j \theta_j + c_l \theta_l + c_s \theta_s = \theta_{jls} \geq \bar{\theta}. \end{aligned}$$

On the other hand, by the definition of $\bar{\theta}$, there exist $\{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}$ such that $\bar{\theta} = \theta_{ikr}$. Using the barycentric coordinates (c_i, c_k, c_r) of (α, β) with respect to the triangle Δ_{ikr}, we obtain

$$\begin{aligned} f(\alpha, \beta) &= f(c_i P_i + c_k P_k + c_r P_r) = c_i f(P_i) + c_k f(P_k) + c_r f(P_r) \\ &\leq c_i \theta_i + c_k \theta_k + c_r \theta_r = \theta_{ikr} = \bar{\theta}. \end{aligned}$$

This completes the proof of $f(\alpha, \beta) = \bar{\theta}$. Hence f is admissible.

A similar construction, based on the upper end-point of $l_{(\alpha,\beta)} \cap \hat{\Pi}$, gives g. $\square$

Now let two non-constant admissible functions f, g be fixed. There exist real numbers $\gamma_1, \gamma_2, \gamma_3, \mu_1, \mu_2, \mu_3$ such that

$$f(x,y) = \gamma_1 x + \gamma_2 y + \gamma_3 \quad , \quad g(x,y) = \mu_1 x + \mu_2 y + \mu_3 .$$

Take any affine bijection R of type

$$R \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and let $\Pi' = R(\Pi) = \overline{P_1' \cdots P_N'}$ and $(\alpha', \beta') = R(\alpha, \beta)$.

Lemma 3.4 *With notation introduced above, we have*

(a) $f' = f \circ R^{-1}$ *and* $g' = g \circ R^{-1}$ *are admissible functions for* Π' *and* (α', β').

Let $f'(x,y) = \gamma_1' x + \gamma_2' y + \gamma_3'$ *and* $g'(x,y) = \mu_1' x + \mu_2' y + \mu_3'$. *Then*

(b)

$$\begin{pmatrix} \gamma_1' & \gamma_2' \\ \mu_1' & \mu_2' \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \mu_1 & \mu_2 \end{pmatrix} A^{-1} \quad \textit{and} \quad \begin{pmatrix} \gamma_3' \\ \mu_3' \end{pmatrix} = \begin{pmatrix} \gamma_3 \\ \mu_3 \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \mu_1 & \mu_2 \end{pmatrix} A^{-1} \begin{pmatrix} \xi \\ \eta \end{pmatrix} .$$

Proof. The barycentric coordinates of (α, β) with respect to the triangle $\overline{P_i P_k P_r}$ coincide with the barycentric coordinates of (α', β') with respect to the triangle $\overline{P_i' P_k' P_r'}$. Therefore, the number $\bar{\theta}$ is the same for both polygons and the same happens for $\breve{\theta}$. Hence,

$$f' = f \circ R^{-1} \quad \text{and} \quad g' = g \circ R^{-1}$$

are admissible functions for Π' and (α', β'). Moreover,

$$\begin{pmatrix} f'(x,y) \\ g'(x,y) \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \mu_1 & \mu_2 \end{pmatrix} A^{-1} \begin{pmatrix} x - \xi \\ y - \eta \end{pmatrix} + \begin{pmatrix} \gamma_3 \\ \mu_3 \end{pmatrix}$$

which yields (b) and finishes the proof. $\square$

Lemma 3.5 *There exists an affine bijection* R *such that*

(i) $\mu_1' = \mu_2' = 1$,
(ii) $\gamma_1', \gamma_2' > 0$ *or* $\gamma_1' = \gamma_2' < 0$,
(iii) $y_j' \geq 0$ *for* $1 \leq j \leq N$.

Proof. Since both of the functions f and g are non-constant, the vectors (γ_1, γ_2) and (μ_1, μ_2) are non-zero. Hence,

$$\lambda = \operatorname{rank} \begin{pmatrix} \gamma_1 & \gamma_2 \\ \mu_1 & \mu_2 \end{pmatrix} \geq 1 \, .$$

If $\lambda = 1$ there is a regular matrix B with

$$B \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad B \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma \end{pmatrix} \quad \text{with} \quad \gamma \neq 0.$$

Put $A = (B^t)^{-1}$ and let

$$\begin{pmatrix} \bar{x}_j \\ \bar{y}_j \end{pmatrix} = A \begin{pmatrix} x_j \\ y_j \end{pmatrix} \quad , \quad j = 1, ..., N \, .$$

With $y_0 = \min\{\bar{y}_1, ..., \bar{y}_N\}$ we define the affine bijection R via

$$R \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ y_0 \end{pmatrix}$$

and obtain immediately (iii). To see (i) and (ii) we apply Lemma 3.4/(b) and obtain

$$\begin{pmatrix} \gamma_1' & \gamma_2' \\ \mu_1' & \mu_2' \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \mu_1 & \mu_2 \end{pmatrix} B^t = \begin{pmatrix} \gamma & \gamma \\ 1 & 1 \end{pmatrix} .$$

If $\lambda = 2$ we simply put

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \mu_1 & \gamma_1 \\ \mu_2 & \gamma_2 \end{pmatrix}^{-1} ,$$

and proceed as above to derive that $\gamma_1' = 1, \gamma_2' = 2, \mu_1' = \mu_2' = 1$ and $y_j' \geq 0$ for $1 \leq j \leq N$. This finishes the proof. $\square$

We are now ready to establish the main result of this section. It shows that if an N-tuple is formed by spaces of class $\mathcal{C}(\theta_j; X, Y)$ then the spaces associated to a polygon can be compared with sums and intersections of real interpolation spaces.

Theorem 3.6 *Let* $\Pi = \overline{P_1 \cdots P_N}$ *be a convex polygon with* $P_j = (x_j, y_j)$ *and let* $(\alpha, \beta) \in$ *Int* Π. *Suppose that* $\{X, Y\}$ *is a Banach couple and that* $\bar{A} =$

$\{A_1, ..., A_N\}$ is a Banach N-tuple formed by spaces A_j of class $\mathcal{C}(\theta_j; X, Y)$ with $0 \le \theta_j \le 1$, $j = 1, ..., N$. Assume also that for each $\{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}$, where $(\alpha, \beta) \in Int\ \overline{P_i P_k P_r}$, the numbers $\theta_i, \theta_k, \theta_r$ are not all equal. If $(\alpha, \beta) \in \overline{P_i P_k}$ we assume that $\theta_i \neq \theta_k$. Then for any $1 \le q \le \infty$ we have the embeddings

(i)

$$\bar{A}_{(\alpha,\beta),q;K} \hookrightarrow (X, Y)_{\bar{\theta},q} + (X, Y)_{\breve{\theta},q} \ ,$$

(ii)

$$(X, Y)_{\bar{\theta},q} \cap (X, Y)_{\breve{\theta},q} \hookrightarrow \bar{A}_{(\alpha,\beta),q;J} \ ,$$

where $0 < \bar{\theta}, \breve{\theta} < 1$ are given in Definition 3.1.

Proof. Let us apply Lemma 3.3 to find admissible functions

$$f(x, y) = \gamma_1 x + \gamma_2 y + \gamma_3 \quad , \quad g(x, y) = \mu_1 x + \mu_2 y + \mu_3 \, .$$

According to Lemma 3.5 and the invariance under affine bijections, we can assume without restriction that we have $y_j \ge 0$ for $j = 1, ..., N$, $\mu_1 = \mu_2 = 1$ and either

(Case A) $\gamma_1, \gamma_2 > 0$ or
(Case B) $\gamma_1 = \gamma_2 = \gamma < 0$.

Now we shall modify the arguments in [16, Lemma 2] to prove the embeddings in (i) and (ii). We start with some estimates for K- and J-functionals. Consider first the Case A. Let $s \ge t$ and $0 < t \le 1$. Since $f(x_j, y_j) \le \theta_j$ and A_j is of class $\mathcal{C}(\theta_j; X, Y)$ (see, in particular, (2.3)), for $a \in \Sigma(\bar{A})$ we obtain

$$\begin{aligned} K(t^{\gamma_1}, s^{\gamma_2}; a) &= \inf\left\{ \sum_{j=1}^{N} t^{\gamma_1 x_j} s^{\gamma_2 y_j} \|a_j\|_{A_j} \ : \ a = \sum_{j=1}^{N} a_j \ , \ a_j \in A_j \right\} \\ &= \inf\left\{ \sum_{j=1}^{N} t^{\gamma_1 x_j + \gamma_2 y_j} (s/t)^{\gamma_2 y_j} \|a_j\|_{A_j} \right\} \\ &\ge \inf\left\{ \sum_{j=1}^{N} t^{f(x_j,y_j) - \gamma_3} \|a_j\|_{A_j} \right\} \\ &\ge \inf\left\{ \sum_{j=1}^{N} t^{\theta_j - \gamma_3} \|a_j\|_{A_j} \right\} \\ &\gtrsim \inf\left\{ \sum_{j=1}^{N} t^{-\gamma_3} \bar{K}(t, a_j) \right\} \\ &\ge t^{-\gamma_3} \bar{K}(t, a) \, . \end{aligned} \tag{3.1}$$

A similar calculation gives (3.1) for $0 < s \le t \le 1$ in Case B.

In both cases, using that $g(x_j, y_j) = x_j + y_j + \mu_3 \geq \theta_j$, for $s \geq t \geq 1$ we obtain analogously

$$K(t, s; a) \gtrsim t^{-\mu_3} \bar{K}(t, a)\,. \tag{3.2}$$

Let us turn to estimates for J-functionals. In Case A, using (2.4), we obtain for $t \geq 1$, $0 < s \leq t$ and $u \in X \cap Y$

$$\begin{aligned} J(t^{\gamma_1}, s^{\gamma_2}; u) &= \max\{t^{f(x_j,y_j)-\gamma_3}(s/t)^{\gamma_2 y_j}\|u\|_{A_j} \;:\; 1 \leq j \leq N\} \\ &\leq \max\{t^{f(x_j,y_j)-\gamma_3}\|u\|_{A_j} \;:\; 1 \leq j \leq N\} \\ &\lesssim t^{-\gamma_3} \max\{t^{f(x_j,y_j)-\theta_j} \bar{J}(t, u) \;:\; 1 \leq j \leq N\} \\ &\leq t^{-\gamma_3} \bar{J}(t, u)\,. \end{aligned} \tag{3.3}$$

A similar calculation gives (3.3) for $t \geq 1$ and $s \geq t$ in Case B.

In both cases, since $g(x_j, y_j) = x_j + y_j + \mu_3 \geq \theta_j$, for $0 < s \leq t \leq 1$ we get

$$J(t, s; u) \lesssim t^{-\mu_3} \bar{J}(t, u)\,. \tag{3.4}$$

To prove the embedding (i) we start with $a \in \bar{A}_{(\alpha,\beta),q;K}$. By change of variable we get

$$\|a\|^q_{\bar{A}_{(\alpha,\beta),q;K}} \sim \int_0^\infty \int_0^\infty [t^{-\gamma_1\alpha} s^{-\gamma_2\beta} K(t^{\gamma_1}, s^{\gamma_2}; a)]^q \,\frac{ds}{s}\frac{dt}{t}\,. \tag{3.5}$$

In Case A, using (3.5) and (3.1), we obtain

$$\begin{aligned} \|a\|^q_{\bar{A}_{(\alpha,\beta),q;K}} &\gtrsim \int_0^1 \int_t^{2t} [t^{-\gamma_1\alpha} s^{-\gamma_2\beta} K(t^{\gamma_1}, s^{\gamma_2}; a)]^q \,\frac{ds}{s}\frac{dt}{t} \\ &\gtrsim \int_0^1 \int_t^{2t} [t^{-\gamma_1\alpha} s^{-\gamma_2\beta} t^{-\gamma_3} \bar{K}(t, a)]^q \,\frac{ds}{s}\frac{dt}{t} \\ &= \int_0^1 [t^{-\gamma_1\alpha} t^{-\gamma_3} \bar{K}(t, a)]^q \Big(\int_t^{2t} s^{-\gamma_2\beta q}\,\frac{ds}{s}\Big)\,\frac{dt}{t} \\ &\sim \int_0^1 [t^{-f(\alpha,\beta)} \bar{K}(t, a)]^q \,\frac{dt}{t} \\ &= \int_0^1 [t^{-\bar{\theta}} \bar{K}(t, a)]^q \,\frac{dt}{t}\,. \end{aligned} \tag{3.6}$$

In Case B we start with

$$\|a\|^q_{\bar{A}_{(\alpha,\beta),q;K}} \gtrsim \int_0^1 \int_{t/2}^{t} [t^{-\gamma_1\alpha} s^{-\gamma_2\beta} K(t^{\gamma_1}, s^{\gamma_2}; a)]^q \, \frac{ds}{s} \frac{dt}{t}$$

and end up with (3.6) as well.

Furthermore, since $\mu_1 = \mu_2 = 1$ and (3.2), in both cases we derive

$$\begin{aligned} \|a\|^q_{\bar{A}_{(\alpha,\beta),q;K}} &\geq \int_1^\infty \int_t^{2t} [t^{-\alpha} s^{-\beta} K(t, s; a)]^q \, \frac{ds}{s} \frac{dt}{t} \\ &\gtrsim \int_1^\infty [t^{-g(\alpha,\beta)} \bar{K}(t, a)]^q \, \frac{dt}{t} \\ &= \int_1^\infty [t^{-\breve{\theta}} \bar{K}(t, a)]^q \, \frac{dt}{t} \,. \end{aligned} \tag{3.7}$$

Using inequalities (3.6) and (3.7) and Holmstedt's formula (see [3, Theorem 3.6.1] or [2, Theorem 5.2.1]) we see that $a \in (X, Y)_{\bar{\theta},q} + (X, Y)_{\breve{\theta},q}$ and we derive embedding (i).

Next we proceed with the embedding (ii). Take any $a \in (X, Y)_{\bar{\theta},q} \cap (X, Y)_{\breve{\theta},q}$. According to the fundamental lemma (see [3, Lemma 3.3.2]), there is a representation of $a = \int_0^\infty u(t)\, dt/t$ such that for all $0 < t < \infty$

$$\bar{J}(t, u(t)) \lesssim \bar{K}(t, a)\,. \tag{3.8}$$

In Case A we define the function $v : (0, \infty) \times (0, \infty) \to X \cap Y$ by

$$v(t, s) = \begin{cases} u(t) & \text{if } 0 < t \leq 1 \text{ and } t/e \leq s \leq t \\ \frac{1}{\gamma_1\gamma_2} u(t^{1/\gamma_1}) & \text{if } 1 < t < \infty \text{ and } t^{1/\gamma_1}/e \leq s^{1/\gamma_2} \leq t^{1/\gamma_1} \\ 0 & \text{otherwise}\,. \end{cases}$$

Then

$$\int_0^\infty\int_0^\infty v(t,s)\,\frac{ds}{s}\frac{dt}{t} = \int_0^1\int_0^\infty v(t,s)\,\frac{ds}{s}\frac{dt}{t} + \int_1^\infty\int_0^\infty \gamma_1\gamma_2 v(t^{\gamma_1},s^{\gamma_2})\,\frac{ds}{s}\frac{dt}{t}$$
$$= \int_0^1\int_{t/e}^t u(t)\,\frac{ds}{s}\frac{dt}{t} + \int_1^\infty\int_{t/e}^t u(t)\,\frac{ds}{s}\frac{dt}{t}$$
$$= \int_0^\infty u(t)\frac{dt}{t} = a\,.$$

We obtain by using (3.3)

$$\begin{aligned}\int_1^\infty [t^{-\bar\theta}\bar J(t,u(t))]^q\,\frac{dt}{t} &= \int_1^\infty [t^{-\gamma_1\alpha-\gamma_2\beta-\gamma_3}\bar J(t,u(t))]^q\,\frac{dt}{t}\\ &\gtrsim \int_1^\infty\int_{t/e}^t [t^{-\gamma_1\alpha}s^{-\gamma_2\beta}J(t^{\gamma_1},s^{\gamma_2};u(t))]^q\,\frac{ds}{s}\frac{dt}{t}\\ &\sim \int_1^\infty\int_0^\infty [t^{-\gamma_1\alpha}s^{-\gamma_2\beta}J(t^{\gamma_1},s^{\gamma_2};v(t^{\gamma_1},s^{\gamma_2}))]^q\,\frac{ds}{s}\frac{dt}{t}\\ &\sim \int_1^\infty\int_0^\infty [t^{-\alpha}s^{-\beta}J(t,s;v(t,s))]^q\,\frac{ds}{s}\frac{dt}{t}\,. \end{aligned} \tag{3.9}$$

Similarly, we estimate by using (3.4)

$$\int_0^1 [t^{-\breve\theta}\bar J(t,u(t))]^q\,\frac{dt}{t} \gtrsim \int_0^1\int_0^\infty [t^{-\alpha}s^{-\beta}J(t,s;v(t,s))]^q\,\frac{ds}{s}\frac{dt}{t}\,. \tag{3.10}$$

Finally, (3.8), (3.9) and (3.10) yield

$$\begin{aligned}\|a\|^q_{(X,Y)_{\bar\theta,q}\cap(X,Y)_{\breve\theta,q}} &\sim \int_0^\infty [t^{-\breve\theta}\bar K(t,a)]^q\,\frac{dt}{t} + \int_0^\infty [t^{-\bar\theta}\bar K(t,a)]^q\,\frac{dt}{t}\\ &\gtrsim \int_0^1 [t^{-\breve\theta}\bar J(t,u(t))]^q\,\frac{dt}{t} + \int_1^\infty [t^{-\bar\theta}\bar J(t,u(t))]^q\,\frac{dt}{t}\\ &\gtrsim \int_0^\infty\int_0^\infty [t^{-\alpha}s^{-\beta}J(t,s;v(t,s))]^q\,\frac{ds}{s}\frac{dt}{t}\,, \end{aligned} \tag{3.11}$$

which proves (ii) in Case A.

For Case B we modify the definition of $v(t,s)$ as follows

$$v(t,s) = \begin{cases} u(t) & \text{if } 0 < t \le 1 \ \text{ and } \ t/e^2 \le s < t/e \\ -\frac{1}{\gamma} u(t^{1/\gamma}) & \text{if } 0 < t \le 1 \ \text{ and } \ t^{1/\gamma} \le s^{1/\gamma} \le e^{-1/\gamma} t^{1/\gamma} \\ 0 & \text{otherwise}. \end{cases}$$

Again, we have

$$\begin{aligned} \int_0^\infty \int_0^\infty v(t,s) \frac{ds}{s}\frac{dt}{t} &= \int_0^1 \int_{t/e^2}^{t/e} u(t) \frac{ds}{s}\frac{dt}{t} - \frac{1}{\gamma}\int_0^1 \int_{t/e}^{t} u(t^{1/\gamma}) \frac{ds}{s}\frac{dt}{t} \\ &= \int_0^1 u(t) \frac{dt}{t} - \frac{1}{\gamma}\int_0^1 u(t^{1/\gamma}) \frac{dt}{t} \\ &= \int_0^\infty u(t) \frac{dt}{t} = a\,. \end{aligned}$$

Using (3.3) we obtain

$$\begin{aligned} \int_1^\infty [t^{-\bar\theta} \bar J(t,u(t))]^q \frac{dt}{t} &= \int_1^\infty [t^{-\gamma\alpha-\gamma\beta-\gamma_3} \bar J(t,u(t))]^q \frac{dt}{t} \\ &\gtrsim \int_1^\infty \int_t^{te^{-1/\gamma}} [t^{-\gamma\alpha} s^{-\gamma\beta} J(t^\gamma, s^\gamma; u(t))]^q \frac{ds}{s}\frac{dt}{t} \\ &\sim \int_1^\infty \int_t^{te^{-1/\gamma}} [t^{-\gamma\alpha} s^{-\gamma\beta} J(t^\gamma, s^\gamma; v(t^\gamma, s^\gamma))]^q \frac{ds}{s}\frac{dt}{t} \\ &\sim \int_1^\infty \int_{t^\gamma/e}^{t^\gamma} [t^{-\gamma\alpha} s^{-\beta} J(t^\gamma, s; v(t^\gamma, s))]^q \frac{ds}{s}\frac{dt}{t} \\ &\sim \int_0^1 \int_{t/e}^{t} [t^{-\alpha} s^{-\beta} J(t, s; v(t, s))]^q \frac{ds}{s}\frac{dt}{t}\,. \end{aligned} \tag{3.12}$$

With the aid of (3.4) we get similarly

$$\int_0^1 [t^{-\breve\theta} \bar J(t,u(t))]^q \frac{dt}{t} \gtrsim \int_0^1 \int_{t/e^2}^{t/e} [t^{-\alpha} s^{-\beta} J(t,s;v(t,s))]^q \frac{ds}{s}\frac{dt}{t}\,. \tag{3.13}$$

The same computations as done in (3.11) but using now (3.12) and (3.13) yield (ii) in Case B. The proof is complete. □

Next we derive the reiteration result. We require a stronger assumption than in Theorem 3.6. Namely, (α, β) must not lie in any diagonal of Π. As we shall show in the next section, that assumption is essential for the result.

Theorem 3.7 *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$ and let $(\alpha, \beta) \in Int\, \Pi$ such that (α, β) does not lie in any diagonal of Π. Suppose that $\{X, Y\}$ is a Banach couple and that $\bar{A} = \{A_1, ..., A_N\}$ is a Banach N-tuple formed by spaces A_j of class $\mathcal{C}(\theta_j; X, Y)$ with $0 \leq \theta_j \leq 1$, $j = 1, ..., N$. Assume also that for each $\{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}$ the numbers $\theta_i, \theta_k, \theta_r$ are not all equal. Then, for any $1 \leq q \leq \infty$, we have with equivalent norms*

(i)

$$\bar{A}_{(\alpha,\beta),q;K} = (X, Y)_{\bar{\theta},q} + (X, Y)_{\breve{\theta},q} \ , \tag{3.14}$$

(ii)

$$\bar{A}_{(\alpha,\beta),q;J} = (X, Y)_{\bar{\theta},q} \cap (X, Y)_{\breve{\theta},q} \, , \tag{3.15}$$

where $0 < \bar{\theta}, \breve{\theta} < 1$ are defined in Definition 3.1.

Proof. It follows from (2.5) and (2.6) that

$$\bar{A}_{(\alpha,\beta),q;J} \hookrightarrow (X, Y)_{\bar{\theta},q} \cap (X, Y)_{\breve{\theta},q}$$

and

$$(X, Y)_{\bar{\theta},q} + (X, Y)_{\breve{\theta},q} \hookrightarrow \bar{A}_{(\alpha,\beta),q;K} \, .$$

The converse inclusions follow by Theorem 3.6. □

Remark 3.8 *Applying Theorem 3.7 to Π equal to the unit square we get an improvement of [7, Theorem 3.1] by relaxing the conditions on $\theta_1, \theta_2, \theta_3, \theta_4$.*

Next, we write down two concrete applications of Theorem 3.7. Let (Ω, μ) be a σ-finite measure space. For $1 < p < \infty$ and $1 \leq q \leq \infty$, the Lorentz function space $L_{p,q}$ consists of all (equivalence classes of) measurable functions f on Ω which have a finite norm

$$\|f\|_{L_{p,q}} = \Big(\int_0^{\mu(\Omega)} \Big[t^{1/p-1} \int_0^t f^*(s)\, ds \Big]^q \frac{dt}{t} \Big)^{1/q} \, ,$$

(with the usual modification if $q = \infty$) where f^* stands for the non-increasing rearrangement of f

$$f^*(s) = \inf\{\gamma > 0 \ : \ \mu(\{x \in \Omega : |f(x)| > \gamma\}) \leq s\} \, .$$

Since $(L_\infty, L_1)_{\theta,q} = L_{p,q}$ for $1/p = \theta$ (see [2,3,24]), according to Theorem 3.7 we obtain the following.

Corollary 3.9 *Let* $\Pi = \overline{P_1 \cdots P_N}$ *be a convex polygon with* $P_j = (x_j, y_j)$ *and let* $(\alpha, \beta) \in$ *Int* Π *such that* (α, β) *does not lie in any diagonal of* Π. *Assume that* $1 < p_j < \infty$, $1 \le q_j, q \le \infty$, $j = 1, ..., N$, *such that for each* $\{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}$ *the numbers* p_i, p_k, p_r *are not all equal. We put*

$$1/p_{ikr} = c_i/p_i + c_k/p_k + c_r/p_r \quad , \quad \{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}$$

where (c_i, c_k, c_r) *are the barycentric coordinates of* (α, β) *with respect to* P_i, P_k, P_r. *Let*

$$1/\bar{p} = \min\{1/p_{ikr} \ : \ \{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}\}, \quad 1/\breve{p} = \max\{1/p_{ikr} \ : \ \{i, k, r\} \in \mathcal{P}_{(\alpha,\beta)}\} .$$

Then we have, with equivalent norms,

$$(L_{p_1,q_1}, ..., L_{p_N,q_N})_{(\alpha,\beta),q;K} = L_{\breve{p},q} + L_{\bar{p},q}$$

and

$$(L_{p_1,q_1}, ..., L_{p_N,q_N})_{(\alpha,\beta),q;J} = L_{\breve{p},q} \cap L_{\bar{p},q} .$$

Next, we consider Besov spaces. Let $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing complex-valued infinitely differentiable functions on $\mathbb{R}^d$, and the space of tempered distributions on $\mathbb{R}^d$, respectively. For $f \in \mathcal{S}'(\mathbb{R}^d)$, we denote by $\hat{f}$ the Fourier transform and by $\check{f}$ the inverse Fourier transform. Let φ_0 be a $C^\infty(\mathbb{R}^d)$-function with

$$\varphi_0(x) = \begin{cases} 1 \text{ if } \|x\|_{\mathbb{R}^d} \le 1 \\ 0 \text{ if } \|x\|_{\mathbb{R}^d} > 2 . \end{cases}$$

For $j \in \mathbb{N}$, we put $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$.

Let $1 < p < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$. The Besov space $B^s_{p,q} = B^s_{p,q}(\mathbb{R}^d)$ is formed by all $f \in \mathcal{S}'(\mathbb{R}^d)$ having a finite norm

$$\|f\|_{B^s_{p,q}} = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|(\varphi_j \hat{f})^\vee\|^q_{L_p(\mathbb{R}^d)} \Big)^{1/q}$$

(see [24,25]). If $s_0 < s_1$, then $B^{s_1}_{p,q} \hookrightarrow B^{s_0}_{p,q}$. Moreover, if $-\infty < s_0 \ne s_1 < \infty$, $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $1 < p < \infty$ and $1 \le q_0, q_1, q \le \infty$, then

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})_{\theta,q} = B^s_{p,q} .$$

Hence, as a direct consequence of Theorem 3.7 we derive the following.

Corollary 3.10 *Let* $\Pi = \overline{P_1 \cdots P_N}$ *be a convex polygon with* $P_j = (x_j, y_j)$ *and let* $(\alpha, \beta) \in$ *Int* Π *such that* (α, β) *does not lie in any diagonal of* Π. *Suppose* *that* $1 < p < \infty$, $1 \le q_j, q \le \infty, -\infty < s_j < \infty, j = 1, ..., N$, *such that for each* $\{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)}$ *the numbers* s_i, s_k, s_r *are not all equal. We put*

$$s_{ikr} = c_i s_i + c_k s_k + c_r s_r \quad , \quad \{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)} \quad ,$$

where (c_i, c_k, c_r) *are the barycentric coordinates of* (α, β) *with respect to* P_i, P_k, P_r*. Let*

$$\bar{s} = \min\{s_{ikr} \ : \ \{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)}\} \quad , \quad \check{s} = \max\{s_{ikr} \ : \ \{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)}\} \, .$$

Then we have with equivalent norms

$$(B^{s_1}_{p,q_1}, ..., B^{s_N}_{p,q_N})_{(\alpha,\beta),q;K} = B^{\bar{s}}_{p,q}$$

and

$$(B^{s_1}_{p,q_1}, ..., B^{s_N}_{p,q_N})_{(\alpha,\beta),q;J} = B^{\check{s}}_{p,q} \, .$$

4 Counterexamples

Suppose now that (α, β) lies in any diagonal of Π, say $\overline{P_i P_k}$ (see Figure 4.1).

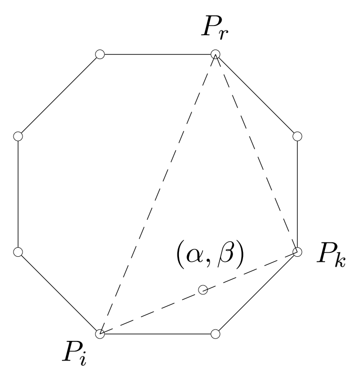

Figure 4.1

Then for any triangle $\overline{P_i P_k P_r}$ we have $\{i,k,r\} \in \mathcal{P}_{(\alpha,\beta)}$ but now (α, β) is not in the interior of $\overline{P_i P_k P_r}$. As a consequence, see [10, Lemma 1.1], if $q < \infty$ then $(A_i, A_k, A_r)_{(\alpha,\beta),q;K} = \{0\}$, and (2.6) fails in this case. However, for the K-method with $q = \infty$ and the J-method with $q = 1$, it has been shown by Cobos, Fernández-Cabrera and Martín in [7, Theorem 4.1] that the conclusion of Theorem 3.7 is still valid under the assumptions of Theorem 3.6. We consider now the other cases. By means of examples, we shall show that embeddings

$$(X,Y)_{\bar{\theta},q} + (X,Y)_{\check{\theta},q} \hookrightarrow \bar{A}_{(\alpha,\beta),q;K} \tag{4.1}$$

and

$$\bar{A}_{(\alpha,\beta),q;J} \hookrightarrow (X,Y)_{\bar{\theta},q} \cap (X,Y)_{\breve{\theta},q} \tag{4.2}$$

may not hold. Hence, Theorem 3.7 fails in general if $1 < q < \infty$ and (α, β) is in any diagonal.

Counterexample 4.1 Take any $1 \le q < \infty$ and assume that Π is the unit square $\Pi = \overline{(0,0)(1,0)(0,1)(1,1)}$. Consider the interior point (α, α) with $1/2 < \alpha < 1$, and take the 4-tuple $\{X, X, X, Y\}$, where X, Y are Banach spaces with $X \hookrightarrow Y$ (see Figure 4.2). Hence, $\theta_1 = \theta_2 = \theta_3 = 0$ and $\theta_4 = 1$.

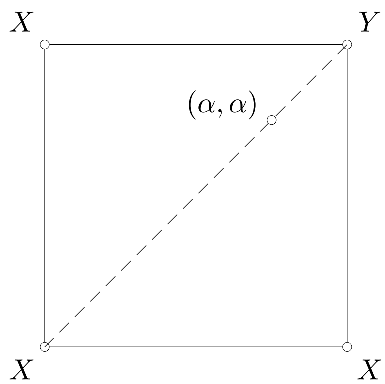

Figure 4.2

For any $\{i, k, r\} \in \mathcal{P}_{(\alpha,\alpha)}$ the numbers $\theta_i, \theta_k, \theta_r$ are not all equal. Moreover, $\theta_1 \neq \theta_4$. Since $\theta_{124} = \theta_{134} = \alpha$ and $\theta_{234} = 2\alpha - 1$, we have $\bar{\theta} = 2\alpha - 1$ and $\breve{\theta} = \alpha$. Assumption $X \hookrightarrow Y$ implies that $(X,Y)_{\bar{\theta},q} \hookrightarrow (X,Y)_{\breve{\theta},q}$. Therefore

$$(X,Y)_{\bar{\theta},q} + (X,Y)_{\breve{\theta},q} = (X,Y)_{\alpha,q} .$$

To determine the space $(X, X, X, Y)_{(\alpha,\alpha),q;K}$ we put

$$\omega_j = \iint_{\Omega_j} [(ts)^{-\alpha} K(t,s;a)]^q \, \frac{dt}{t}\frac{ds}{s} \quad , \quad j = 1, 2, 3 ,$$

and define ω'_j similarly. Here Ω_j and Ω'_j are the sets described in Figure 4.3.

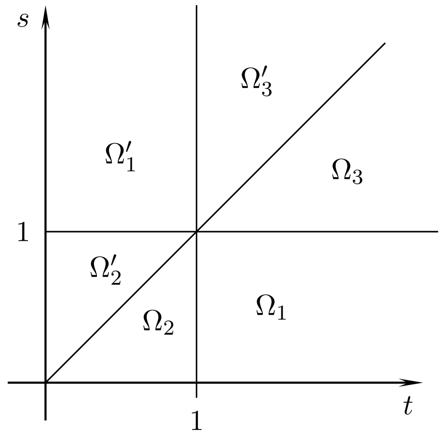

Figure 4.3

We have

$$\|a\|^q_{(X,X,X,Y)_{(\alpha,\alpha),q;K}} = \sum_{j=1}^{3}(\omega_j + \omega'_j)\,.$$

One can estimate each of those integrals by using that

$$K(t,s;a) = \min\{1,t,s\}\bar{K}\Big(\frac{st}{\min\{1,t,s\}}, a\Big), \quad a \in Y.$$

For ω_1 we get

$$\begin{aligned}\omega_1 &= \int_1^\infty \int_0^1 [(ts)^{-\alpha} s\bar{K}(t,a)]^q \,\frac{ds}{s}\frac{dt}{t}\\ &\sim \int_1^\infty [t^{-\alpha}\bar{K}(t,a)]^q \,\frac{dt}{t}\,.\end{aligned}$$

For ω_2, using that $\bar{K}(t,a) \sim t\|a\|_Y$ if $t \leq 1$, we obtain

$$\begin{aligned}\omega_2 &= \int_0^1 \int_0^t [(ts)^{-\alpha} s\bar{K}(t,a)]^q \,\frac{ds}{s}\frac{dt}{t}\\ &\sim \int_0^1 [t^{1-2\alpha}\bar{K}(t,a)]^q \,\frac{dt}{t}\\ &\sim \|a\|_Y^q \lesssim \int_1^\infty [t^{-\alpha}\bar{K}(t,a)]^q \,\frac{dt}{t}\,.\end{aligned}$$

For ω_3 we derive

$$\omega_3 = \int_1^\infty \int_s^\infty [(ts)^{-\alpha} \bar{K}(st,a)]^q \frac{dt}{t}\frac{ds}{s}$$
$$= \int_1^\infty \int_{s^2}^\infty [w^{-\alpha} \bar{K}(w,a)]^q \frac{dw}{w}\frac{ds}{s}$$
$$= \int_1^\infty [w^{-\alpha} \bar{K}(w,a)]^q \int_1^{\sqrt{w}} \frac{ds}{s}\,\frac{dw}{w}$$
$$\sim \int_1^\infty [w^{-\alpha}(\log w)^{1/q} \bar{K}(w,a)]^q \frac{dw}{w}$$
$$\sim \int_1^\infty [w^{-\alpha}(1+\log w)^{1/q} \bar{K}(w,a)]^q \frac{dw}{w}\,.$$

Due to the symmetry between Ω_j and Ω'_j, the remaining terms can be estimated similarly. Consequently,

$$\|a\|_{(X,X,X,Y)_{(\alpha,\alpha),q;K}} \sim \Big(\int_1^\infty [t^{-\alpha}(1+\log t)^{1/q} \bar{K}(t,a)]^q \frac{dt}{t}\Big)^{1/q}. \tag{4.3}$$

Now, consider $[0,1]$ with the usual Lebesgue measure and take $X = L_\infty$ and $Y = L_1$. Then $(L_\infty, L_1)_{\alpha,q} = L_{1/\alpha,q}$. By interpolating the 4-tuple it follows from (4.3) and the well-known equality $\bar{K}(t,f) = t\int_0^{1/t} f^*(s)\,ds$ that

$$(L_\infty, L_\infty, L_\infty, L_1)_{(\alpha,\alpha),q;K}$$
$$= \Big\{ f \ :\ \Big(\int_0^1 \Big[t^{\alpha-1}(1+|\log t|)^{1/q} \int_0^t f^*(s)\,ds \Big]^q \frac{dt}{t}\Big)^{1/q} < \infty \Big\}.$$

The last space is the Lorentz-Zygmund function space $L_{1/\alpha,q}(\log L)_{1/q}$ (see [2,15]) . Clearly, $L_{1/\alpha,q} \not\subseteq L_{1/\alpha,q}(\log L)_{1/q}$, so (4.1) fails in this case.

Counterexample 4.2 Take any $1 < q \le \infty$ and the unit square $\Pi = \overline{(0,0)(1,0)(0,1)(1,1)}$. Consider now the point (α,α) with $0 < \alpha < 1/2$ and the 4-tuple $\{X,Y,Y,Y\}$ where $X \hookrightarrow Y$. Then we have $\theta_1 = 0, \theta_2 = \theta_3 = \theta_4 = 1$. For any $\{i,k,r\} \in \mathcal{P}_{(\alpha,\alpha)}$ the numbers $\theta_i, \theta_k, \theta_r$ are not all equal and $\theta_1 \ne \theta_4$. Since $\theta_{123} = 2\alpha$ and $\theta_{124} = \theta_{134} = \alpha$, we obtain $\bar{\theta} = \alpha, \breve{\theta} = 2\alpha$ and

$$(X,Y)_{\bar{\theta},q} \cap (X,Y)_{\breve{\theta},q} = (X,Y)_{\alpha,q}\,.$$

We are going to show that

$$(X,Y)_{\rho,q} \hookrightarrow (X,Y,Y,Y)_{(\alpha,\alpha),q;J}\ , \tag{4.4}$$

where $\rho(t) = t^\alpha(1+|\log t|)^{1/q'}$, $1/q+1/q'=1$, and $(X,Y)_{\rho,q}$ is the J-space with function parameter ρ, that is to say the collection of all those $a \in Y$ for which there is a strongly measurable X-valued function $u(t)$ such that $a = \int_1^\infty u(t)\,dt/t$ (convergence in Y) and $(\int_1^\infty [(1/\rho(t))\bar{J}(t,u(t))]^q\,dt/t)^{1/q} < \infty$. The norm is defined in the usual way

$$\|a\|_{(X,Y)_{\rho,q}} = \inf\Big\{\Big(\int_1^\infty [\frac{1}{\rho(t)}\bar{J}(t,u(t))]^q\,\frac{dt}{t}\Big)^{1/q} \;:\; a = \int_1^\infty u(t)\,\frac{dt}{t}\Big\}.$$

(See [21,22] for details on the real method with a function parameter; since $X \hookrightarrow Y$, we consider only integrals on $(1,\infty)$.)

Let $a = \int_1^\infty u(t)\,dt/t$ with

$$\Big(\int_1^\infty [\frac{1}{\rho(t)}\bar{J}(t,u(t))]^q\,\frac{dt}{t}\Big)^{1/q} \le 2\|a\|_{(X,Y)_{\rho,q}}\,.$$

We put

$$v(t,s) = \begin{cases} \frac{u(ts)}{1+\log(ts)} & \text{if } 1/e \le t \le e \text{ and } \max\{1/t,1\} \le s \le e/t \\ 2\frac{u(ts)}{1+\log(ts)} & \text{if } 1 < t < \infty \text{ and } \max\{e/t,1\} < s \le et \\ 0 & \text{otherwise}. \end{cases}$$

A change of variable for s yields

$$\int_0^\infty\int_0^\infty v(t,s)\,\frac{ds}{s}\frac{dt}{t} = \int_{1/e}^1\int_1^e \frac{u(w)}{1+\log w}\,\frac{dw}{w}\,\frac{dt}{t} + \int_1^e\int_t^e \frac{u(w)}{1+\log w}\,\frac{dw}{w}\,\frac{dt}{t}$$
$$+2\int_1^e\int_e^{et^2} \frac{u(w)}{1+\log w}\,\frac{dw}{w}\,\frac{dt}{t} + 2\int_e^\infty\int_t^{et^2} \frac{u(w)}{1+\log w}\,\frac{dw}{w}\,\frac{dt}{t}.$$

Changing the order of integration and combining the first two integrals as well as the second two integrals we obtain

$$\int_0^\infty\int_0^\infty v(t,s)\,\frac{ds}{s}\frac{dt}{t} = \int_1^e \frac{u(w)}{1+\log w}\int_{1/e}^w \frac{dt}{t}\frac{dw}{w} + \int_e^\infty 2\frac{u(w)}{1+\log w}\int_{\sqrt{w/e}}^w \frac{dt}{t}\frac{dw}{w}$$
$$= \int_1^\infty u(w)\,\frac{dw}{w} = a\,.$$

Consequently,

$$\|a\|^q_{(X,Y,Y,Y)_{(\alpha,\alpha),q;J}} \leq \Big(\int_{1/e}^{1}\int_{1/t}^{e/t} + \int_{1}^{e}\int_{1}^{e/t} + \int_{1}^{e}\int_{e/t}^{et} + \int_{e}^{\infty}\int_{1}^{et}\Big)[(ts)^{-\alpha}J(t,s;v(t,s))]^q \frac{ds}{s}\frac{dt}{t}.$$

In the domains of the last three integrals we have $\max\{t,s,ts\} = ts$. Thus, we get

$$J(t,s;v(t,s)) \sim \frac{1}{1+\log(ts)}\bar{J}(ts,u(ts))\,.$$

In the domain of the first integral we have $t \leq ts \leq s \leq ets$. Hence, we derive

$$J(t,s;v(t,s)) \sim \frac{1}{1+\log(ts)}\bar{J}(s,u(ts)) \sim \frac{1}{1+\log(ts)}\bar{J}(ts,u(ts))\,.$$

Using this estimate we obtain similar as above after a change of variable and a change of the order of integration (again we combine the first two and the last two integrals)

$$\begin{aligned}\|a\|^q_{(X,Y,Y,Y)_{(\alpha,\alpha),q;J}} &\lesssim \int_1^e [w^{-\alpha}\bar{J}(w,u(w))]^q \frac{1}{(1+\log w)^q}\int_{1/e}^{w}\frac{dt}{t}\frac{dw}{w}\\ &\quad + \int_e^\infty [w^{-\alpha}\bar{J}(w,u(w))]^q \frac{1}{(1+\log w)^q}\int_{\sqrt{w/e}}^{w}\frac{dt}{t}\frac{dw}{w}\\ &\sim \int_1^\infty [\frac{1}{\rho(w)}\bar{J}(w,u(w))]^q \frac{dw}{w}\,.\end{aligned}$$

This establishes (4.4). The converse embedding to (4.4) holds as well, but we do not need it here.

Take now $[0,1]$ with the usual Lebesgue measure and put $X = L_\infty$, $Y = L_1$. Then $(L_\infty, L_1)_{\alpha,q} = L_{1/\alpha,q}$. On the other hand, since $X \hookrightarrow Y$ and the equivalence theorem holds for the function parameter ρ (see [21, Theorem 2.2]), we have for the Lorentz-Zygmund space $L_{1/\alpha,q}(\log L)_{-1/q'}$ that

$$\begin{aligned}&L_{1/\alpha,q}(\log L)_{-1/q'}\\ &= \Big\{f : \Big(\int_0^1 \Big[t^{\alpha-1}(1+|\log t|)^{-1/q'}\int_0^t f^*(s)\,ds\Big]^q \frac{dt}{t}\Big)^{1/q} < \infty\Big\}\\ &= \Big\{f : \Big(\int_1^\infty \Big[t^{-\alpha+1}(1+\log t)^{-1/q'}\int_0^{1/t} f^*(s)\,ds\Big]^q \frac{dt}{t}\Big)^{1/q} < \infty\Big\}\\ &= (L_\infty, L_1)_{\rho,q}\,.\end{aligned}$$

Hence,

$$L_{1/\alpha,q}(\log L)_{-1/q'} \hookrightarrow (L_\infty, L_1, L_1, L_1)_{(\alpha,\alpha),q;J}.$$

Since

$$L_{1/\alpha,q}(\log L)_{-1/q'} \nsubseteq L_{1/\alpha,q}$$

it follows that (4.2) fails in this case.

We finish the paper by recalling that interpolation by the methods associated to the unit square of the 4-tuple $\{X, Y, Y, X\}$ with $X \hookrightarrow Y$ and (α, β) in the diagonals results in extrapolation spaces of the type $(X, Y)_{0,q}$ or $(X, Y)_{1,q}$ (see [7,6]).

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