THE ROLE OF FROLOV'S CUBATURE FORMULA FOR FUNCTIONS WITH BOUNDED MIXED DERIVATIVE

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Abstract. We prove upper bounds on the order of convergence of Frolov's cubature formula for numerical integration in function spaces of dominating mixed smoothness on the unit cube with homogeneous boundary condition. More precisely, we study worst-case integration errors for Besov $\mathbf{B}^s_{p,\theta}$ and Triebel-Lizorkin spaces $\mathbf{F}^s_{p,\theta}$ and our results treat the whole range of admissible parameters ($s \geq 1/p$). In particular, we treat the case of small smoothness which is given for Triebel-Lizorkin spaces $\mathbf{F}^s_{p,\theta}$ in case $1 < \theta < p < \infty$ with $1/p < s \leq 1/\theta$. The presented upper bounds show a completely different behavior of the worst-case error compared to "large" smoothness $s > 1/\theta$. In the latter case the presented upper bounds are optimal, i.e., they can not be improved by any other cubature formula. The optimality for "small" smoothness is open. Moreover, we present a modification of the algorithm which lead to the same bounds (including small smoothness) also for the (larger) spaces of periodic functions.

1. Introduction

The efficient integration of multivariate functions is a crucial task for the numerical treatment of many multi-parameter real-world problems. The computation of the integral can almost never be done analytically since often the available information of the signal or function f is highly incomplete or simply no closed-form solution exists. A cubature rule approximates the integral $I(f) = \int_{[0,1]^d} f(x)\,dx$ by computing a weighted sum of finitely many function values and the d-variate function f is assumed to belong to some (quasi-)normed function space $\mathbf{F}_d \subset L_1([0,1]^d)$. The optimal worst-case error with respect to the class $\mathbf{F}_d$ is given by

$$\text{Int}_n(\mathbf{F}_d) := \inf_{x^1,\ldots,x^n \in [0,1]^d} \; \inf_{\lambda^1,\ldots,\lambda^n \in \mathbb{R}} \; \sup_{\|f\|_{\mathbf{F}_d} \leq 1} \left| I(f) - \frac{1}{n} \sum_{i=1}^n \lambda^i f(x^i) \right| . \tag{1.1}$$

In this paper we (almost) completely answer the question for the correct asymptotical behavior of $\text{Int}_n(\mathbf{F}_d)$ for several classes of functions with dominating mixed smoothness. The by now classical research topic of numerically integrating d-variate functions goes back to the work of Korobov [20], Hlawka [16], and Bakhvalov [2] in the 1960s and was continued later by numerous authors including Temlyakov [32, 33, 34, 36], Dubinin [6, 7], Skriganov [29], Triebel [38], Hinrichs [14, 15], Novak and Woźniakowski [24], Dick and Pillichshammer [5], and Markhasin [22] to mention just a few. In contrast to the quadrature of univariate functions, where equidistant point grids lead to optimal formulas, the multivariate problem is much more involved. In fact, the choice of proper sets $X_n \subset [0,1]^d$ of integration nodes in the d-dimensional unit cube is the essence of "discrepancy theory" and connected with deep problems in number theory, already for $d = 2$.

We study Besov-Triebel-Lizorkin classes $\mathbf{B}^s_{p,\theta}$ and $\mathbf{F}^s_{p,\theta}$ on the unit cube $[0,1]^d$ and provide lower and upper bounds for $\text{Int}_n(\mathbf{A}^s_{p,\theta})$ which are sharp in order. Here and in the following

Date: March 31, 2015.

Key words and phrases. Numerical integration, Besov-Triebel-Lizorkin space, Sobolev space, Frolov's rule.

$\mathbf{A}$ stands for $\mathbf{B}$ or $\mathbf{F}$. As a motivation let us emphasize that Besov regularity is the correct framework when it comes to so-called kink functions, which often occur in mathematical finance, e.g. the pricing of a European call option, whose pay-off function possesses a kink at the strike price [11]. In general, one can not expect Sobolev regularity higher than $r = 3/2$ for f. However, when considering Besov regularity we can achieve smoothness $r = 2$. Indeed, the simple example $f(t) = |\sin(2\pi t)|$ belongs to $\mathbf{B}^2_{1,\infty}(\mathbb{T})$ while its Sobolev regularity in $\mathbf{W}^s_2(\mathbb{T})$ is below $s = 3/2$. In a sense, one sacrifices integrability for gaining regularity. Looking at the bounds (1.3) and (1.4) below we see that certain cubature rules benefit from higher Besov regularity while the integrability p does not enter the picture.

The main goal of this paper is to show that there exists one universal cubature formula which provides the optimal order of convergence for all the different function classes. Universality has to be understood in the following sense: The nodes and weights of the cubature formulas are constructed independently of the parameters of the spaces, i.e. independent of s, p and θ. This method was introduced by Frolov [10] in 1976, see also [39]. For the definition of the method, let $T \in \mathbb{R}^{d\times d}$ be a suitable matrix with unit determinant, see Section 2, and define the lattices $\mathbb{X}_n = n^{-1/d}T(\mathbb{Z}^d)$, $n > 1$. Frolov's cubature formula is then rather simply defined as

$$Q_n(f) \;=\; \frac{1}{n} \sum_{x\in\mathbb{X}_n\cap[0,1)^d} f(x) \tag{1.2}$$

and it is well known that $|\mathbb{X}_n \cap [0,1)^d| \asymp n$. For the detailed definition of the cubature formula Q_n and the spaces $\mathbf{A}^s_{p,\theta}$ we refer to Section 2 and Section 4, respectively. We define the *worst-case error* of the cubature rule Q_n in a class $\mathbf{F}_d$ by

$$e(Q_n, \mathbf{F}_d) \;:=\; \sup_{\|f\|_{\mathbf{F}_d}\le 1} \Big| \int_{[0,1]^d} f(x)\,\mathrm{d}x - Q_n(f)\Big|.$$

Furthermore, we define the spaces $\mathring{\mathbf{A}}^s_{p,\theta}$ as the collection of all d-variate functions from $\mathbf{A}^s_{p,\theta}$ with support in the unit cube, cf. (4.2). It seems that the cubature formula Q_n of Frolov itself is suitable only for functions from $\mathring{\mathbf{A}}^s_{p,\theta}$, see Remark 5.4. However, in Section 6, we will present a universal modification of Q_n that has the same order of convergence also for periodic functions $(\mathbf{A}^s_{p,\theta}(\mathbb{T}^d))$. Note that **all** upper bounds on $\mathrm{Int}_n(\mathbf{A}^s_{p,\theta})$ that will be proven in this article are constructive, i.e. we prove all upper bounds via $\mathrm{Int}_n(\mathbf{F}_d) \lesssim e(Q_n, \mathbf{F}_d)$ for a specific cubature formula Q_n. In fact, these cubature formulas will be Q_n, cf. (1.2), for functions from $\mathring{\mathbf{A}}^s_{p,\theta}$.

The main results of this article are the following Theorems 1.1 & 1.2.

Theorem 1.1. Let $1 \le p, \theta \le \infty$ (with $p < \infty$ in the $\mathbf{F}$-case).
If $s > 1/p$, then

$$\mathrm{Int}_n\big(\mathbf{B}^s_{p,\theta}(\mathbb{T}^d)\big) \;\asymp\; n^{-s}(\log n)^{(d-1)(1-1/\theta)} \quad , \quad n \ge 2\,. \tag{1.3}$$

If $s > \max\{1/p, 1/\theta\}$, then

$$\mathrm{Int}_n\big(\mathbf{F}^s_{p,\theta}(\mathbb{T}^d)\big) \;\asymp\; n^{-s}(\log n)^{(d-1)(1-1/\theta)} \quad , \quad n \ge 2\,. \tag{1.4}$$

This theorem complements earlier results in the literature, see the list below, and verifies a widely believed conjecture.

In addition, we particularly pay attention to the case of small smoothness which appears for the spaces $\mathbf{F}^s_{p,\theta}$ if $p > \theta$ and $1/p < s \le 1/\theta$. It has been already observed by Temlyakov [35, Thm. IV.2.5] that for the Fibonacci lattice rule on $\mathbb{T}^2$ the asymptotical worst-case error order with respect to the Sobolev class $\mathbf{W}^s_p$ differs essentially in the log-power from (1.3) in the critical range of parameters $p > 2$ and $1/p < s \le 1/2$. We were able to show this effect for $\mathbf{F}^s_{p,\theta}$ also in higher dimensions for the Frolov cubature rule.

Theorem 1.2. Let $1 \le p < \infty$ and $1 \le \theta < p < \infty$.
(i) If $1/p < s < 1/\theta$ then

$$\mathrm{Int}_n\big(\mathbf{F}^s_{p,\theta}(\mathbb{T}^d)\big) \lesssim n^{-s}(\log n)^{(d-1)(1-s)} \quad , \quad n \ge 2\,.$$

(ii) If $s = 1/\theta$ then

$$\mathrm{Int}_n\big(\mathbf{F}^s_{p,\theta}(\mathbb{T}^d)\big) \lesssim n^{-s}(\log n)^{(d-1)(1-s)}(\log\log n)^{1-s} \quad , \quad n \ge 3\,.$$

Concerning matching lower bounds there is so far only hope for $e(Q_n, \mathring{\mathbf{F}}^s_{p,\theta})$. This will be discussed in a forthcoming paper by the authors. It is an interesting open problem to ask the same question for $\mathrm{Int}_n(\mathbf{F}^s_{p,\theta})$ in the critical range of parameters. In any case, we strongly conjecture that the given convergence rates are sharp.

Spaces of the above type have a long history in the former Soviet Union, see [1, 23, 28, 35] and the references therein. The scale of spaces $\mathbf{B}^s_{p,\theta}$ contains two important special cases of spaces with mixed smoothness: the Hölder-Zygmund spaces ($p = \theta = \infty$) and the classical Nikol'skij spaces ($\theta = \infty$). Note that Sobolev spaces $\mathbf{W}^s_p$ with integrability $1 < p < \infty$ and $s > 0$ are not contained in the Besov scale. They represent special cases of Triebel-Lizorkin spaces $\mathbf{F}^s_{p,\theta}$ if $\theta = 2$. The upper bounds in Theorem 1.1 have been already observed for

- $\mathbf{W}^s_p(\mathbb{T}^2)$, $s > 1/p$, see [35, Thm. IV.2.1 & IV.2.5],
- $\mathbf{B}^s_{p,\infty}(\mathbb{T}^2)$, $s > 1/p$, see [35, Thm. IV.2.6],
- $\mathbf{B}^s_{p,\theta}(\mathbb{T}^2)$ for $s > 1/p$, see [42] and [8],
- $\mathbf{B}^s_{p,\theta}([0,1]^2)$ for $1/p < s < 1 + 1/p$, see [42] and [8],
- $\mathbf{B}^s_{p,\theta}([0,1]^d) \cap \{f\colon f(x) = 0 \text{ if } x_i = 1 \text{ for some } i\}$ for $s \le 1$, see [22],
- $\mathring{\mathbf{W}}^s_p([0,1]^d)$ for $s \in \mathbb{N}$, see [10, 29],
- $\mathbf{W}^s_p(\mathbb{T}^d)$ for $2 \le p < \infty$ and $s \ge 1$, see [35, Thm. IV.4.4],
- $\mathbf{B}^s_{p,\infty}(\mathbb{T}^d)$ if $s > 1$ and $1 < p \le \infty$, see [35, Thm. IV.4.6], and
- $\mathbf{B}^s_{p,\theta}(\mathbb{T}^d)$ for $1 \le p, \theta \le \infty$ and $s > 1/p$, see [6, 7]; $1/p < s < 2$, see [15].

The last four results were obtained by using Frolov(-type) cubature rules. The other bounds are achieved for cubature formulas that use (digital) nets and the Fibonacci lattice rule, respectively, where the latter is restricted to $d = 2$. For the Besov spaces with $d > 2$ there are also some (not optimal) upper bounds by Triebel [38] for $1/p < s < 1 + 1/p$ using integration knots from Smolyak grids or by Temlyakov [32] using quasi-Monte Carlo lattice rules of Korobov type. Additionally, some results for Triebel-Lizorkin spaces can be easily obtained by embedding, see Lemma 4.6 below. Note that the last item above already gives our result for the Besov spaces $\mathbf{B}^s_{p,\theta}$. However, our method is different and gives a unified proof for Besov and Triebel-Lizorkin spaces. That is why we give also the (relatively short) proof for this case.

Moreover, we will present asymptotically optimal results for

1) $\mathrm{Int}_n\big(\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)\big)$ in the quasi-Banach cases $\min\{p,\theta\} < 1$, see Section 7.1, and
2) $\mathrm{Int}_n\big(\mathbf{A}^{1/p}_{p,\theta}(\mathbb{T}^d)\big)$ in the limiting case ($s = 1/p$), see Section 7.2.

Note that the results of Section 7.2 require additional assumptions on θ to assure contnuity of the functions. For the precise statement consider the mentioned section.

The paper is organized as follows. In Section 2 we introduce the Frolov cubature rule in detail. In addition, we give a direct construction of the Frolov lattice matrix without matrix inversion. Afterwards, in Section 3, we collect several tools from harmonic analysis. In particular, Poisson's summation formula and Calderon's reproducing formula in connection with local mean characterizations of function spaces turn out to be crucial for the error analysis of the cubature formula. Based on that we introduce the function spaces of interest in Section 4. In Sections 5 and 7 we prove the upper bounds for Frolov's cubature rule in the spaces $\mathring{\mathbf{A}}^s_{p,\theta}$ for the Banach and the quasi-Banach situation, respectively. Section 6 contains the modification of the cubature rule for periodic functions, and hence, finally proves the upper bounds in Theorems 1.1 and 1.2. The lower bounds will be proven in Section 8.

Notation. As usual $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the integers, $\mathbb{R}$ the real numbers, and $\mathbb{C}$ the complex numbers. By $\mathbb{T}$ we denote the torus represented by the interval $[0,1]$, where the end points are identified. The letter d is always reserved for the underlying dimension in $\mathbb{R}^d, \mathbb{Z}^d, \mathbb{T}^d$ etc. We denote with $\langle x, y\rangle$ the usual Euclidean inner product in $\mathbb{R}^d$. For $a \in \mathbb{R}$ we denote $a_+ := \max\{a, 0\}$ and $a = \lfloor a\rfloor + \{a\}$ with $\lfloor a\rfloor \in \mathbb{Z}$ and $0 \le \{a\} < 1$. For $0 < p \le \infty$ and $x \in \mathbb{R}^d$ we denote $|x|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ with the usual modification in the case $p = \infty$. By $(x_1, \ldots, x_d) > 0$ we mean that each coordinate is positive and $\lfloor x\rfloor$ (resp. $\{x\}$) are meant componentwise. By $\mathbb{T}$ we denote the torus represented by the interval $[0,1]$. If X and Y are two (quasi-)normed spaces, the (quasi-)norm of an element x in X will be denoted by $\|x\|_X$. The symbol $X \hookrightarrow Y$ indicates that the identity operator is continuous. For two sequences a_n and b_n we will write $a_n \lesssim b_n$ if there exists a constant $c > 0$ such that $a_n \le c\, b_n$ for all n. We will write $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$.

2. Frolov's construction

In this section we give the matrix T that is used in the construction of the lattice $\mathbb{X}_n$ for the cubature rule Q_n from (1.2). We follow to a large extend the original work of Frolov [10], see also [35, 39]. Let

$$P_d(t) := \prod_{j=1}^d \big(t - 2j + 1\big) - 1 = \prod_{j=1}^d (t - \xi_j), \qquad t \in \mathbb{R}, \tag{2.1}$$

for suitable $\xi_1, \ldots, \xi_d \in \mathbb{R}$, i.e. the ξ_j are the roots of P_d. It is easy to see that the ξ_j are indeed real numbers and all different. Now define the matrix

$$\widetilde{T} = \big(\xi_i^{j-1}\big)_{i,j=1}^d \tag{2.2}$$

and let $T_n = \big(n \det(\widetilde{T})\big)^{-1/d}\, \widetilde{T}$ such that $\det(T_n) = 1/n$. The lattice for the cubature formula Q_n is given by $\mathbb{X}_n = T_n(\mathbb{Z}^d)$.

Later we will see that the error of the cubature formula Q_n applied to some function f is given by a sum over function evaluation of the Fourier transform $\mathcal{F}f$ at the points $B_n(\mathbb{Z}^d)$, where B_n is given by

$$B_n = \big(T_n^{-1}\big)^\top. \tag{2.3}$$

The lattice $B_n(\mathbb{Z}^d)$ is usually called the *dual lattice* of $\mathbb{X}_n$. The following result contains the most important property of B_n.

Lemma 2.1. Let B_n be as above. Then, for each $z \in B_n(\mathbb{Z}^d) \setminus \{0\}$, we have

$$\prod_{j=1}^d |z_j| \gtrsim n.$$

Proof. First note that it is known that $|\prod_{j=1}^d (\widetilde{T}m)_j| \ge 1$ for each $m \in \mathbb{Z}^d \setminus \{0\}$ with $\widetilde{T}$ from (2.2), see [35, Lemma IV.4.3] or [39, Lemma 8]. Since $\widetilde{T}$ is a Vandermonde matrix we can write $\widetilde{T}^{-1}$ as the product $H\widetilde{T}^\top D^{-1}$, where H is Hankel matrix with integer entries and $\det(H) = -1$, and D is a diagonal matrix with $\det(D) = \det(\widetilde{T})^2$, see [21, Section 4]. Hence,

$$\begin{aligned}\inf_{m\in\mathbb{Z}^d\setminus\{0\}} \prod_{j=1}^d |(B_n m)_j| \;&=\; n\det(\widetilde{T}) \inf_{m\in\mathbb{Z}^d\setminus\{0\}} \prod_{j=1}^d |(D^{-1}\widetilde{T}H^\top m)_j| \\ &= \frac{n}{\det(\widetilde{T})} \inf_{m\in\mathbb{Z}^d\setminus\{0\}} \prod_{j=1}^d |(\widetilde{T}H^\top m)_j| \;\ge\; \frac{n}{\det(\widetilde{T})} \inf_{m\in\mathbb{Z}^d\setminus\{0\}} \prod_{j=1}^d |(\widetilde{T}m)_j| \\ &\ge \frac{n}{\det(\widetilde{T})},\end{aligned}$$

where we have used that $H^\top(\mathbb{Z}^d) \subset \mathbb{Z}^d$. □

Note that lattices which satisfy the conclusion of Lemma 2.1 satisfy $|\mathbb{X}_n \cap [0,1)^d| = n + O\big(\ln^{d-1}(n)\big)$, see e.g. [29].

Remark 2.2. The original construction of the lattice $\mathbb{X}_n$ of Frolov [10], see also [35], uses the generator $n^{-1/d}B_1$ instead of T_n. The advantage of (2.2) is that we do not have to invert the matrix $\widetilde{T}$ for the construction of the nodes. However, this "trick" only works if the generator $\widetilde{T}$ is (a multiple of) a Vandermonde matrix.

The striking advantage of the property in Lemma 2.1 is reflected best by the numbers

$$Z_n(m) \;=\; \Big|(B_n(\mathbb{Z}^d) \setminus \{0\}) \cap I_m\Big|, \tag{2.4}$$

with some dyadic rectangles

$$I_m := \{x \in \mathbb{R}^d \colon C_1 2^{m_j - 1} \le |x_j| < C_2 2^{m_j} \text{ for } j = 1,\dots,d\}, \quad m \in \mathbb{N}_0^d, \tag{2.5}$$

such that the numbers $0 < C_1 \le C_2 < \infty$ are independent of m. That is, the number of points from $B_n(\mathbb{Z}^d)$ (excluding 0) in certain axes-parallel boxes with volume around $2^{|m|_1}$.

We will need the following properties of $Z_n(m)$.

Lemma 2.3. There exists an absolute constant $c < \infty$ such that with

$$r_n := \log_2(n) - c \tag{2.6}$$

the numbers $Z_n(m)$ from (2.4), $n \in \mathbb{N}$, $m \in \mathbb{N}_0^d$, satisfy

(i) $Z_n(m) \;=\; 0,$ if $|m|_1 \le r_n$, and
(ii) $Z_n(m) \;\lesssim\; 2^{|m|_1}/n,$ otherwise.

Proof. From Lemma 2.1 we have $\prod_{j=1}^d |z_j| \ge C_0\, n$ for $z \in B_n(\mathbb{Z}^d) \setminus \{0\}$ and some $C_0 > 0$. For $x \in I_m$ we have that $\prod_{j=1}^d |x_j| < \delta^d 2^{|m|_1}$ with $\delta = C_2 - C_1/2 > 0$, cf. (2.5). This shows that $Z_n(m) = 0$ if $|m|_1 \le \log_2(n) + \log_2(C_0/\delta^d) =: r_n$.

To prove part (ii) consider axis-parallel boxes $\Omega \in \mathbb{R}^d$. Let us assume that Ω contains at least two points from $B_n(\mathbb{Z}^d)$, say $z', z'' \in \Omega$. Then, since $B_n(\mathbb{Z}^d)$ is a lattice, $z := z' - z''$ is also in $B_n(\mathbb{Z}^d)$. We obtain that $\mathrm{vol}_d(\Omega) \ge \prod_{j=1}^d |z'_j - z''_j| = \prod_{j=1}^d |z_j| \ge C_0\, n$. Hence, axis-parallel boxes Ω with volume less than $C_0\, n$ contain at most one point. Boxes of larger volume can now be devided into $\lfloor \mathrm{vol}_d(\Omega)/(C_0 n) + 1 \rfloor$ boxes of volume less than $C_0 n$, each with at most one point from $B_n(\mathbb{Z}^d)$. This shows $|B_n(\mathbb{Z}^d) \cap \Omega| \lesssim \mathrm{vol}_d(\Omega)/n$ for axis-parallel boxes Ω. □

Remark 2.4. The choice of the polynomial (2.1) is quite flexible. One could replace P_d by every irreducible (over $\mathbb{Q}$) polynomial with integer coefficients that has d different real roots. Another example, for $d = 2^\ell$, is the Chebychev-type polynomial $P_{2^\ell}(t) = 2\cos(2^\ell \arccos(t/2))$, see [35, p, 242]. The roots of these polynomials are given by $\xi_i = 2\cos(\pi(2i-1)/2^{n+1})$, $i = 1, \dots, d$.

Remark 2.5. It is worth noting that the results of this paper do not rely on the specific construction. In fact, every lattice for which the the dual lattice satisfies the bound of Lemma 2.1 would lead to the same results. See e.g. [29] for a detailed study of such lattices.

3. Tools from Fourier analysis

3.1. **Preliminaries.** Let $L_p = L_p(\mathbb{R}^d)$, $0 < p \le \infty$, be the space of all functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_p := \Big(\int_{\mathbb{R}^d} |f(x)|^p \,\mathrm{d}x \Big)^{1/p} < \infty$$

with the usual modification if $p = \infty$. In addition, we denote by $C(\mathbb{R}^d)$ the space of all bounded and coontinuous complex-valued functions on $\mathbb{R}^d$.

We will also need L_p-spaces on compact domains $\Omega \subset \mathbb{R}^d$ instead of $\mathbb{R}^d$. We write $\|f\|_{L_p(\Omega)}$ for the corresponding (restricted) L_p-norm. For $f \in L_1(\mathbb{R}^d)$ we define the Fourier transform

$$\mathcal{F}f(\xi) \;=\; \int_{\mathbb{R}^d} f(y) e^{-2\pi i \langle \xi, y\rangle} \,\mathrm{d}y, \qquad \xi \in \mathbb{R}^d,$$

and the corresponding inverse Fourier transform $\mathcal{F}^{-1}f(\xi) = \mathcal{F}f(-\xi)$. Additionally, we define the spaces of continuous functions $C(\mathbb{R}^d)$, infinitely differentiable functions $C^\infty(\mathbb{R}^d)$ and infinitely differentiable functions with compact support $C_0^\infty(\mathbb{R}^d)$ as well as the *Schwartz space* $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ of all rapidly decaying infinitely differentiable functions on $\mathbb{R}^d$, i.e.,

$$\mathcal{S} := \left\{ \varphi \in C^\infty(\mathbb{R}^d) \colon \|\varphi\|_{k,\ell} < \infty \;\text{ for all } k, \ell \in \mathbb{N} \right\},$$

where

$$\|\varphi\|_{k,\ell} := \Big\| (1 + |\cdot|)^k \sum_{m=0}^{\ell} |\varphi^{(m)}(\cdot)| \Big\|_\infty .$$

The space $\mathcal{S}'(\mathbb{R}^d)$, the topological dual of $\mathcal{S}(\mathbb{R}^d)$, is also referred to as the set of tempered distributions on $\mathbb{R}^d$. Indeed, a linear mapping $f : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ if and only if there exist numbers $k, \ell \in \mathbb{N}$ and a constant $c = c_f$ such that

$$|f(\varphi)| \le c_f \|\varphi\|_{k,\ell} \tag{3.1}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The space $\mathcal{S}'(\mathbb{R}^d)$ is equipped with the weak*-topology. The convolution $\varphi * \psi$ of two square integrable functions φ, ψ is defined via the integral

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^d} \varphi(x - y)\psi(y)\, dy \,.$$

If $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ then $\varphi * \psi$ still belongs to $\mathcal{S}(\mathbb{R}^d)$. In fact, we have $\varphi * \psi \in \mathcal{S}(\mathbb{R}^d)$ even if $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $f \in L_1(\mathbb{R}^d)$.

The convolution can be extended to $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$ via $(\varphi * f)(x) = f(\varphi(x - \cdot))$. It makes sense pointwise and is a C^∞-function in $\mathbb{R}^d$. As usual, the Fourier transform can be extended to $\mathcal{S}'(\mathbb{R}^d)$ by $(\mathcal{F}f)(\varphi) := f(\mathcal{F}\varphi)$, where $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The mapping $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is a bijection.

3.2. **Periodization and Poisson's summation formula.** The analysis of the error of cubature formulas that use nodes from a lattice is naturally related to an application of Poisson's summation formula and variations thereof, see (5.2). A more detailed treatment and a proof of the following theorem can be found, e.g., in [30, Thm. VII.2.4 & Cor. VII.2.6].

Theorem 3.1. Let $f \in L_1(\mathbb{R})$. Then its periodization $\sum_{\ell\in\mathbb{Z}^d} f(\ell+\cdot)$ converges in the norm of $L_1([0,1]^d)$. The resulting (1-periodic) function in $L_1([0,1]^d)$ has the formal Fourier expansion

$$\sum_{k\in\mathbb{Z}^d} \mathcal{F}f(k)\, e^{2\pi i k x}.$$

Moreover, if $f \in \mathcal{S}(\mathbb{R}^d)$, then both sums converge absolutely, and hence pointwise.

Note that absolute convergence in the above sums actually holds under weaker assumptions on f. However, this version is enough for our purposes.

In what follows, cf. (5.2), we will need a pointwise version of Poisson's summation formula, like Theorem 3.1, in cases where we cannot guarantee that the sums converge absolutely. In such cases we have to specify what is meant by convergence. This is provided by the corollary below.

Corollary 3.2. Let $f \in L_1(\mathbb{R}^d)$ be continuous with compact support, $T \in \mathbb{R}^{d\times d}$ be an invertible matrix, and $B = (T^{-1})^\top$. Furthermore, let $\varphi_0 \in C_0^\infty(\mathbb{R})$ with $\varphi_0(0) = 1$ and define $\varphi_j(t) := \varphi_0(2^{-j}t) - \varphi_0(2^{-j+1}t)$, $j \in \mathbb{N}$, $t \in \mathbb{R}$, as well as the (tensorized) functions $\varphi_m(x) := \varphi_{m_1}(x_1)\cdot\ldots\cdot\varphi_{m_d}(x_d)$, $m \in \mathbb{N}_0^d$, $x \in \mathbb{R}^d$. Then

$$\det(T)\sum_{\ell\in\mathbb{Z}^d} f(T\ell) \;=\; \lim_{N\to\infty} \sum_{m\colon |m|_\infty\le N}\; \sum_{k\in\mathbb{Z}^d} \varphi_m(Bk)\, \mathcal{F}f(Bk).$$

In particular, the limit on the right hand side exists.

Proof. The proof is based on [30, Theorem VII.2.11]. We put $\Phi_B(\cdot) := \varphi_0(B\cdot)$ and note, that Φ_B satisfies the assumptions [30, (2.10)], in particular $\Phi_B(0) = 1$. Moreover, f is continuous and has compact support, which implies that the periodization $\det(T)\sum_{\ell\in\mathbb{Z}^d} f(T(\ell+x))$ belongs to $C(\mathbb{T}^d)$. Applying [30, Thm. VII.2.11] together with Theorem 3.1 above we obtain

$$\det(T)\sum_{\ell\in\mathbb{Z}^d} f(T(\ell+x)) \;=\; \lim_{N\to\infty} \sum_{k\in\mathbb{Z}^d} \Phi_B(2^{-N}k)\, \mathcal{F}f(Bk)\, e^{2\pi i k x}$$

in $C(\mathbb{T}^d)$. Setting $x = 0$ gives

$$\det(T)\sum_{\ell\in\mathbb{Z}^d} f(T\ell) = \lim_{N\to\infty} \sum_{k\in\mathbb{Z}^d} \Phi_B(2^{-N}k)\, \mathcal{F}f(Bk). \tag{3.2}$$

By construction we note that

$$\Phi_B(2^{-N}k) = \varphi_0(2^{-N}(Bk)_1)\cdot\ldots\cdot\varphi_0(2^{-N}(Bk)_d) = \sum_{m\colon |m|_\infty\le N} \varphi_m(Bk)\,.$$

Plugging this into (3.2) and interchanging the order of summation yields the desired result. □

To prove our main results we will also have to bound the norm of certain series of functions. In fact, we treat two different kinds of functions: The first bound requires that the functions itself have compact support, while the second is for function with compactly supported Fourier transform.

Lemma 3.3. Let $B \in \mathbb{R}^{d\times d}$ be an invertible matrix and $\Omega \in \mathbb{R}^d$ be a bounded set. Furthermore, let $\{f_m\}_{m\in\mathbb{N}_0^d} \subset \mathcal{S}(\mathbb{R}^d)$ be functions with $\mathrm{supp}(f_m) \subset \Omega$ for all $m \in \mathbb{N}_0^d$ and define

$$M_{B,\Omega} := \Big|\{\ell \in \mathbb{Z}^d \colon (\ell + [0,1)^d) \cap B^\top(\Omega)\}\Big|.$$

Then, for $1 \le \theta, p \le \infty$, we have

$$\Big\|\Big(\sum_{m\in\mathbb{N}_0^d}\Big|\sum_{\ell\in\mathbb{Z}^d}\mathcal{F}f_m(B\ell)\,\mathrm{e}^{2\pi i\ell\cdot}\Big|^\theta\Big)^{1/\theta}\Big\|_{L_p([0,1]^d)} \le \Big(\frac{M_{B,\Omega}}{\det(B)}\Big)^{1-1/p}\Big\|\Big(\sum_{m\in\mathbb{N}_0^d}|f_m|^\theta\Big)^{1/\theta}\Big\|_p.$$

In particular,

$$\Big\|\sum_{\ell\in\mathbb{Z}^d}\mathcal{F}f(B\ell)\,\mathrm{e}^{2\pi i\ell\cdot}\Big\|_{L_p([0,1]^d)} \le \Big(\frac{M_{B,\Omega}}{\det(B)}\Big)^{1-1/p}\|f\|_p.$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ with $\mathrm{supp}(f) \subset \Omega$.

Proof. Let p' be given by $1/p' = 1 - 1/p$. Let $h_m(x) = f_m(Tx)$ with $T = (B^{-1})^\top$ and note that $\mathrm{supp}(h_m) \subset B^\top(\Omega)$. Clearly, $\mathcal{F}f_m(B\ell) = \det(T)\mathcal{F}h_m(\ell)$ and, hence, by Theorem 3.1 and Hölder's inequality we obtain

$$\begin{aligned}\Big\|\Big(\sum_{m\in\mathbb{N}_0^d}\Big|\sum_{\ell\in\mathbb{Z}^d}\mathcal{F}f_m(B\ell)\,\mathrm{e}^{2\pi i\ell\cdot}\Big|^\theta\Big)^{1/\theta}\Big\|_{L_p([0,1]^d)} &= \det(T)\Big\|\Big(\sum_{m\in\mathbb{N}_0^d}\Big|\sum_{\ell\in\mathbb{Z}^d}h_m(\ell+\cdot)\Big|^\theta\Big)^{1/\theta}\Big\|_{L_p([0,1]^d)}\\ &\le \det(T)\Big\|\sum_{\ell\in\mathbb{Z}^d}\Big(\sum_{m\in\mathbb{N}_0^d}\Big|h_m(\ell+\cdot)\Big|^\theta\Big)^{1/\theta}\Big\|_{L_p([0,1]^d)}\\ &\le \det(T)\Big\|M_{B,\Omega}^{1/p'}\Big(\sum_{\ell\in\mathbb{Z}^d}\Big(\sum_{m\in\mathbb{N}_0^d}\Big|h_m(\ell+\cdot)\Big|^\theta\Big)^{p/\theta}\Big)^{1/p}\Big\|_{L_p([0,1]^d)}.\end{aligned}$$

Performing the integration and interchanging sum and integral yields

$$\begin{aligned}&\Big\|\Big(\sum_{m\in\mathbb{N}_0^d}\Big|\sum_{\ell\in\mathbb{Z}^d}\mathcal{F}f_m(B\ell)\,\mathrm{e}^{2\pi i\ell\cdot}\Big|^\theta\Big)^{1/\theta}\Big\|_{L_p([0,1]^d)}\\ &\qquad\le \det(T)\,M_{B,\Omega}^{1/p'}\Big(\int_{[0,1]^d}\sum_{\ell\in\mathbb{Z}^d}\Big(\sum_{m\in\mathbb{N}_0^d}\Big|h_m(\ell+x)\Big|^\theta\Big)^{p/\theta}\,\mathrm{d}x\Big)^{1/p}\\ &\qquad= \det(T)\,M_{B,\Omega}^{1/p'}\Big(\sum_{\ell\in\mathbb{Z}^d}\int_{\ell+[0,1]^d}\Big(\sum_{m\in\mathbb{N}_0^d}\Big|h_m(x)\Big|^\theta\Big)^{p/\theta}\,\mathrm{d}x\Big)^{1/p}\\ &\qquad= \det(T)\,M_{B,\Omega}^{1-1/p}\Big(\int_{\mathbb{R}^d}\Big(\sum_{m\in\mathbb{N}_0^d}\Big|h_m(x)\Big|^\theta\Big)^{p/\theta}\,\mathrm{d}x\Big)^{1/p}\\ &\qquad= \Big(\det(T)\,M_{B,\Omega}\Big)^{1-1/p}\Big\|\Big(\sum_{m\in\mathbb{N}_0^d}|f_m|^\theta\Big)^{1/\theta}\Big\|_p.\end{aligned}$$

The second statement follows if we set $f_0 = f$ and $f_m = 0$, $m \neq 0$.

□

Remark 3.4. Note that $M_{B,\Omega}$ is the number of unit cubes that are necessary to cover the set $B^\top(\Omega)$, while $\det(B)$ equals the volume of $B^\top([0,1]^d)$. This shows that with a matrix of the form $B_n = n^{1/d}B$, $n \geq 1$, cf. (2.3), we obtain

$$\lim_{n\to\infty} \frac{M_{B_n,\Omega}}{\det(B_n)} \;=\; \mathrm{vol}_d(\Omega)$$

for every Jordan measurable set Ω.

Lemma 3.5. Let $B \in \mathbb{R}^{d\times d}$ be an invertible matrix and $\Omega \in \mathbb{R}^d$ be a bounded set. Furthermore, let $g \in \mathcal{S}(\mathbb{R}^d)$ with $\mathrm{supp}(\mathcal{F}g) \subset \Omega$. Then, for $1 \leq p \leq \infty$, we have

$$\Big\| \sum_{k\in\mathbb{Z}^d} \mathcal{F}g(Bk)\, \mathrm{e}^{2\pi ik\cdot} \Big\|_{L_p([0,1]^d)} \;\leq\; \Big|B(\mathbb{Z}^d)\cap\Omega\Big|^{1-1/p}\, \|g\|_1.$$

Proof. Clearly, the proof of Lemma 3.3 for $p = 1$ works also for functions f_m without compact support and, hence, already proves the statement for $p = 1$. (Set $f_0 = g$ and $f_m = 0$ for $m \neq 0$.) For $p = \infty$ we easily obtain the upper bound $|B(\mathbb{Z}^d)\cap\Omega| \cdot \|\mathcal{F}g\|_\infty \leq |B(\mathbb{Z}^d)\cap\Omega| \cdot \|g\|_1$. Hence, the lemma follows by interpolation. □

3.3. **A discrete version of Calderon's reproducing formula.** Our analysis heavily relies on a discrete version of Calderon's reproducing formula [4, eq. (3.1)]. A "continuous" and homogeneous version of the Lemma below has been proved in [19]. This principle has been used by several authors [3], [9], [27], [26] to prove equivalent (local mean) characterizations for Besov-Triebel-Lizorkin spaces, see Section 4.

Lemma 3.6. Let $\Psi_0, \Psi_1 \in \mathcal{S}(\mathbb{R})$ be functions with

$$|\mathcal{F}\Psi_0(\xi)| > 0 \quad \text{for } |\xi| < \varepsilon \tag{3.3}$$

and

$$|\mathcal{F}\Psi_1(\xi)| > 0 \quad \text{for } \frac{\varepsilon}{2} < |\xi| < 2\varepsilon \tag{3.4}$$

for some $\varepsilon > 0$. Then there exist $\Lambda_0, \Lambda_1 \in \mathcal{S}(\mathbb{R})$ such that

(i) $\mathrm{supp}\,\mathcal{F}\Lambda_0 \subset \{t \in \mathbb{R} : |t| \leq \varepsilon\}$
(ii) $\mathrm{supp}\,\mathcal{F}\Lambda_1 \subset \{t \in \mathbb{R} : \varepsilon/2 \leq |t| \leq 2\varepsilon\}$ and
(iii) for all $\xi \in \mathbb{R}$,

$$\sum_{j=0}^{\infty} \mathcal{F}\Lambda_j(\xi)\, \mathcal{F}\Psi_j(\xi) \;=\; 1, \tag{3.5}$$

where $\Psi_j(x) = 2^{j-1}\Psi_1(2^{j-1}x)$ and $\Lambda_j(x) = 2^{j-1}\Lambda_1(2^{j-1}x)$ for $j \in \mathbb{N}$.

Proof. Following [43, Thm. 1.20] we use the special dyadic decomposition of unity with $\varphi(t) = 1$ if $|t| \leq 4/3$ and $\varphi(t) = 0$ if $|t| > 3/2$. Put $\Phi_0 := \mathcal{F}^{-1}\varphi$ and $\Phi_1 := 2\Phi_0(2\cdot) - \Phi_0$, i.e. $\mathcal{F}\Phi_1 = \Phi_0(\cdot/2) - \Phi_0$. With $\Phi_j := 2^{j-1}\Phi_1(2^{j-1}\cdot)$ for $j \geq 1$ we define Λ_0, Λ_1 through

$$\mathcal{F}\Lambda_j(t) := \frac{\mathcal{F}\Phi_j(2t/\varepsilon)}{\mathcal{F}\Psi_j(t)} \quad , \quad t \in \mathbb{R}\,.$$

□

We define the d-fold tensorized functions

$$\Lambda_m(x) \;:=\; \prod_{i=1}^{d} \Lambda_{m_i}(x_i) \quad \text{and} \quad \Psi_m(x) \;:=\; \prod_{i=1}^{d} \Psi_{m_i}(x_i) \quad , \quad x \in \mathbb{R}^d. \tag{3.6}$$

where Λ_j, Ψ_j, $j \in \mathbb{N}$, are defined in Lemma 3.6. We obtain from (3.5) the identity

$$\sum_{m\in\mathbb{N}_0^d} \mathcal{F}\Lambda_m(\xi)\,\mathcal{F}\Psi_m(\xi) \;=\; 1 \quad , \quad \xi \in \mathbb{R}^d\,. \tag{3.7}$$

By the construction of the tensorized functions (and Lemma 3.6) we know that the support of $\mathcal{F}\Lambda_m$ is of the form (2.5) and we will write in the sequel

$$I_m \;:=\; \operatorname{supp}\mathcal{F}\Lambda_m, \qquad m \in \mathbb{N}_0^d.$$

3.4. **The up-function.** We make use of a special infinitely many times differentiable compactly supported function. This (univariate) function is known as the *up-function*, see e.g. Rvachev [25] or [17, Section 6.1], and has many useful properties.

Let $g(t) = \mathbb{1}_{[-1/2,1/2]}(t)$, $t \in \mathbb{R}$, be the characteristic function of the interval $[-1/2, 1/2]$ and define $g_k(t) = 2^k g(2^k t)$. The up-function φ is now given by the infinite convolution

$$\varphi \;=\; g * g_1 * g_2 * \dots .$$

Obviously, $\varphi \in C_0^\infty(\mathbb{R})$, $0 \le \varphi \le 1$, $\|\varphi\|_1 = \|g\|_1 = 1$ and $\operatorname{supp}(\varphi) = [-1, 1]$. It is known that $\mathcal{F}g(\xi) = \operatorname{sinc}(\xi) := \sin(\pi\xi)/(\pi\xi)$ and hence, φ could be equivalently defined by

$$\mathcal{F}\varphi(\xi) \;=\; \prod_{k=0}^{\infty} \operatorname{sinc}(2^{-k}\xi), \qquad \xi \in \mathbb{R}.$$

Moreover, φ is a solution to the equation $\varphi'(t) = 2\varphi(2t+1) - 2\varphi(2t-1)$ and satisfies therefore

$$\varphi(t) + \varphi(t-1) \;=\; 1 \qquad \text{for all } t \in [0, 1],$$

see [25, Paragraph 1]. To see this, observe that $\int_0^1 \varphi(t) + \varphi(t-1)\,\mathrm{d}t = \int_{-1}^1 \varphi(t)\,\mathrm{d}t = 1$ and that the differential equation (together with $\operatorname{supp}(\varphi) = [-1, 1]$) implies $\varphi'(t) + \varphi'(t-1) = 2\varphi(2t+1) - 2\varphi(2t-3) = 0$ for $t \in [0, 1]$.

4. Function spaces with dominating mixed smoothness

In this section we introduce the function spaces under consideration, namely, the Besov and Triebel-Lizorkin spaces of dominating mixed smoothness. Note that the Sobolev spaces of mixed smoothness appear as a special case of the Triebel-Lizorkin spaces. There are several equivalent characterizations of these spaces, see e.g. [43]. For our purposes, the most suitable is the characterization by local means (see [43, Theorem 1.23] or [41, Definition 2.5]).

4.1. **Spaces on $\mathbb{R}^d$.** Let $\Psi_0, \Psi_1 \in C_0^\infty(\mathbb{R})$ be such that

(i) $|\mathcal{F}\Psi_0(\xi)| > 0$ for $|\xi| < \varepsilon$,
(ii) $|\mathcal{F}\Psi_1(\xi)| > 0$ for $\frac{\varepsilon}{2} < |\xi| < 2\varepsilon$ and
(iii) $D^\alpha\mathcal{F}\Psi_1(0) = 0$ for all $0 \le \alpha \le L$

for some $\varepsilon > 0$. A suitable L will be chosen in Definitions 4.1 & 4.2. Such functions obviously exist. Our choice will be based on the *up-function* φ from Subsection 3.4. We define $\Psi_0(x) := 2\varphi(2x)$. The function $\Psi_0 \in C_0^\infty(\mathbb{R})$ satisfies $\operatorname{supp}(\Psi_0) = [-1/2, 1/2]$ and $\mathcal{F}\Psi_0(\xi) = \prod_{k=1}^\infty \operatorname{sinc}(2^{-k}\xi)$, $\xi \in \mathbb{R}$. Then, we define $\Psi_1 \in C_0^\infty(\mathbb{R})$ by

$$\Psi_1(x) \;=\; \frac{d^L}{dx^L}\big(2\Psi_0(2\cdot) - \Psi_0(\cdot)\big)(x), \qquad x \in \mathbb{R}.$$

It follows that $\mathcal{F}\Psi_1(\xi) \;=\; (2\pi i\xi)^L\Big(\mathcal{F}\Psi_0(\xi/2) - \mathcal{F}\Psi_0(\xi)\Big)$. It is easily checked that these function satisfy the conditions above.

As usual, we define

$$\Psi_j(x) \,=\, 2^{j-1}\,\Psi_1(2^{j-1}x), \qquad j\in\mathbb{N},$$

and the (d-fold) tensorization

$$\Psi_m(x) \,=\, \prod_{i=1}^{d}\Psi_{m_i}(x_i), \tag{4.1}$$

where $m=(m_1,\ldots,m_d)\in\mathbb{N}_0^d$ and $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$.

Let us start with the definition of the function spaces $\mathbf{A}^s_{p,\theta}=\mathbf{A}^s_{p,\theta}(\mathbb{R}^d)$ with $\mathbf{A}\in\{\mathbf{B},\mathbf{F}\}$ defined on the entire $\mathbb{R}^d$.

Definition 4.1 (Besov space)**.** Let $0<p,\theta\le\infty$, $s\in\mathbb{R}$, and $\{\Psi_m\}_{m\in\mathbb{N}_0^d}$ be as above with $L+1>s$. The *Besov space of dominating mixed smoothness* $\mathbf{B}^s_{p,\theta}=\mathbf{B}^s_{p,\theta}(\mathbb{R}^d)$ is the set of all $f\in\mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{\mathbf{B}^s_{p,\theta}} \,:=\, \Big(\sum_{m\in\mathbb{N}_0^d} 2^{s|m|_1\theta}\,\|\Psi_m * f\|_p^\theta\Big)^{1/\theta} \,<\, \infty$$

with the usual modification for $\theta=\infty$.

Definition 4.2 (Triebel-Lizorkin space)**.** Let $0<p<\infty$, $0<\theta\le\infty$, $s\in\mathbb{R}$, and $\{\Psi_m\}_{m\in\mathbb{N}_0^d}$ be as above with $L+1>s$. The *Triebel-Lizorkin space of dominating mixed smoothness* $\mathbf{F}^s_{p,\theta}=\mathbf{F}^s_{p,\theta}(\mathbb{R}^d)$ is the set of all $f\in\mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{\mathbf{F}^s_{p,\theta}} \,:=\, \Big\|\Big(\sum_{m\in\mathbb{N}_0^d} 2^{s|m|_1\theta}\,|\Psi_m * f(\cdot)|^\theta\Big)^{1/\theta}\Big\|_p \,<\, \infty$$

with the usual modification for $\theta=\infty$.

Remark 4.3. In the special case $\theta=2$ and $1<p<\infty$ we put $\mathbf{W}^s_p:=\mathbf{F}^s_{p,2}$ which denotes the *Sobolev spaces of dominating mixed smoothness*. It is well-known (cf. [28, Chapt. 2] or [43]), that in case $s\in\mathbb{N}_0$ the spaces $\mathbf{W}^s_p$ can be equivalently normed by

$$\|f\|_{\mathbf{W}^s_p} \,\asymp\, \Big(\sum_{\substack{\alpha\in\mathbb{N}_0^d\\ |\alpha|_\infty\le s}} \|D^\alpha f\|_p^p\Big)^{1/p}.$$

Remark 4.4. Different choices of Ψ_0,Ψ_1 lead to equivalent (quasi-)norms. In fact, it is not even necessary that Ψ_0 and Ψ_1 have compact support. However, for the proof of our results this specific choice is crucial.

Remark 4.5. Note, that the spaces $\mathbf{B}^s_{p,\theta}(\mathbb{R}^d)$ and $\mathbf{F}^s_{p,\theta}(\mathbb{R}^d)$ are usually defined via a dyadic decompostion of unity on the Fourier side (which represents a special case of the above framework) like the one we used in the proof of Lemma 3.6, see [43]. Here we used a different characterization which was proven to be equivalent, see Vybíral [43] and Ullrich [41]. Let us comment on the recent and non-trivial history of those characterizations.

- In 1992 Triebel proved those characterizations for the isotropic spaces see [37, 2.4.2, 2.5.1]. He obtained characterizations for $p,\theta\ge 1$.
- Later, in 1999, Rychov [26] extended it to the quasi-Banach case. However, there is a gap in his proof (observed 2007 by M. Hansen, see [12]) and Remark 4.4 in [41].

- In 2005 Vybíral modified Rychkov's method for the dominating mixed case. Vybíral's proof also contains Rychkov's gap. However, Vybíral was the first who did the local means for dominating mixed spaces.
- The proof in [41] differs from Vybíral's proof and fixes the mentioned gap, see also Hansen [12]. The proof is based on a maximal function technique, see [41, pp. 20], due to Stromberg, Torchinsky [31, Chapt. V] which has been already proposed by Rychkov [27, Thm. 3.2].
- The proof in [41] is a bit more general, namely for function spaces on semi-infinite rectangular domains.

The next lemma collects some frequently used embedding properties of the spaces.

Lemma 4.6. Let $0 < p < u \leq \infty$ ($p, u < \infty$ in the $\mathbf{F}$-case), $s, t \in \mathbb{R}$, $0 < \theta, \eta \leq \infty$.
(i) For equal p we have the chain of embeddings

$$\mathbf{B}^s_{p,\min\{p,\theta\}} \hookrightarrow \mathbf{F}^s_{p,\theta} \hookrightarrow \mathbf{B}^s_{p,\max\{p,\theta\}}$$

(ii) In addition, whenever $s - 1/p = t - 1/u$ the following "diagonal embeddings" hold true

$$\mathbf{B}^s_{p,\theta} \hookrightarrow \mathbf{B}^t_{u,\theta} \quad , \quad \mathbf{F}^s_{p,\theta} \hookrightarrow \mathbf{F}^t_{u,\eta}$$

(iii) as well as the embeddings of Jawerth-Franke type

$$\mathbf{F}^s_{p,\theta} \hookrightarrow \mathbf{B}^t_{u,p} \quad , \quad \mathbf{B}^s_{p,u} \hookrightarrow \mathbf{F}^t_{u,\eta} \,.$$

Proof. For a proof we refer to [28, Chapt. 2] and [13].

□

Throughout this article we will assume that $s > 1/p$ (or $s = 1/p$ with additional assumptions on θ). This assures that the functions in $\mathbf{B}^s_{p,\theta}$, $\mathbf{F}^s_{p,\theta}$ and $\mathbf{W}^s_p$, respectively, are continuous, see [28, Chapt. 2]. With the same reasoning we obtain that $\mathbf{B}^{1/p}_{p,\theta} \hookrightarrow C(\mathbb{R}^d)$ for $\theta \leq 1$, and $\mathbf{F}^{1/p}_{p,\theta} \hookrightarrow C(\mathbb{R}^d)$ for $p \leq 1$.

4.2. **Spaces on the cube.** Here we turn to the subclasses of functions which are supported in the unit cube $[0,1]^d$. We define for $s > \max\{1/p - 1, 0\}$

$$\mathring{\mathbf{A}}^s_{p,\theta} := \left\{ f \in \mathbf{A}^s_{p,\theta}(\mathbb{R}^d) \colon \operatorname{supp}(f) \subset [0,1]^d \right\} . \tag{4.2}$$

The spaces $\mathring{\mathbf{A}}^s_{p,\theta}$ can be interpreted as subspaces of $\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)$ which denotes the space of periodic functions on $\mathbb{T}^d = [0,1]^d$. Indeed, a function $f : \mathbb{T}^d \to \mathbb{C}$ is defined on $\mathbb{R}^d$ and supposed to be 1-periodic in every component. We will define periodicity in a wider sense. A distribution $f \in \mathcal{S}'(\mathbb{R})$ is a periodic distribution from $\mathcal{S}'_\pi(\mathbb{R}^d)$ if and only if

$$f(\varphi) = f(\varphi(\cdot + k)) \,, \quad \varphi \in \mathcal{S}(\mathbb{R}^d) \,, k \in \mathbb{Z}^d \,,$$

Now we define

$$\mathbf{A}^s_{p,\theta}(\mathbb{T}^d) = \{ f \in \mathcal{S}'_\pi(\mathbb{R}^d) \,:\, \|f\|_{\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)} < \infty \}$$

with

$$\|f\|_{\mathbf{B}^s_{p,\theta}(\mathbb{T}^d)} := \Big(\sum_{m \in \mathbb{N}_0^d} 2^{s|m|_1\theta} \, \|\Psi_m * f\|^\theta_{L_p(\mathbb{T}^d)} \Big)^{1/\theta}$$

and

$$\|f\|_{\mathbf{F}^s_{p,\theta}(\mathbb{T}^d)} := \Big\| \Big(\sum_{m \in \mathbb{N}_0^d} 2^{s|m|_1\theta} \, |\Psi_m * f(\cdot)|^\theta \Big)^{1/\theta} \Big\|_{L_p(\mathbb{T}^d)},$$

respectively. Note, that we replace the $L_p(\mathbb{R}^d)$-norms by $L_p(\mathbb{T}^d)$-norms.

In addition to the embeddings in Lemma 4.6 we will need the following inclusions based on Hölder's inequality.

Lemma 4.7. If $0 < p < u \le \infty$, $0 < \theta \le \infty$ and $s > \max\{1/p - 1, 0\}$ then

$$\mathring{\mathbf{A}}^s_{u,\theta} \hookrightarrow \mathring{\mathbf{A}}^s_{p,\theta} \quad \text{and} \quad \mathbf{A}^s_{u,\theta}(\mathbb{T}^d) \hookrightarrow \mathbf{A}^s_{p,\theta}(\mathbb{T}^d)\,,$$

where we need $u < \infty$ in the $\mathbf{F}$-case.

Proof. For the periodic spaces $\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)$ the proof is a simple consequence of the definition and the embedding $L_u(\mathbb{T}^d) \hookrightarrow L_p(\mathbb{T}^d)$ due to the compact domain. If f is compactly supported inside $[0,1]^d$ then the convolution $\Psi_m * f$ with Ψ_m from (4.1) is supported in a universal set Ω for all $m \in \mathbb{N}_0^d$. Hence

$$\Big(\sum_{m\in\mathbb{N}_0^d} 2^{s|m|_1\theta}\,|\Psi_m * f(\cdot)|^\theta\Big)^{1/\theta}$$

is supported in Ω. Hölder's inequality finally yields $\|f\|_{\mathbf{A}^s_{p,\theta}} \le C_\Omega \|f\|_{\mathbf{A}^s_{u,\theta}}$.

□

5. Integration of functions from $\mathring{\mathbf{A}}^s_{p,\theta}$

Here, we study the cubature formula Q_n from (1.2), see Section 2, for the spaces $\mathring{\mathbf{A}}^s_{p,\theta}$ defined in (4.2). Our main results read as follows.

Theorem 5.1. For each $1 \le p, \theta \le \infty$ and $s > 1/p$, we have

$$e(Q_n, \mathring{\mathbf{B}}^s_{p,\theta}) \asymp n^{-s}(\log n)^{(d-1)(1-1/\theta)_+}.$$

Theorem 5.2. For each $1 \le p < \infty$, $1 \le \theta \le \infty$ and $s > 1/p$ the following bounds hold true.

(i) If $s > 1/\theta$ then

$$e(Q_n, \mathring{\mathbf{F}}^s_{p,\theta}) \asymp n^{-s}\,(\log n)^{(d-1)(1-1/\theta)}\,.$$

(ii) If $s < 1/\theta$ then

$$e(Q_n, \mathring{\mathbf{F}}^s_{p,\theta}) \lesssim n^{-s}\,(\log n)^{(d-1)(1-s)}\,.$$

(iii) If $s = 1/\theta$ then

$$e(Q_n, \mathring{\mathbf{F}}^s_{p,\theta}) \lesssim n^{-s}\,(\log n)^{(d-1)(1-s)}\,(\log\log n)^{1-s}\,.$$

Setting $\theta = 2$ immediately implies the following.

Corollary 5.3. For each $1 < p < \infty$ and $s > 1/p$ the following bounds hold true.

(i) If $s > 1/2$ then

$$e(Q_n, \mathring{\mathbf{W}}^s_p) \asymp n^{-s}\,(\log n)^{(d-1)/2}\,.$$

(ii) If $s < 1/2$ then

$$e(Q_n, \mathring{\mathbf{W}}^s_p) \lesssim n^{-s}\,(\log n)^{(d-1)(1-s)}\,.$$

(iii) If $s = 1/2$ then

$$e(Q_n, \mathring{\mathbf{W}}^s_p) \lesssim n^{-s}\,(\log n)^{(d-1)/2}\,\sqrt{\log\log n}\,.$$

Remark 5.4. The subclasses $\mathring{\mathbf{A}}^s_{p,\theta}$ of functions with support inside the unit cube $[0,1]^d$ represent the "natural domain" for the Frolov cubature rule. This comes from the fact that the proof heavily relies on the use of Poisson's summation formula (Corollary 3.2), which requires that we can replace the summation in (1.2) over $\mathbb{X}_n \cap [0,1)^d$ by a summation over $\mathbb{X}_n$ without changing the value of the sum, see (5.1).

5.1. **The proofs.** We begin this section with the derivation of an error formula that is the starting point for the proofs in the specific cases. This explicit formula for $|I(f) - Q_n(f)|$ follows immediately from the pointwise version of Poisson's summation formula for general lattices, see Corollary 3.2.

In this section we consider $f \in \mathring{\mathbf{A}}^s_{p,\theta}$, i.e. functions with support in the unit cube. Hence, we can rewrite our cubature rule Q_n from (1.2) as

$$Q_n(f) \,=\, \frac{1}{n} \sum_{x\in\mathbb{X}_n\cap[0,1)^d} f(x) \,=\, \det(T_n) \sum_{x\in\mathbb{X}_n} f(x). \tag{5.1}$$

Now we take some tensorized admissible kernels Ψ_m, $m \in \mathbb{N}_0^d$, e.g. the compactly supported functions constructed in Section 4, and the corresponding admissible kernels Λ_m, $m \in \mathbb{N}_0^d$, that are given by Calderon's reproducing formula, see (3.6). The construction of these functions, cf. the proof of Lemma 3.6, assures that the functons $\varphi_m := \mathcal{F}[\Lambda_m * \Psi_m]$ satisfy the assumptions of Corollary 3.2 and hence,

$$Q_n(f) \,=\, \sum_{m\in\mathbb{N}_0^d} \sum_{k\in\mathbb{Z}^d} \mathcal{F}[\Lambda_m * \Psi_m](B_n k)\, \mathcal{F}f(B_n k).$$

Note that the inner sum is finite and that, actually, the outer sum is defined as a certain limit, see Corollary 3.2. However, we will see this sum converges absolutely in all cases under consideration.

Note that $\langle \mathrm{e}^{2\pi i k\cdot}, \mathrm{e}^{2\pi i \ell\cdot}\rangle = 1$, if $k = \ell$, and 0 otherwise, where $\langle\cdot,\cdot\rangle$ is usual inner product in $L_2([0,1]^d)$. Using this together with $I(f) = \mathcal{F}f(0)$ we obtain

$$\begin{aligned} |I(f) - Q_n(f)| \,&=\, \Big| \sum_{m\in\mathbb{N}_0^d} \sum_{k\neq 0} \mathcal{F}\Lambda_m(B_n k)\, \mathcal{F}\Psi_m(B_n k)\, \mathcal{F}f(B_n k)\Big| \\ &=\, \Big| \sum_{m\in\mathbb{N}_0^d} \sum_{k\neq 0} \sum_{\ell\neq 0} \mathcal{F}\Lambda_m(B_n k)\, \mathcal{F}[\Psi_m * f](B_n \ell)\, \Big\langle \mathrm{e}^{2\pi i k\cdot}, \mathrm{e}^{2\pi i \ell\cdot}\Big\rangle\Big| \\ &=\, \Big| \sum_{m\in\mathbb{N}_0^d} \Big\langle \sum_{k\neq 0} \mathcal{F}\Lambda_m(B_n k)\, \mathrm{e}^{2\pi i k\cdot}, \sum_{\ell\neq 0} \mathcal{F}[\Psi_m * f](B_n \ell)\, \mathrm{e}^{2\pi i \ell\cdot}\Big\rangle\Big|. \end{aligned} \tag{5.2}$$

Now we have to proceed differently depending on the space under consideration. In fact, we will perform in either case Hölder's inequality twice, but in a different order.

The result for $\mathring{\mathbf{B}}^s_{p,\theta}$. We now prove Theorem 5.1.

Using (5.2) we obtain by Hölder's inequality

$$\begin{aligned} |I(f) - Q_n(f)| \,&=\, \Big| \sum_{m\in\mathbb{N}_0^d} \Big\langle \sum_{k\neq 0} \mathcal{F}\Lambda_m(B_n k)\, \mathrm{e}^{2\pi i k\cdot}, \sum_{\ell\neq 0} \mathcal{F}[\Psi_m * f](B_n \ell)\, \mathrm{e}^{2\pi i \ell\cdot}\Big\rangle\Big| \\ &\le\, \sum_{m\in\mathbb{N}_0^d} \Big\| \sum_{k\neq 0} \mathcal{F}\Lambda_m(B_n k)\, \mathrm{e}^{2\pi i k\cdot}\Big\|_{L_{p'}([0,1]^d)} \Big\| \sum_{\ell\neq 0} \mathcal{F}[\Psi_m * f](B_n \ell)\, \mathrm{e}^{2\pi i \ell\cdot}\Big\|_{L_p([0,1]^d)} \end{aligned}$$

with $1/p + 1/p' = 1$.

Using Lemma 3.5 for the first and Lemma 3.3 for the second factor, we obtain

$$|I(f)-Q_n(f)| \le \sum_{m\in\mathbb{N}_0^d} Z_n(m)^{1-1/p'}\Big(\frac{M_{B_n,\mathrm{supp}(\Psi_m*f)}}{\det(B_n)}\Big)^{1-1/p}\,\|\Lambda_m\|_1\,\|\Psi_m*f\|_p$$

By construction, the third factor is bounded by a constant. The second factor converges, as $n\to\infty$, to $\mathrm{vol}_d(\mathrm{supp}(\Psi_m*f))^{1-1/p} \le \mathrm{vol}_d([-1/2,3/2]^d)^{1-1/p}$, see Remark 3.4. Hence, we obtain with Lemma 2.3 that

(5.3)

$$|I(f)-Q_n(f)| \lesssim \sum_{\substack{m\in\mathbb{N}_0^d:\\ |m|_1>r_n}} (2^{|m|_1}/n)^{1/p}\,\|\Psi_m*f\|_p = n^{-1/p}\sum_{\substack{m\in\mathbb{N}_0^d:\\ |m|_1>r_n}} 2^{|m|_1(1/p-s)}\,2^{s|m|_1}\|\Psi_m*f\|_p$$

with $r_n=\log_2(n)-c$ from (2.6). Applying Hölder's inequality one more time, with $1/\theta+1/\theta'=1$, we finally obtain

$$\begin{aligned} e(Q_n,\mathring{\mathbf{B}}^s_{p,\theta}) &\lesssim n^{-1/p}\Big(\sum_{m\colon |m|_1>r_n} 2^{\theta'|m|_1(1/p-s)}\Big)^{1/\theta'} < n^{-1/p}\Big(\sum_{\ell>r_n}(\ell+1)^{d-1}2^{\theta'\ell(1/p-s)}\Big)^{1/\theta'}\\ &\lesssim n^{-s}(\log n)^{(d-1)(1-1/\theta)}, \end{aligned}$$

since $s>1/p$. This proves Theorem 5.1.

The result for $\mathring{\mathbf{F}}^s_{p,\theta}$. We prove Theorem 5.2. If $s>\max\{1/p,1/\theta\}$ Theorem 5.2,(i) directly follows from Theorem 5.1. In fact, in case $p\le\theta$ we have $\mathbf{F}^s_{p,\theta}\hookrightarrow\mathbf{B}^s_{p,\theta}$ for all s. If $p>\theta$ we use the embedding $\mathring{\mathbf{F}}^s_{p,\theta}\hookrightarrow\mathring{\mathbf{B}}^s_{\theta,\theta}$ due to the compact support and the definition via local means. To apply the results in Theorem 5.1 we need $s>1/\theta$. It remains to deal with the situation $p>\theta$ and $1/p<s\le 1/\theta$.

Again, let $1/p+1/p'=1/\theta+1/\theta'=1$. We obtain from (5.2) that

$$\begin{aligned} |I(f)-Q_n(f)| &= \Big|\sum_{m\in\mathbb{N}_0^d}\Big\langle \sum_{k\neq 0}\mathcal{F}\Lambda_m(B_nk)\,\mathrm{e}^{2\pi ik\cdot},\sum_{\ell\neq 0}\mathcal{F}[\Psi_m*f](B_n\ell)\,\mathrm{e}^{2\pi i\ell\cdot}\Big\rangle\Big|\\ &= \Big|\sum_{m\in\mathbb{N}_0^d}\int_{[0,1]^d}\Big(\sum_{k\neq 0}\mathcal{F}\Lambda_m(B_nk)\,\mathrm{e}^{2\pi ikx}\Big)\overline{\Big(\sum_{\ell\neq 0}\mathcal{F}[\Psi_m*f](B_n\ell)\,\mathrm{e}^{2\pi i\ell x}\Big)}\,\mathrm{d}x\Big| \end{aligned}$$

Interchanging summation and integration and applying Hölder's inequality twice yields

$$\begin{aligned} |I(f)-Q_n(f)| \le &\int_{[0,1]^d}\Big(\sum_{m\in\mathbb{N}_0^d}2^{-\theta' s|m|_1}\Big|\sum_{k\neq 0}\mathcal{F}\Lambda_m(B_nk)\,\mathrm{e}^{2\pi ikx}\Big|^{\theta'}\Big)^{1/\theta'}\\ &\qquad\cdot\Big(\sum_{m\in\mathbb{N}_0^d}2^{\theta s|m|_1}\Big|\sum_{\ell\neq 0}\mathcal{F}[\Psi_m*f](B_n\ell)\,\mathrm{e}^{2\pi i\ell x}\Big|^{\theta}\Big)^{1/\theta}\,\mathrm{d}x\\ \le &\,\Big\|\Big(\sum_{m\in\mathbb{N}_0^d}2^{-\theta' s|m|_1}\Big|\sum_{k\neq 0}\mathcal{F}\Lambda_m(B_nk)\,\mathrm{e}^{2\pi ikx}\Big|^{\theta'}\Big)^{1/\theta'}\Big\|_{L_{p'}([0,1]^d)}\\ &\qquad\cdot\Big\|\Big(\sum_{m\in\mathbb{N}_0^d}2^{\theta s|m|_1}\Big|\sum_{\ell\in\mathbb{Z}^d}\mathcal{F}[\Psi_m*f](B_n\ell)\,\mathrm{e}^{2\pi i\ell x}\Big|^{\theta}\Big)^{1/\theta}\Big\|_{L_p([0,1]^d)}. \end{aligned}$$

At first, we bound the second factor. For this recall that the supports of Ψ_m*f are subsets of $[-1/2,3/2]^d$ and set $M_n=M_{B_n,[-1/2,3/2]^d}$ in Lemma 3.3. Hence, we obtain from Lemma 3.3

(with $f_m = 2^{s|m|_1}\Psi_m * f$) that

$$\Big\|\Big(\sum_{m\in\mathbb{N}_0^d} 2^{\theta s|m|_1}\Big|\sum_{\ell\in\mathbb{Z}^d}\mathcal{F}[\Psi_m * f](B_n\ell)\,\mathrm{e}^{2\pi i\ell\cdot}\Big|^{\theta}\Big)^{1/\theta}\Big\|_{L_p([0,1]^d)}$$
$$\le \big(\det(T_n)\,M_n\big)^{1-1/p}\Big\|\Big(\sum_{m\in\mathbb{N}_0^d} 2^{\theta s|m|_1}\Big|\Psi_m * f\Big|^{\theta}\Big)^{1/\theta}\Big\|_p \lesssim \|f\|_{F^s_{p,\theta}}.$$

It remains to bound the first factor. For this let

$$\widetilde{\Lambda}_{m,n}(x) := \sum_{k\neq 0}\mathcal{F}\Lambda_m(B_nk)\,\mathrm{e}^{2\pi ikx}$$

and note that $p>\theta$ implies $p'<\theta'$.

We prove the upper bound by splitting the sum into two parts. This approach was already used in [35] to prove the result for the Fibonacci cubature rule in $\mathbf{W}^s_p(\mathbb{T}^2)$. Let $L_n := (d-1)\log\log n$ and bound the first factor from above by

$$\Big\|\Big(\sum_{m\colon |m|_1\le r_n+L_n} 2^{-\theta' s|m|_1}\Big|\widetilde{\Lambda}_{m,n}(\cdot)\Big|^{\theta'}\Big)^{1/\theta'}\Big\|_{L_{p'}([0,1]^d)}$$
$$+\ \Big\|\Big(\sum_{m\colon |m|_1> r_n+L_n} 2^{-\theta' s|m|_1}\Big|\widetilde{\Lambda}_{m,n}(\cdot)\Big|^{\theta'}\Big)^{1/\theta'}\Big\|_{L_{p'}([0,1]^d)}.$$

Using $p'\le\theta'$ and $s<1/\theta$ we use Lemma 3.5 and Lemma 2.3 to bound the first summand by

$$\Big(\sum_{m\colon |m|_1\le r_n+L_n} 2^{-\theta' s|m|_1}\Big\|\widetilde{\Lambda}_{m,n}\Big\|^{\theta'}_{L_{\theta'}([0,1]^d)}\Big)^{1/\theta'} \lesssim n^{-1/\theta}\Big(\sum_{\ell=r_n}^{r_n+L_n}(\ell+1)^{d-1}\,2^{-\theta'\ell(s-1/\theta)}\Big)^{1/\theta'}$$
$$\lesssim n^{-s}\,(\log n)^{(d-1)(1-s)}.$$

In the case $s=1/\theta$ this sum is bounded by $n^{-s}\,(\log n)^{(d-1)(1-s)}\,(\log\log n)^{(1-s)}$.

To bound the second summand we replace the $\ell_{\theta'}$-norm inside by a $\ell_{p'}$-norm. We obtain for $s>1/p$ again by Lemma 3.5 and Lemma 2.3 the upper bound

$$\Big(\sum_{m\colon |m|_1> r_n+L_n} 2^{-p' s|m|_1}\Big\|\widetilde{\Lambda}_{m,n}\Big\|^{p'}_{L_{p'}([0,1]^d)}\Big)^{1/p'} \lesssim n^{-1/p}\Big(\sum_{\ell=r_n+L_n}^{\infty}(\ell+1)^{d-1}\,2^{-p'\ell(s-1/p)}\Big)^{1/p'}$$
$$\lesssim n^{-s}\,(\log n)^{(d-1)(1-s)}.$$

This finally proves Theorem 5.2.

6. Integration of functions from $\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)$

For the following considerations it is important to note that all upper bounds from above hold also if we change the domain of integration to an arbitrary bounded, Jordan measurable set, say $\Omega\subset\mathbb{R}^d$, as long as the functions under consideration have support in Ω. That is, the cubature formula $Q_n(f)=\frac{1}{n}\sum_{x\in\mathbb{X}_n} f(x)$, $f\in\mathbf{A}^s_{p,\theta}$, satisfies the bounds from above, if $\mathrm{supp}(f)$ is bounded. In this form, Q_n has formally infinitely many nodes, but all $x\notin\mathrm{supp}(f)$ could obviously be removed from the summation.

In this section we consider periodic functions from $\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)$. For this let $\varepsilon > 0$ be given, φ is the up-function from Section 3.4 and define the one-dimensional function

$$\psi(t) = \begin{cases} \varphi(t/\varepsilon - 1), & \text{if } t < \varepsilon, \\ 1, & \text{if } \varepsilon \le t \le 1, \\ \varphi((t-1)/\varepsilon), & \text{otherwise.} \end{cases}$$

This function satisfies $\operatorname{supp}(\psi) = [0, 1+\varepsilon]$, $\psi(t) + \psi(t-1) = 1$ for all $t \in [0,1]$ and $\psi \in C_0^\infty(\mathbb{R})$. To treat the d-variate situation we define the d-fold tensor products $\psi_d(x) = \prod_{i=1}^d \psi(x_i)$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, which obviously satisfy the natural substitutes for higher dimensions of the properties of ψ from above. Given a 1-periodic function $f\colon \mathbb{R}^d \to \mathbb{R}$ we obtain that

$$\begin{aligned} \int_{\mathbb{R}^d} \psi(x)\, f(x)\,\mathrm{d}x \;&=\; \sum_{m\in\{0,1\}^d} \int_{[0,1]^d} \psi(x-m)\, f(x-m)\,\mathrm{d}x \;=\; \int_{[0,1]^d} \sum_{m\in\{0,1\}^d} \psi(x-m)\, f(x)\,\mathrm{d}x \\ &= \int_{[0,1]^d} f(x)\,\mathrm{d}x \end{aligned}$$

and $\operatorname{supp}(\psi f) \subset [0, 1+\varepsilon]^d$. Recall that the nodes of our cubature formula Q_n are given by (a subset of) a lattice $\mathbb{X}_n = n^{-1/d} T(\mathbb{Z}^d)$, $n > 1$, for a suitable invertible matrix $T \in \mathbb{R}^{d\times d}$. Hence, if we define the cubature formula

$$\widetilde{Q}_n^\psi(f) \;:=\; Q_n(\psi f) \;=\; \sum_{x\in\mathbb{X}_n\cap[0,1+\varepsilon)^d} \frac{\psi(x)}{n}\, f\big(\{x\}\big)$$

for $f \in \mathbf{A}^s_{p,\theta}(\mathbb{T}^d)$, we obtain

$$\Big| \int_{[0,1]^d} f(x)\,\mathrm{d}x - \widetilde{Q}_n^\psi(f) \Big| \;=\; \Big| \int_{[0,1+\varepsilon]^d} \psi(x)\, f(x)\,\mathrm{d}x - Q_n(\psi f) \Big|.$$

The latter is upper bounded by $e(Q_n, \mathring{\mathbf{A}}^s_{p,\theta}) \cdot \|\psi f\|_{\mathbf{A}^s_{p,\theta}}$ by taking the comments from the beginning of this section into account.

To prove that the order of convergence of $\widetilde{Q}_n^\psi$ in $\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)$ is upper bounded by the order of Q_n in $\mathring{\mathbf{A}}^s_{p,\theta}$ it remains to show boundedness of the pointwise multiplication

$$\widetilde{T}_d^\psi : f \mapsto \psi f.$$

For a proof of this fact we refer to [40]. Hence, we obtain the following results, which finally prove Theorems 1.1 and 1.2.

Theorem 6.1. With ψ from above we have for each $1 \le p, \theta \le \infty$ and $s \ge 1/p$ ($\theta = 1$ if $s = 1/p$) that

$$e(\widetilde{Q}_n^\psi, \mathbf{B}^s_{p,\theta}(\mathbb{T}^d)) \;\lesssim\; e(Q_n, \mathring{\mathbf{B}}^s_{p,\theta}).$$

Theorem 6.2. With ψ from above we have for each $1 \le p < \infty$, $1 \le \theta \le \infty$ and $s \ge 1/p$ ($p = 1$ if $s = 1/p$) that

$$e(\widetilde{Q}_n^\psi, \mathbf{F}^s_{p,\theta}(\mathbb{T}^d)) \;\lesssim\; e(Q_n, \mathring{\mathbf{F}}^s_{p,\theta}).$$

Remark 6.3. Clearly, the lower bounds for $\operatorname{Int}_n(\mathring{\mathbf{A}}^s_{p,\theta})$ are also valid for $\operatorname{Int}_n\big(\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)\big)$. Hence, we have shown that the algorithm $\widetilde{Q}_n^\psi$ has the optimal order of convergence for periodic functions from $\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)$ whenever Q_n is proven to be optimal in $\mathring{\mathbf{A}}^s_{p,\theta}$, see Section 8.

Remark 6.4. Using another modification of the cubature formula Q_n it is also possible to treat the spaces $\mathbf{A}^s_{p,\theta}([0,1]^d)$ of (non-periodic) functions on the cube. This modification was already studied by Temlyakov, see [35, Theorem VI.4.1], for Sobolev spaces. Boundedness of the corresponding transformation will be studied in the forthcoming paper [40].

It is worth mentioning that $\widetilde{Q}_n^{\psi}$ is, in a certain sense, only a minor modification of Q_n. The set of nodes is now given by $\widetilde{\mathbb{X}}_n := \big\{\{x\} \colon x \in \mathbb{X}_n \cap [0, 1+\varepsilon)^d\big\}$, where the weight of the node $\{x\}$ is $\psi(x)$. This means that the nodes and the weights of $\widetilde{Q}_n^{\psi}$ and Q_n coincide away from the coordinate axes, i.e. $\widetilde{\mathbb{X}}_n \cap (\varepsilon,1)^d = \mathbb{X}_n \cap (\varepsilon,1)^d$. Only in $[0,1)^d \setminus (\varepsilon,1)^d$ there is a different structure. In this region we have a lot more nodes (about twice as many as in $\mathbb{X}_n$), but the weights are smaller than $1/n$ and no longer equal. We leave it as an open problem, whether this modification is really needed or if the cubature formula Q_n already works for periodic functions. But, note that in general $n \neq \big|\mathbb{X}_n \cap [0,1)^d\big|$ and hence, there is already some issue for Q_n if one considers the integration of the constant function.

7. Quasi-Banach and limiting cases

In this section we deal with the remaining cases of the Besov and Triebel-Lizorkin scales, that are not treated in the previous sections. We are interested in numerical integration and hence only in classes of continuous functions. Besides the quasi-Banach cases $\min\{p,\theta\} < 1$ with $s > 1/p$ we will also consider the limiting case $s = 1/p$. In the latter case additional assumption are needed to assure continuity.

We will state the results for $\mathring{\mathbf{A}}^s_{p,\theta}$. But, however, note that all results of this section are proven by embedding into a space with $p, \theta \geq 1$, which also works for the spaces $\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)$. Moreover, we have the same upper bounds on $e(\widetilde{Q}_n^{\psi}, \mathbf{A}^s_{p,\theta}(\mathbb{T}^d))$ as we have for $e(Q_n, \mathring{\mathbf{A}}^s_{p,\theta})$, see Theorems 6.1 and 6.2, in the cases $p, \theta \geq 1$. Hence, the results below could be stated also with $e(Q_n, \mathring{\mathbf{A}}^s_{p,\theta})$ replaced by $e(\widetilde{Q}_n^{\psi}, \mathbf{A}^s_{p,\theta}(\mathbb{T}^d))$.

7.1. **The situation** $\min\{p,\theta\} < 1$ **and** $s > 1/p$**.** In this section we deal with the classes $\mathring{\mathbf{B}}^s_{p,\theta}$ and $\mathring{\mathbf{F}}^s_{p,\theta}$ with $p, \theta < 1$, i.e. the quasi-Banach cases. Here, $p < 1$ affects the asymptotical error order negatively, while $\theta < 1$ does not. The presented lower bounds are given in the upcoming Section 8.

Corollary 7.1. Let $1 \leq p \leq \infty$ (with $p < \infty$ in the $\mathbf{F}$-case), $0 < \theta < 1$ and $s > 1/p$.
(i) Then
$$e(Q_n, \mathring{\mathbf{B}}^s_{p,\theta}) \asymp n^{-s}.$$
(ii) If $s \geq 1$ then
$$e(Q_n, \mathring{\mathbf{F}}^s_{p,\theta}) \asymp n^{-s}.$$
(iii) If $1/p < s < 1$ then
$$e(Q_n, \mathring{\mathbf{F}}^s_{p,\theta}) \lesssim n^{-s} (\log n)^{(d-1)(1-s)}.$$

Proof. The upper bounds follow from the embeddings $\mathbf{A}^s_{p,\theta} \hookrightarrow \mathbf{A}^s_{p,1}$ for $\theta < 1$, together with Theorems 5.1 & 5.2.

□

Corollary 7.2. Let $0 < p < 1$, $0 < \theta \leq \infty$ and $s > 1/p$.
(i) Then
$$e(Q_n, \mathring{\mathbf{B}}^s_{p,\theta}) \asymp n^{-s+1/p-1} (\log n)^{(d-1)(1-1/\theta)_+},$$

(ii) and

$$e(Q_n, \mathring{\mathbf{F}}^s_{p,\theta}) \asymp n^{-s+1/p-1}.$$

Proof. The stated bounds are a direct consequence of Theorem 5.1 and Theorem 5.2. In fact, from Lemma 4.6 we know that we have for $p<1$ the embeddings

$$\mathring{\mathbf{B}}^s_{p,\theta} \hookrightarrow \mathring{\mathbf{B}}^{s+1-1/p}_{1,\theta} \quad \text{and} \quad \mathring{\mathbf{F}}^s_{p,\theta} \hookrightarrow \mathring{\mathbf{F}}^{s+1-1/p}_{1,1} = \mathring{\mathbf{B}}^{s+1-1/p}_{1,1}.$$

□

Remark 7.3. *(i)* In contrast to the case $p \geq 1$, there is no dependence on θ in (ii). In particular, there is no effect of "small smoothness".

7.2. **Limiting cases.** In this section we deal with the limiting situation $s = 1/p$. When $\theta \leq 1$ in the $\mathbf{B}$-case or $p \leq 1$ in the $\mathbf{F}$-case we have the continuous embedding $\mathbf{A}^s_{p,\theta}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$.

Theorem 7.4. *(i)* Let $0 < p \leq \infty$ and $0 < \theta \leq 1$. Then

$$e(Q_n, \mathring{\mathbf{B}}^{1/p}_{p,\theta}) \asymp n^{-1/\max\{p,1\}}.$$

(ii) Let $0 < p < 1$ and $0 < \theta \leq \infty$. Then

$$e(Q_n, \mathring{\mathbf{F}}^{1/p}_{p,\theta}) \asymp n^{-1}.$$

Proof. Due to the embeddings $\mathring{\mathbf{F}}^{1/p}_{p,\theta}, \mathring{\mathbf{B}}^{1/p}_{p,\theta} \hookrightarrow \mathring{\mathbf{F}}^1_{1,1} = \mathring{\mathbf{B}}^1_{1,1}$ if $p<1$ it suffices to prove (i) in case $1 \leq p \leq \infty$, $s = 1/p$ and $\theta = 1$. It follows the same line as the proof in the case $s > 1/p$. In fact, (5.3) for $s = 1/p$ shows

$$|I(f) - Q_n(f)| \lesssim n^{-1/p} \sum_{\substack{m \in \mathbb{N}_0^d: \\ |m|_1 > r_n}} 2^{|m|_1/p} \|\Psi_m * f\|_p \leq n^{-1/p} \|f\|_{\mathbf{B}^{1/p}_{p,1}}.$$

□

Unfortunately, our proof techniques do not seem to work for $p = 1$ in (ii). The following result, which is probably not sharp, follows from the Jawerth-Franke embedding, see Lemma 4.6, together with Theorem 7.4(i). We conjecture that $e(Q_n, \mathring{\mathbf{F}}^1_{1,\theta}) \asymp n^{-1}(\log n)^{(d-1)(1-1/\theta)}$. We leave this as an open problem.

Corollary 7.5. Let $0 < \theta \leq \infty$. For any $\varepsilon > 0$ there is a constant c_ε such that

$$e(Q_n, \mathring{\mathbf{F}}^1_{1,\theta}) \lesssim c_\varepsilon n^{-(1-\varepsilon)}.$$

Proof. For an arbitrary $p > 1$ we have the embedding $\mathring{\mathbf{F}}^1_{1,\theta} \hookrightarrow \mathring{\mathbf{B}}^{1/p}_{p,1}$, see Lemma 4.6(iii). By Theorem 7.4 we obtain the rate $n^{-1/p}$.

□

7.3. **The classes $\mathbf{W}^s_1(\mathbb{T}^d)$ and $\mathbf{W}^s_\infty(\mathbb{T}^d)$.** Let us finally comment on the classes $\mathbf{W}^s_{1,\alpha}(\mathbb{T}^d)$ and $\mathbf{W}^s_{\infty,\alpha}(\mathbb{T}^d)$ which have been frequently used by Temlyakov and several other authors from the former Soviet Union, [35, Chapt. I.3]. Let for $s > 0$ and $\alpha \in \mathbb{R}$

$$F_s(t,\alpha) := 1 + 2\sum_{k=1}^{\infty} k^{-s}\cos(2\pi k t - \alpha\pi/2) \quad , \quad t \in \mathbb{T},$$

denote the univariate Bernoulli kernel of order s. Let further

$$F_s(x,\alpha) := \prod_{i=1}^{d} F_s(x_i,\alpha) \quad , \quad x = (x_1,...,x_d) \in \mathbb{R}^d,$$

its d-fold tensor product. For $s > 0$ and $1 \le p \le \infty$ we define the classes

$$\mathbf{W}^s_{p,\alpha}(\mathbb{T}^d) := \{f = F_s(\cdot,\alpha) * \varphi \ : \ \|\varphi\|_p \le 1\}\,.$$

In the case $1 < p < \infty$ and $s > 0$ the class $\mathbf{W}^s_{p,\alpha}(\mathbb{T}^d)$ is essentially the unit ball in $\mathbf{W}^s_p(\mathbb{T}^d)$ considered above. However, the classes $\mathbf{W}^s_{1,\alpha}(\mathbb{T}^d)$ and $\mathbf{W}^s_{\infty,\alpha}$ are neither contained in the $\mathbf{B}$ nor in the $\mathbf{F}$-scale. Nevertheless, we have some useful embeddings. Due to $F_s(\cdot,\alpha) \in \mathbf{B}^s_{1,\infty}(\mathbb{T}^d)$ we immediately obtain the embedding $\mathbf{W}^s_{p,\alpha} \hookrightarrow \mathbf{B}^s_{p,\infty}(\mathbb{T}^d)$ for all $1 \le p \le \infty$, see for instance [35, I.3, III.3]. Moreover, as a simple consequence of Hölder's inequality we have $\mathbf{W}^s_{\infty,\alpha}(\mathbb{T}^d) \hookrightarrow \mathbf{W}^s_p(\mathbb{T}^d)$. Note, that there is a slight abuse of notation since the spaces on the left-hand side are rather classes (unit balls) of functions than spaces.

Theorem 7.6. For $s > 1$ we have

$$e(\widetilde{Q}^\psi_n, \mathbf{W}^s_{1,0}(\mathbb{T}^d)) \asymp n^{-s}(\log n)^{d-1}\,.$$

Proof. The upper bound holds for any $\alpha \in \mathbb{R}$ and the corresponding class $\mathbf{W}^s_{1,\alpha}(\mathbb{T}^d)$. It follows immediately from the embedding $\mathbf{W}^s_{1,\alpha}(\mathbb{T}^d) \hookrightarrow \mathbf{B}^s_{1,\infty}(\mathbb{T}^d)$ and Theorem 5.1 together with Theorem 6.1. The lower bound is a consequence of the fact that for any $\widetilde{Q}^\psi_n$ the sum over the cubature weights $\sum_i |\lambda^i|$ is universally bounded with respect to n, see [33]. $\square$

Note that the lower bound only holds for cubature formulas if the sum of the absolute values of the weights is bounded independent of n. Clearly, this is fulfilled for $\widetilde{Q}^\psi_n$. A matching lower bound for arbitrary cubature formulas is not known in this situation. We conjecture that the provided upper bound is sharp.

Theorem 7.7. Let $\alpha \in \mathbb{R}$.
(i) If $0 < s < 1/2$ then

$$e(\widetilde{Q}^\psi_n, \mathbf{W}^s_{\infty,\alpha}(\mathbb{T}^d)) \lesssim n^{-s}(\log n)^{(d-1)(1-s)}\,.$$

(ii) If $s = 1/2$ then

$$e(\widetilde{Q}^\psi_n, \mathbf{W}^s_{\infty,\alpha}(\mathbb{T}^d)) \lesssim n^{-1/2}(\log n)^{(d-1)/2}\sqrt{\log\log n}\,.$$

(ii) If $s > 1/2$ then

$$e(\widetilde{Q}^\psi_n, \mathbf{W}^s_{\infty,\alpha}(\mathbb{T}^d)) \lesssim n^{-s}(\log n)^{(d-1)/2}\,.$$

Proof. The result is a consequence of Corollary 5.3 together with Theorem 6.2. If $s > 1/2$ we use the embedding $\mathbf{W}^s_{\infty,\alpha}(\mathbb{T}^d) \hookrightarrow \mathbf{W}^s_2(\mathbb{T}^d)$. If $s \le 1/2$ choose $2 < p < \infty$ such that $1/p < s \le 1/2$. Then we see the "small smoothness effect". $\square$

We do not know any matching lower bounds in this situation, neither for the specific rule $\widetilde{Q}^\psi_n$ nor for the general situation.

8. Lower bounds

The lower bounds that we want to present are valid for *arbitrary* cubature formulas. For this we study the quantity $\mathrm{Int}_n(\mathbf{F}_d)$ from (1.1) for the spaces $\mathbf{A}^s_{p,\theta}$. Moreover, since $\mathring{\mathbf{A}}^s_{p,\theta}$ is continuously embedded in $\mathbf{A}^s_{p,\theta}(\mathbb{T}^d)$, it is sufficient to present the lower bounds for the "smallest" class $\mathring{\mathbf{A}}^s_{p,\theta}$. There are already lower bounds for Besov classes in the literature see [33] and the recent work [8]. None of those references gives lower bounds for $p < 1$. We will provide them using an approach which is close to the one in [8]. We will use the modern tool of atomic decompositions [43], see Section 8.1 below, to construct appropriate fooling functions.

8.1. **Atomic decomposition.** We will describe the notion of an atom first. For $j \in \mathbb{N}_0^d$ and $k \in \mathbb{Z}^d$ let $Q_{j,k}$ denote the cube with center $(2^{-j_1}m_1, ..., 2^{-j_d}m_d)$ and with sides parallel to the coordinate axes of length $2^{-j_1}, ..., 2^{-j_d}$. For $\gamma > 0$ we denote with $\gamma Q_{j,k}$ the cube concentric with $Q_{j,k}$ with sides also parallel to the axes and length $\gamma 2^{-j_1}, ..., \gamma 2^{-j_d}$.

Definition 8.1. Let $K \in \mathbb{N}_0$, $L+1 \in \mathbb{N}_0$ and $\gamma > 1$. A K-times differentiable complex-valued function $a_{j,k}$ is called (K,L)-atom centered at $Q_{j,k}$ if
(i) $\operatorname{supp} a_{j,k} \subset \gamma Q_{j,k}$,
(ii) $|D^\alpha a_{j,k}(x)| \leq 2^{\alpha \cdot j}$, for all $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d,\ 0 \leq \alpha_i \leq K$,
(iii) and there are the coordinate-wise moment conditions

$$\int_{-\infty}^{\infty} x_i^\ell a_{j,k}(x) dx_i = 0 \quad \text{if} \quad i = 1, ..., d,\ \ell = 0, ..., L,\ j_i \geq 1 .$$

The following Proposition is due to Vybíral [43]. Recall, that $\sigma_p := \max\{0, 1/p - 1\}$ for $0 < p < \infty$.

Proposition 8.2. Let $0 < p, \theta \leq \infty$ and $r \in \mathbb{R}$. Fix $K \in \mathbb{N}_0$ and $L+1 \in \mathbb{N}_0$ with

$$K \geq (1 + \lfloor r \rfloor)_+ \quad \text{and} \quad L \geq \max\{-1, \lfloor \sigma_p - r \rfloor\} .$$

If $\{\lambda_{j,k}\}_{j,k}$ is a sequence of complex-valued coefficients and $\{a_{j,k}(x)\}_{j,k}$ a collection of (K,L)-atoms centered at $Q_{j,k}$ then the function

$$f := \sum_{j \in \mathbb{N}_0^d} \sum_{k \in \mathbb{Z}^d} \lambda_{j,k} a_{j,k}(x) \tag{8.1}$$

belongs to $\mathbf{B}^s_{p,\theta}$ if the right-hand side in (8.2) below is finite. Then it holds

$$\|f\|_{\mathbf{B}^s_{p,\theta}} \lesssim \Big(\sum_{j \in \mathbb{N}_0^d} 2^{|j|_1 (r - 1/p)\theta} \Big[\sum_{k \in \mathbb{Z}^d} |\lambda_{j,k}|^p \Big]^{\theta/p} \Big)^{1/\theta} . \tag{8.2}$$

Consider the up-function φ from Section 3.4. Now we define

$$a_{j,k}(x_1, ..., x_d) := \varphi(2^{j_1+1}x_1 - k_1) \cdot ... \cdot \varphi(2^{j_d+1}x_d - k_d) \quad , \quad x \in \mathbb{R}^d,\ j \in \mathbb{N}_0^d,\ k \in \mathbb{Z}^d$$

and observe thet $a_{j,k}$ is $(K,-1)$-atom for any $K \in \mathbb{N}_0$ centered at $Q_{j,k}$ with $\gamma = 1$ according to Definition 8.1. Note, that we do not have moment conditions. According to Proposition 8.2 we do not need moment conditions if $r > \sigma_p$.

8.2. **Test functions.** Now we are in a position to define test functions of type (8.1) in order to prove the Theorem below. By (8.2) we are able to control the norm $\|\cdot\|_{\mathbf{B}^s_{p,\theta}}$. Following [8, Thm. 4.1] we will use test functions of type

$$g^1_{s,\theta} := C 2^{-sm} m^{-(d-1)/\theta} \sum_{|j|_1 = m+1} \sum_{k \in K_j(X_n)} a_{j,k} , \tag{8.3}$$

where $D_\ell := \{1, ..., 2^\ell - 1\}$, $\ell \in \mathbb{N}$, and $K_j(X_n) \subset \mathbb{D}_j := D_{j_1} \times ... \times D_{j_d}$ depends on the set of integration nodes $X_n := \{x^1, ..., x^n\}$.

Let $\mathring{\mathbf{A}}^s_{p,\theta} \cap C(\mathbb{R}^d)$ denote the function class $\mathring{\mathbf{A}}^s_{p,\theta} \cap C(\mathbb{R}^d)$ equipped with the norm $\|\cdot\|_{\mathbf{A}^s_{p,\theta}}$.

Theorem 8.3. Let $0 < p, \theta \leq \infty$ and $s > \sigma_p := \max\{0, 1/p - 1\}$.
(i) Then

$$\mathrm{Int}_n(\mathring{\mathbf{B}}^s_{p,\theta} \cap C(\mathbb{R}^d)) \gtrsim n^{-s + (1/p - 1)_+} (\log n)^{(d-1)(1 - 1/\theta)_+} .$$

(ii) If $1 \le p < \infty$ then

$$\mathrm{Int}_n(\mathring{\mathbf{F}}^s_{p,\theta} \cap C(\mathbb{R}^d)) \gtrsim n^{-s}(\log n)^{(d-1)(1-1/\theta)_+}\,.$$

(iii) If $0 < p < 1$ then

$$\mathrm{Int}_n(\mathring{\mathbf{F}}^s_{p,\theta} \cap C(\mathbb{R}^d)) \gtrsim n^{-s+1/p-1}\,.$$

Proof. We only need to prove (i). In fact, (ii) follows from (i) and the embeddings $\mathring{\mathbf{B}}^s_{\theta,\theta} \hookrightarrow \mathring{\mathbf{F}}^s_{p,\theta}$ if $\theta \ge p$ and $\mathring{\mathbf{B}}^s_{p,\theta} \hookrightarrow \mathring{\mathbf{F}}^s_{p,\theta}$ if $\theta < p$. The relation in (iii) follows from (i) and the embedding $\mathring{\mathbf{B}}^s_{p,\min\{p,\theta\}} \hookrightarrow \mathring{\mathbf{F}}^s_{p,\theta}$ where $\min\{p,\theta\} < 1$.
Let us prove (i): We follow the arguments in [8, Thm. 4.1]. Let n be given and $X_n = \{x^1,...,x^n\} \subset [0,1]^d$ be an arbitrary set of n points. Without loss of generality we assume that $n = 2^m$. Since $Q_{j,k} \cap Q_{j,k'} = \emptyset$ for $k \neq k'$ we have for every $|j|_1 = m+1$ a set $\mathbb{D}_j(X_n) \subset \mathbb{D}_j$ with $\#\mathbb{D}_j(X_n) \gtrsim 2^{|j|_1}$ and $X_n \cap \mathbb{D}_j(X_n) = \emptyset$.

(a) Let $p, \theta \ge 1$. We choose the test function (8.3) with $K_j(X_n) := \mathbb{D}_j(X_n)$. Note, that for every $k \in K_j(X_n)$ we have that $\operatorname{supp} a_{j,k} \subset [0,1]^d$. The function $g^1_{s,\theta}$ in (8.3) is defined via a finite sum of continuous functions with support contained in $[0,1]^d$. Therefore, $g^1_{s,\theta}$ is continuous and $\operatorname{supp} g^1_{s,\theta} \subset [0,1]^d$. By Proposition 8.2 and (8.2) we can arrange $C > 0$ such that $\|g^1_{s,\theta}\|_{\mathbf{B}^s_{p,\theta}} \le 1$. Clearly, $g^1_{s,\theta}$ belongs to $\mathring{\mathbf{B}}^s_{p,\theta} \cap C(\mathbb{R}^d)$. It is obvious that

$$\int_{[0,1]^d} g^1_{s,\theta}\,dx \asymp 2^{-ms} m^{(d-1)(1-1/\theta)}\,.$$

Of course, a cubature rule admitted in (1.1) that uses the points X_n produces a zero output. This proves (i) in case $p, \theta \ge 1$.

(b) Let $p \ge 1$ and $\theta < 1$. Let us choose a $j \in \mathbb{N}^d$ with $|j|_1 = m+1$ and define the function

$$g^2_s := C 2^{-sm} \sum_{k \in \mathbb{D}_j(X_n)} a_{j,k}. \tag{8.4}$$

Again, by Proposition 8.2 there is a $C > 0$ such that $\|g^2_s\|_{\mathbf{B}^s_{p,\theta}} \le 1$. Moreover, we have

$$\int_{[0,1]^d} g^2_s\,dx \asymp 2^{-sm}\,.$$

With the same reasoning as in (a) this proves (i) in case $\theta < 1$.

(c) Let $p < 1$ and $\theta \ge 1$. We define the test function

$$g^3_{s,p,\theta} := C 2^{-sm} m^{-(d-1)/\theta} 2^{m/p} \sum_{|j|_1 = m+1} a_{j,k_j}\,,$$

where k_j is chosen from $\mathbb{D}_j(X_n)$. By (8.2) we find a $C > 0$ such that $\|g^3_{s,\theta}\|_{\mathbf{B}^s_{p,\theta}} \le 1$. Computing the integral gives

$$\int_{[0,1]^d} g^3_{s,p,\theta}\,dx \asymp 2^{(-s+1/p-1)m} m^{(d-1)(1-1/\theta)}\,.$$

With the same reasoning as in (a) and (b) this proves (i) in case $p < 1$ and $\theta \ge 1$.

(d) Finally, if $0 < p, \theta < 1$ we simply take one single atom $g^4_{s,p} := 2^{-ms} 2^{m/p} a_{j,k}$ with $j \in \mathbb{N}^d$ and $k \in \mathbb{D}_j(X_n)$. Again, there is a $C > 0$ such that $\|g^4_{s,p}\|_{\mathbf{B}^s_{p,\theta}} \le 1$ and

$$\int_{[0,1]^d} g^4_{s,p}\,dx \asymp 2^{(-s+1/p-1)m}\,,$$

which finishes the proof. □

Remark 8.4. Note that the lower bound in (ii) differs from our upper bounds in the case of "small smoothness" $p > \theta$ and $1/p < s \leq \min\{1, 1/\theta\}$. Hence, to prove optimality of Frolov's cubature formula for each $\mathring{\mathbf{A}}^s_{p,\theta}$ it remains to prove the corresponding lower bound for "small" smoothness in Triebel-Lizorkin and Sobolev spaces. This seems to be very delicate and we leave it as an open problem. However, we conjecture that our upper bounds are tight. This is supported by the fact that for $d = 2$ the corresponding lower bound was proven for the Fibonacci cubature formula, see [35, Theorem 2.5], which is conjectured to be optimal, cf. [18].

Acknowledgement. The authors would like to thank the organizers of the conference "Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing" (Leuven, 2014), where this work has been initiated, for providing a pleasent and fruitful working atmosphere. They would further like to thank Dauren Bazarkhanov for pointing out the reference [7], and Glenn Byrenheid, Aicke Hinrichs, Dinh Dũng, Erich Novak, Jens Oettershagen, Winfried Sickel and Vladimir N. Temlyakov for several helpful remarks and comments on earlier versions of this manuscript.

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