

Sampling on energy-norm based sparse grids for the optimal recovery of Sobolev type functions in H^γ

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Abstract

We investigate the rate of convergence of linear sampling numbers of the embedding $H^{\alpha,\beta}(\mathbb{T}^d) \hookrightarrow H^\gamma(\mathbb{T}^d)$. Here α governs the mixed smoothness and β the isotropic smoothness in the space $H^{\alpha,\beta}(\mathbb{T}^d)$ of hybrid smoothness, whereas $H^\gamma(\mathbb{T}^d)$ denotes the isotropic Sobolev space. If $\gamma > \beta$ we obtain sharp polynomial decay rates for the first embedding realized by sampling operators based on “energy-norm based sparse grids” for the classical trigonometric interpolation. This complements earlier work by Griebel, Knapek and Dũng, Ullrich, where general linear approximations have been considered. In addition, we study the embedding $H_{\text{mix}}^\alpha(\mathbb{T}^d) \hookrightarrow H_{\text{mix}}^\gamma(\mathbb{T}^d)$ and achieve optimality for Smolyak’s algorithm applied to the classical trigonometric interpolation. This can be applied to investigate the sampling numbers for the embedding $H_{\text{mix}}^\alpha(\mathbb{T}^d) \hookrightarrow L_q(\mathbb{T}^d)$ for $2 < q \leq \infty$ where again Smolyak’s algorithm yields the optimal order. The precise decay rates for the sampling numbers in the mentioned situations always coincide with those for the approximation numbers, except probably in the limiting situation $\beta = \gamma$ (including the embedding into $L_2(\mathbb{T}^d)$). The best what we could prove there is a (probably) non-sharp results with a logarithmic gap between lower and upper bound.

1 Introduction

The efficient approximation of multivariate functions is a crucial task for the numerical treatment of several real-world problems. Typically the computation time of approximating algorithms grows dramatically with the number of variables d . Therefore, one is interested in reasonable model assumptions and corresponding efficient algorithms. In fact, a large class of solutions of the electronic Schrödinger equation in quantum chemistry does not only belong to a Sobolev spaces with mixed regularity, one also knows additional information in terms of isotropic smoothness properties, see Yserentant’s recent lecture notes [40] and the references therein. This type of regularity is precisely expressed by the spaces $H^{\alpha,\beta}(\mathbb{T}^d)$, defined in Section 2 below. Here, the parameter α reflects the smoothness in the dominating mixed sense and the parameter β reflects the smoothness in the isotropic sense. We aim at approximating such functions in an energy-type norm, i.e., we measure the approximation error in

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an isotropic Sobolev space $H^\gamma(\mathbb{T}^d)$. This is motivated by the use of Galerkin methods for the $H^1(\mathbb{T}^d)$ -approximation of the solution of general elliptic variational problems see, e.g., [1, 2, 11, 10, 12, 24]. The present paper can be seen as a continuation of [9], where finite-rank approximations in the sense of approximation numbers were studied. The latter are defined as

$$a_m(T : X \rightarrow Y) := \inf_{\substack{A: X \rightarrow Y \\ \text{rank } A \leq m}} \sup_{\|f\|_X \leq 1} \|Tf - Af\|_Y \quad , \quad m \in \mathbb{N} ,$$

where X, Y are Banach spaces and $T \in \mathcal{L}(X, Y)$, where $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators $T : X \rightarrow Y$. In contrast to that, we restrict the class of admissible algorithms even further in this paper and deal with the problem of the optimal recovery of $H^{\alpha, \beta}$ -functions from only a finite number of function values, where the optimality in the worst-case setting is commonly measured in terms of linear sampling numbers

$$g_m(T : X \rightarrow Y) := \inf_{(x_j)_{j=1}^m \subset \mathbb{T}^d} \inf_{(\psi_j)_{j=1}^m \subset Y} \sup_{\|f\|_X \leq 1} \left\| Tf - \sum_{j=1}^m f(x_j) \psi_j(\cdot) \right\|_Y \quad , \quad m \in \mathbb{N} .$$

Here, $X \subset C(\mathbb{T}^d)$ denotes a Banach space of functions on \mathbb{T}^d and $T \in \mathcal{L}(X, Y)$. The inclusion of X in $C(\mathbb{T}^d)$ is necessary to give a meaning to function evaluations at single points $x_j \in \mathbb{T}^d$.

We will mainly focus on the situation $X = H^{\alpha, \beta}(\mathbb{T}^d)$ and $Y = H^\gamma(\mathbb{T}^d)$. The condition $\alpha > \gamma - \beta$ ensures a compact embedding

$$I_1 : H^{\alpha, \beta}(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d) \tag{1.1}$$

such that we can ask for the asymptotic decay of the sampling numbers

$$g_m(I_1 : H^{\alpha, \beta}(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d))$$

in m . By investing more isotropic smoothness $\gamma \geq 0$ in the target space $H^\gamma(\mathbb{T}^d)$ than $\beta \in \mathbb{R}$ in the source space $H^{\alpha, \beta}$ we encounter two surprising effects for the sampling numbers $g_m(I_1)$ if $\gamma > \beta$. The main result of the present paper is the following asymptotic order

$$g_m(I_1) \asymp a_m(I_1) \asymp m^{-(\alpha + \beta - \gamma)} \quad , \quad m \in \mathbb{N} , \tag{1.2}$$

which shows, on the one hand, the asymptotic equivalence to the approximation numbers and, on the other hand, the purely polynomial decay rate, i.e., no logarithmic perturbation. In the case $\beta = 0$ sampling numbers for these kind of embeddings were also studied in [13]. The current paper can be considered as a partial periodic counterpart of the recent papers [7, 8] where the author has investigated the nonperiodic situation, namely sampling recovery in L_q -norms as well as corresponding isotropic Sobolev norms of functions on $[0, 1]^d$ from Besov spaces $B_{p, \theta}^{\alpha, \beta}$ with hybrid smoothness of mixed smoothness α and isotropic smoothness β . The asymptotic behavior of the approximation numbers $a_m(I_1 : H^{\alpha, \beta}(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d))$ (including the dependence of all constants on d) has been completely determined in [9], see the Appendix in this paper for a listing of all relevant results. The present paper is intended as a partial extension of the latter reference to the sampling recovery problem. The general observation is the fact that there is no difference in the asymptotic behavior between sampling and general approximation if we impose certain smoothness conditions on the target spaces Y . That is $\gamma > \beta$ if $Y = H^\gamma(\mathbb{T}^d)$ and $\gamma > 0$ if $Y = H_{\text{mix}}^\gamma(\mathbb{T}^d)$.

It turned out, that the critical cases are $\gamma = \beta \geq 0$. We were not able to give the precise decay rate of

$$g_m(I_2 : H^{\alpha, \beta}(\mathbb{T}^d) \rightarrow H^\beta(\mathbb{T}^d)) \tag{1.3}$$

although we are dealing with a Hilbert space setting and additional smoothness in the target space. However, the following statement is true if $\alpha > 1/2$. We have

$$m^{-\alpha}(\log m)^{(d-1)\alpha} \asymp a_m(I_2) \leq g_m(I_2) \lesssim m^{-\alpha}(\log m)^{(d-1)(\alpha+1/2)} \quad , \quad 2 \leq m \in \mathbb{N}.$$

Note, that if $\gamma = \beta = 0$ this includes the classical problem of finding the correct asymptotic behavior of the sampling numbers for the embedding

$$I_3 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d), \quad (1.4)$$

where $H_{\text{mix}}^\alpha(\mathbb{T}^d)$ denotes the Sobolev space of dominating mixed fractional order $\alpha > 1/2$. Originally brought up by Temlyakov [33] in 1985, this problem attracted much attention in multivariate approximation theory, see Dũng [4, 5, 6], Temlyakov [33, 34, 35] and the references therein, Sickel [26, 27], and Sickel, Ullrich [29]-[31]. Temlyakov himself proved for $\alpha > 1/2$ and $2 \leq m \in \mathbb{N}$ the estimate

$$m^{-\alpha}(\log m)^{\alpha(d-1)} \asymp a_m(I_3) \leq g_m(I_3) \lesssim m^{-\alpha}(\log m)^{(d-1)(\alpha+1)}, \quad (1.5)$$

which was later improved by Sickel, Ullrich [29] - [31], Dũng [7], and Triebel [38] to

$$g_m(I_3 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \lesssim m^{-\alpha}(\log m)^{(d-1)(\alpha+1/2)} \quad , \quad 2 \leq m \in \mathbb{N}. \quad (1.6)$$

The estimate for the approximation numbers in (1.5) can be found in [35, Theorem III.4.4]. What concerns the exact d -dependence we refer to Dũng, Ullrich [9, Theorem 4.10] and the recent contribution Kühn, Sickel, Ullrich [16]. There still remains a logarithmic gap of order $(\log m)^{(d-1)/2}$ between the given upper and lower bounds for the sampling numbers. It is a general open problem whether sampling operators can be as good as general linear operators in this particular situation. Let us refer to Hinrichs, Novak, Vybřal [15] and Novak, Woźniakowski [19] for relations between approximation and sampling numbers in an general context. In this paper, we did neither close the gap in (1.5) nor shorten it further. However, we were able to recover these results within our new simplified framework in Subsection 5.3.

Surprisingly, the situation becomes much more easy, when we replace in (1.4) the target space $L_2(\mathbb{T}^d)$ by a Lebesgue space $L_q(\mathbb{T}^d)$ with $q > 2$. In fact, we observed for the embedding

$$I_4 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow L_q(\mathbb{T}^d) \quad (1.7)$$

with $\alpha > 1/2$ the sharp two-sided estimates

$$g_m(I_4) \asymp a_m(I_4) \asymp \begin{cases} m^{-(\alpha-1/2+1/q)}(\log m)^{(d-1)(\alpha-1/2+1/q)} & : \quad 2 < q < \infty, \\ m^{-(\alpha-1/2)}(\log m)^{\alpha(d-1)} & : \quad q = \infty, \end{cases} \quad (1.8)$$

for $2 \leq m \in \mathbb{N}$. The first result of type (1.8) was obtained in [4, 5] for the sampling numbers $g_m(I : B_{p,\infty}^\alpha(\mathbb{T}^d) \rightarrow L_q(\mathbb{T}^d))$ with $1 < p < q \leq 2$, the case $q = \infty$ of (1.8) was observed by Temlyakov [34], we refer to Dũng [7] for nonperiodic results of type (1.8). Our method allowed for a significant extension of these results with a shorter proof. As a vehicle for $2 < q < \infty$ we also took a look to the embedding

$$I_5 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H_{\text{mix}}^\gamma(\mathbb{T}^d) \quad (1.9)$$

with $\alpha > \max\{\gamma, 1/2\}$ and observed

$$g_m(I_5) \asymp a_m(I_5) \asymp m^{-(\alpha-\gamma)}(\log m)^{(d-1)(\alpha-\gamma)} \quad , \quad 2 \leq m \in \mathbb{N}. \quad (1.10)$$

Let us finally mention that the optimal sampling numbers in (1.8) and (1.10) are realized by the well-known Smolyak algorithm. In other words we presented examples where the Smolyak sampling operator yields optimality. It is also used for the upper bound in (1.6), but so far not clear whether it is the optimal choice.

All our proofs are constructive. We explicitly construct sequences of sampling operators that yield the optimal approximation order. Let us briefly describe the framework. The sampling operators will be appropriate sums of tensor products of the classical univariate trigonometric interpolation with respect to the equidistant grid

$$t_\ell^m := \frac{2\pi\ell}{2m+1}, \quad \ell = 0, 1, \dots, 2m,$$

given by

$$I_m f(t) := \frac{1}{2m+1} \sum_{\ell=0}^{2m} f(t_\ell^m) D_m(t - t_\ell^m), \quad (1.11)$$

where

$$D_m(t) := \sum_{|k| \leq m} e^{ikt} = \frac{\sin((m+1/2)t)}{\sin(t/2)}, \quad t \in \mathbb{R}.$$

It is well-known that $I_m f \xrightarrow{m \rightarrow \infty} f$ in $L_2(\mathbb{T})$ for every $f \in H^s(\mathbb{T})$ with $s > 1/2$. Due to telescoping series argument we may also write

$$f = I_1 f + \sum_{k=1}^{\infty} (I_{2^k} - I_{2^{k-1}}) f.$$

Therefore, we put for $m \in \mathbb{N}_0$

$$\eta_m := \begin{cases} I_{2^m} - I_{2^{m-1}} & \text{if } m > 0, \\ I_1 & \text{if } m = 0. \end{cases}$$

The special structure of the η_m immediately admits the following tensorization

$$q_k := \eta_{k_1} \otimes \dots \otimes \eta_{k_d}, \quad k \in \mathbb{N}_0^d. \quad (1.12)$$

Finally, for a given finite $\Delta \subset \mathbb{N}_0^d$ we define the general sampling operator Q_Δ as

$$Q_\Delta := \sum_{k \in \Delta} q_k. \quad (1.13)$$

Our degree of freedom will be the set Δ . We will choose Δ according to the different situations we are dealing with. That means in particular that Δ may depend on the parameters of the function classes of interest. The most interesting case is represented by the index set

$$\Delta(\xi) = \Delta(\alpha, \beta, \gamma; \xi) := \{k \in \mathbb{N}_0^d : \alpha|k|_1 - (\gamma - \beta)|k|_\infty \leq \xi\}, \quad \xi > 0, \quad (1.14)$$

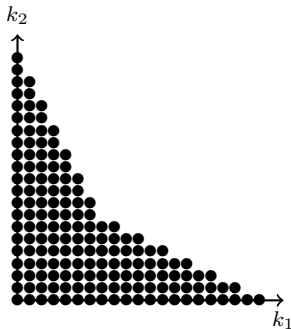


Figure 1: $d = 2$, $\alpha = 2$, $\beta = 0$, $\gamma = 1$, $\xi = 20$

or more exactly, by an ε -modification of it given by

$$\Delta_\varepsilon(\xi) = \Delta(\varepsilon, \alpha, \beta, \gamma; \xi) := \{k \in \mathbb{N}_0^d : (\alpha - \varepsilon)|k|_1 - (\gamma - \beta - \varepsilon)|k|_\infty \leq \xi\} \quad , \quad \xi > 0, \quad (1.15)$$

and $\varepsilon > 0$ chosen sufficiently small (but not close to zero). These index sets will be used in connection with the embedding (1.1). The set of sampling points used by (1.13) will be called “energy-norm based sparse grid”. This phrase stems from the works of Bungartz, Griebel and Knappek [1, 2, 10, 11, 12] and refers to the special case where the error is measured in the “energy space” $H^1(\mathbb{T}^d)$. These authors were the first observing the potential of this modification of the classical “sparse grid”. Here we use the phrase “energy-norm based grids” in the wider sense of being adapted to the smoothness parameter γ of the target space $H^\gamma(\mathbb{T}^d)$ (with α considered to be fixed). These extensions with respect to approximation numbers as well as to sampling numbers have been discussed in [8] (non-periodic case) and [9] (periodic case). In particular, (1.14) in case $\gamma \neq 1$ goes back to [9], and (1.15) in the case $\gamma > 0$ to [8].

The second important example is given by the index set

$$\Delta(\xi) = \Delta(\alpha; \xi) := \{k \in \mathbb{N}_0^d : \alpha|k|_1 \leq \xi\} \quad , \quad \xi > 0, \quad (1.16)$$

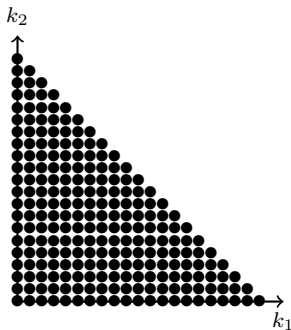


Figure 2: $d = 2$, $\alpha = 1$, $\xi = 20$

and represents the classical Smolyak algorithm, originally introduced in [32]. Although this set represents a special case of (1.14) it has a completely different geometry and leads to structurally different results. The sampling points used by the associated Q_Δ is commonly called “sparse grid”. Putting $\xi = \alpha m$ in (1.16) it is well-known, see [39] and [30, 29], that the

operator $Q_{\Delta(\xi)}$ samples the function f on the grid

$$\mathcal{G}(m) := \left\{ \left(\frac{2\pi\ell_1}{2^{j_1+1} + 1}, \dots, \frac{2\pi\ell_d}{2^{j_d+1} + 1} \right) : \right. \\ \left. 0 \leq \ell_i \leq 2^{j_i}, i = 1, \dots, d, \quad m - d + 1 \leq |j|_1 \leq m \right\}. \quad (1.17)$$

It turned out that the previously defined framework fits very well to the function space setting described above. In Lemma 2.7 below we give the Littlewood-Paley decomposition of $H^{\alpha,\beta}(\mathbb{T}^d)$, i.e.,

$$H^{\alpha,\beta}(\mathbb{T}^d) = \left\{ f \in L_2(\mathbb{T}^d) : \|f\|_{H^{\alpha,\beta}(\mathbb{T}^d)}^2 := \sum_{k \in \mathbb{N}_0^d} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|\delta_k(f)\|_2^2 < \infty \right\}.$$

As usual, $\delta_k(f)$, $k \in \mathbb{N}_0^d$, represents that part of the Fourier series of f supported in a dyadic block

$$\mathcal{P}_k := P_{k_1} \times \dots \times P_{k_d}, \quad (1.18)$$

where $P_j := \{\ell \in \mathbb{Z} : 2^{j-1} \leq |\ell| < 2^j\}$ and $P_0 = \{0\}$. In fact, looking at the approximation scheme in (1.13) it would be desirable to have an equivalent norm where we replace $\delta_k(f)$ by $q_k(f)$ from (1.12). Under additional restrictions on the parameters (one has to at least ensure an embedding in $C(\mathbb{T}^d)$) this is indeed possible as Theorem 3.6 below shows. This gives us convenient characterizations of the function spaces of interest in terms of the sampling operators we are going to analyze.

The paper is organized as follows. In Section 2 we define and discuss the spaces $H_{\text{mix}}^\alpha(\mathbb{T}^d)$ and $H^{\alpha,\beta}(\mathbb{T}^d)$. Section 3 is used to establish our main tool in all proofs involving sampling numbers, the so-called ‘‘sampling representation’’, see Theorem 3.6 below. The next Section 4 deals in a constructive way with estimates from above for the sampling numbers of the embedding (1.1) by evaluating the error norm $\|I - Q_\Delta\|$ with the corresponding Δ from (1.15). With the limiting cases (1.3) leading to the classical Smolyak algorithm we deal in Section 5. Here we also consider the embeddings (1.9) and (1.7). In Section 6 we transfer our approximation results into the notion of sampling numbers and compare them to existing estimates for the approximation numbers. The relevant estimates are collected in the appendix.

Notation. As usual, \mathbb{N} denotes the natural numbers, \mathbb{N}_0 the non-negative integers, \mathbb{Z} the integers and \mathbb{R} the real numbers. With \mathbb{T} we denote the torus represented by the interval $[0, 2\pi]$. The letter d is always reserved for the dimension in \mathbb{Z}^d , \mathbb{R}^d , \mathbb{N}^d , and \mathbb{T}^d . For $0 < p \leq \infty$ and $x \in \mathbb{R}^d$ we denote $|x|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ with the usual modification for $p = \infty$. We write e_j , $j = 1, \dots, d$, for the respective canonical unit vector and $\bar{1} := \sum_{j=1}^d e_j$ in \mathbb{R}^d . If X and Y are two Banach spaces, the norm of an operator $A : X \rightarrow Y$ will be denoted by $\|A : X \rightarrow Y\|$. The symbol $X \hookrightarrow Y$ indicates that there is a continuous embedding from X into Y . The relation $a_n \lesssim b_n$ means that there is a constant $c > 0$ independent of the context relevant parameters such that $a_n \leq cb_n$ for all n belonging to a certain subset of \mathbb{N} , often \mathbb{N} itself. We write $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$ holds.

2 Sobolev-type spaces

In this section we recall the definition of the function spaces under consideration here. They are all of Sobolev-type. In a first subsection we consider the periodic Sobolev spaces $H_{\text{mix}}^\alpha(\mathbb{T}^d)$ of dominating mixed fractional order $\alpha > 0$. In the second subsection the more general classes $H^{\alpha,\beta}(\mathbb{T}^d)$ are discussed.

2.1 Periodic Sobolev spaces of mixed and isotropic smoothness

All results in this paper are stated for function spaces on the d -torus \mathbb{T}^d , which is represented in the Euclidean space \mathbb{R}^d by the cube $\mathbb{T}^d = [0, 2\pi]^d$, where opposite faces are identified. The space $L_2(\mathbb{T}^d)$ consists of all (equivalence classes of) measurable functions f on \mathbb{T}^d such that the norm

$$\|f\|_2 := \left(\int_{\mathbb{T}^d} |f(x)|^2 dx \right)^{1/2}$$

is finite. All information on a function $f \in L_2(\mathbb{T}^d)$ is encoded in the sequence $(c_k(f))_k$ of its Fourier coefficients, given by

$$c_k(f) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z}^d.$$

Indeed, we have Parseval's identity

$$\|f\|_2^2 = (2\pi)^d \sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \tag{2.1}$$

as well as

$$f(x) = \sum_{k \in \mathbb{Z}^d} c_k(f) e^{ikx}$$

with convergence in $L_2(\mathbb{T}^d)$.

The mixed Sobolev space $H_{\text{mix}}^m(\mathbb{T}^d)$ with smoothness vector $m = (m_1, \dots, m_d) \in \mathbb{N}^d$ is the collection of all $f \in L_2(\mathbb{T}^d)$ such that all distributional derivatives $D^\gamma f$ of order $\gamma = (\gamma_1, \dots, \gamma_d)$ with $\gamma_j \leq m_j$, $j = 1, \dots, d$, belong to $L_2(\mathbb{T}^d)$. We put

$$\|f\|_{H_{\text{mix}}^m(\mathbb{T}^d)}^* := \left(\sum_{\substack{0 \leq \gamma_j \leq m_j \\ j=1, \dots, d}} \|D^\gamma f\|_2^2 \right)^{1/2}. \tag{2.2}$$

One can rewrite this definition in terms of Fourier coefficients. However, it is more convenient to use an equivalent norm like

$$\|f\|_{H_{\text{mix}}^m(\mathbb{T}^d)}^\# := \left[\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d (1 + |k_j|^2)^{m_j} \right]^{1/2}. \tag{2.3}$$

For $m \in \mathbb{N}$ we denote with $H^m(\mathbb{T}^d)$ the space $H^{m, \bar{1}}(\mathbb{T}^d)$. Inspired by (2.3) we define Sobolev spaces of dominating mixed smoothness of fractional order α as follows.

Definition 2.1. *Let $\alpha > 0$. The periodic Sobolev space $H_{\text{mix}}^\alpha(\mathbb{T}^d)$ of dominating mixed smoothness α is the collection of all $f \in L_2(\mathbb{T}^d)$ such that*

$$\|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)}^\# := \left[\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \prod_{j=1}^d (1 + |k_j|^2)^\alpha \right]^{1/2} < \infty. \tag{2.4}$$

Remark 2.2. There is different notation in the literature. E.g., Temlyakov and others use $MW_2^\alpha(\mathbb{T}^d)$ instead of $H_{\text{mix}}^\alpha(\mathbb{T}^d)$, whereas Amanov, Lizorkin, Nikol'skij, Schmeisser and Triebel prefer to use $S_2^\alpha W(\mathbb{T}^d)$.

We also need the (isotropic) Sobolev spaces $H^\gamma(\mathbb{T}^d)$.

Definition 2.3. Let $\gamma \geq 0$. The periodic Sobolev space $H^\gamma(\mathbb{T}^d)$ of smoothness γ is the collection of all $f \in L_2(\mathbb{T}^d)$ such that

$$\|f\|_{H^\gamma(\mathbb{T}^d)}^\# := \left[\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 (1 + |k|_2^2)^\gamma \right]^{1/2} < \infty. \quad (2.5)$$

Remark 2.4. It is elementary to check

$$H^{\alpha d}(\mathbb{T}^d) \hookrightarrow H_{\text{mix}}^\alpha(\mathbb{T}^d) \hookrightarrow H^\alpha(\mathbb{T}^d).$$

In addition it is known that $H^\gamma(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$ if and only if $H^\gamma(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d)$ if and only if $\gamma > d/2$, see [28].

2.2 Hybrid type Sobolev spaces

To define the scale $H^{\alpha,\beta}(\mathbb{T}^d)$ we look for subspaces of $H_{\text{mix}}^\alpha(\mathbb{T}^d)$ obtained by adding isotropic smoothness. To make this more transparent we start again with a situation where smoothness can be described exclusively in terms of weak derivatives. It is easy to see that isotropic smoothness of order $n \in \mathbb{N}$ can be achieved by “intersecting” mixed smoothness conditions, i.e.,

$$H^n(\mathbb{T}^d) = H_{\text{mix}}^{(n,0,\dots,0)}(\mathbb{T}^d) \cap H_{\text{mix}}^{(0,n,0,\dots,0)} \cap \dots \cap H_{\text{mix}}^{(0,0,\dots,n)}.$$

Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ such that $m + n \geq 0$. We will use the above principle to “add” an isotropic smoothness of order n to the mixed smoothness of order m . The hybrid type Sobolev space $H^{m,n}(\mathbb{T}^d)$ is the set

$$H^{m,n}(\mathbb{T}^d) = \begin{cases} \bigcap_{j=1}^d H_{\text{mix}}^{m \cdot \bar{1} + ne_j}(\mathbb{T}^d) & : n \geq 0, \\ \sum_{j=1}^d H_{\text{mix}}^{m \cdot \bar{1} + ne_j}(\mathbb{T}^d) & : n < 0. \end{cases}$$

A function $f \in L_2(\mathbb{T}^d)$ belongs to $H^{m,n}(\mathbb{T}^d)$, if and only if the semi-norm

$$|f|'_{H^{m,n}(\mathbb{T}^d)} = \begin{cases} \max_{1 \leq j \leq d} \|f\|_{H_{\text{mix}}^{m \cdot \bar{1} + ne_j}(\mathbb{T}^d)} & : n \geq 0, \\ \min_{1 \leq j \leq d} \|f\|_{H_{\text{mix}}^{m \cdot \bar{1} + ne_j}(\mathbb{T}^d)} & : n < 0, \end{cases}$$

is finite. The norm of f in $H^{m,n}(\mathbb{T}^d)$ is defined as $\|f\|'_{H^{m,n}(\mathbb{T}^d)} := \|f\|_2 + |f|'_{H^{m,n}(\mathbb{T}^d)}$. Hence, one can verify that

$$\|f\|'_{H^{m,n}(\mathbb{T}^d)} \asymp \left[\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \left(\prod_{j=1}^d (1 + |k_j|^2)^m \right) (1 + |k|_2^2)^n \right]^{1/2}.$$

This motivates the following definition.

Definition 2.5. Let $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that $\alpha + \beta \geq 0$. The generalized periodic Sobolev space $H^{\alpha,\beta}(\mathbb{T}^d)$ is the collection of all $f \in L_2(\mathbb{T}^d)$ such that

$$\|f\|_{H^{\alpha,\beta}(\mathbb{T}^d)}^\# := \left[\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \left(\prod_{j=1}^d (1 + |k_j|^2)^\alpha \right) (1 + |k|_2^2)^\beta \right]^{1/2} < \infty. \quad (2.6)$$

Remark 2.6. (i) Obviously we have $H_{\text{mix}}^{\alpha,0}(\mathbb{T}^d) = H_{\text{mix}}^{\alpha}(\mathbb{T}^d)$ and $H_{\text{mix}}^{0,\beta}(\mathbb{T}^d) = H^{\beta}(\mathbb{T}^d)$, $\beta \geq 0$. More important for us will be the embedding

$$H^{\alpha,\beta}(\mathbb{T}^d) \hookrightarrow H^{\gamma}(\mathbb{T}^d) \quad \text{if} \quad 0 \leq \gamma \leq \alpha + \beta. \quad (2.7)$$

(ii) Spaces of such a type have been first considered by Griebel and Knappek [11]. Also in the non-periodic context they play a role in the description of the fine regularity properties of certain eigenfunctions of Hamilton operators in quantum chemistry, see [40]. The periodic spaces $H_{\text{mix}}^{\alpha,\beta}(\mathbb{T}^d)$ also occur in the recent works [9] and [13].

A first step towards the sampling representation in Theorem 3.6 below will be the following equivalent characterization of Littlewood-Paley type. We will work with the dyadic blocks from (1.18) and put for $\ell \in \mathbb{N}_0^d$

$$\delta_{\ell}(f) := \sum_{k \in \mathcal{P}_{\ell}} c_k(f) e^{ikx}.$$

Hence, for all $f \in L_2(\mathbb{T}^d)$ we have the Littlewood-Paley decomposition

$$f = \sum_{\ell \in \mathbb{N}_0^d} \delta_{\ell}(f). \quad (2.8)$$

The following lemma is an elementary consequence of Definition 2.5.

Lemma 2.7. *Let $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that $\alpha + \beta \geq 0$.*

(i) *Then*

$$H^{\alpha,\beta}(\mathbb{T}^d) = \left\{ f \in L_2(\mathbb{T}^d) : \|f\|_{H^{\alpha,\beta}(\mathbb{T}^d)} := \left(\sum_{k \in \mathbb{N}_0^d} 2^{2(\alpha|k|_1 + \beta|k|_{\infty})} \|\delta_k(f)\|_2^2 \right)^{1/2} < \infty \right\}$$

in the sense of equivalent norms.

(ii) *We have*

$$H^{\alpha,\beta}(\mathbb{T}^d) = \begin{cases} \bigcap_{j=1}^d H_{\text{mix}}^{\alpha \cdot \bar{1} + \beta e_j}(\mathbb{T}^d) & : \beta \geq 0, \\ \sum_{j=1}^d H_{\text{mix}}^{\alpha \cdot \bar{1} + \beta_j}(\mathbb{T}^d) & : \beta < 0. \end{cases}$$

We need a few more properties of these spaces. For $\ell \in \mathbb{N}_0^d$ we define the set of trigonometric polynomials

$$\mathcal{T}^{\ell} := \left\{ \sum_{\substack{|k_i| \leq 2^{\ell_i} \\ i=1, \dots, d}} a_k e^{ikx} : a_k \in \mathbb{C} \right\}.$$

Of course, $\delta_{\ell}(f) \in \mathcal{T}^{\ell}$ for all $f \in L_2(\mathbb{T}^d)$.

Lemma 2.8 (Nikol'skij's inequality). *Let $0 < p \leq q \leq \infty$. Then there is a constant $C = C(p, q) > 0$ (independent of g and ℓ) such that*

$$\|g\|_q \leq C 2^{|\ell|_1 \left(\frac{1}{p} - \frac{1}{q}\right)} \|g\|_p$$

holds for every $g \in \mathcal{T}^{\ell}$ and every $\ell \in \mathbb{N}_0^d$.

Proof. A proof can be found in [22, Theorem 3.3.2]. ■

To give a meaning to point evaluations of functions it is essential that the spaces under consideration contain only continuous functions. To be more precise, they contain equivalence classes of functions having one continuous representative.

Theorem 2.9. *Let $\alpha > 0$, $\beta \in \mathbb{R}$ such that $\min\{\alpha + \beta, \alpha + \frac{\beta}{d}\} > \frac{1}{2}$. Then*

$$H^{\alpha, \beta}(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d).$$

Proof. Applying Lemma 2.8 yields

$$\begin{aligned} \sum_{k \in \mathbb{N}_0^d} \|\delta_k(f)\|_\infty &= \sum_{k \in \mathbb{N}_0^d} 2^{\alpha|k|_1 + \beta|k|_\infty} 2^{-(\alpha|k|_1 + \beta|k|_\infty)} \|\delta_k(f)\|_\infty \\ &\lesssim \sum_{k \in \mathbb{N}_0^d} 2^{\alpha|k|_1 + \beta|k|_\infty} 2^{-(\alpha|k|_1 + \beta|k|_\infty)} 2^{\frac{|k|_1}{2}} \|\delta_k(f)\|_2. \end{aligned}$$

Employing Hölder's inequality we find

$$\begin{aligned} \sum_{k \in \mathbb{N}_0^d} \|\delta_k(f)\|_\infty &\leq \left(\sum_{k \in \mathbb{N}_0^d} 2^{-2(\alpha|k|_1 + \beta|k|_\infty)} 2^{|k|_1} \right)^{\frac{1}{2}} \left(\sum_{k \in \mathbb{N}_0^d} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|\delta_k(f)\|_2^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k \in \mathbb{N}_0^d} 2^{-2(\alpha|k|_1 + \beta|k|_\infty)} 2^{|k|_1} \right)^{\frac{1}{2}} \|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}. \end{aligned}$$

Using $|k|_\infty \leq |k|_1 \leq d|k|_\infty$ gives in case $\beta \geq 0$

$$\sum_{k \in \mathbb{N}_0^d} 2^{-2(\alpha|k|_1 + \beta|k|_\infty)} 2^{|k|_1} \leq \sum_{k \in \mathbb{N}_0^d} 2^{-2(\alpha + \frac{\beta}{d} - \frac{1}{2})|k|_1} < \infty,$$

whenever $\alpha + \frac{\beta}{d} > \frac{1}{2}$. For the case $\beta < 0$ observe that

$$\sum_{k \in \mathbb{N}_0^d} 2^{-2(\alpha|k|_1 + \beta|k|_\infty)} 2^{|k|_1} \leq \sum_{k \in \mathbb{N}_0^d} 2^{-2(\alpha + \beta - \frac{1}{2})|k|_1} < \infty$$

if $\alpha + \beta > \frac{1}{2}$. Since $C(\mathbb{T}^d)$ is a Banach space, the sum $\sum_{k \in \mathbb{N}_0^d} \delta_k(f)$ belongs to $C(\mathbb{T}^d)$ due to its absolute convergence. Further

$$f = \sum_{k \in \mathbb{N}_0^d} \delta_k(f)$$

holds in $L_2(\mathbb{T}^d)$. Consequently, the equivalence class $f \in H^{\alpha, \beta}(\mathbb{T}^d)$ has a continuous representative. ■

Remark 2.10. (i) With essentially the same proof technique as above the assertion in Theorem 2.9 can be refined as follows. Let $\alpha \geq 0$ and $\beta \in \mathbb{R}$ such that $\alpha + \beta \geq 0$. Then it holds the embedding

$$H^{\alpha, \beta}(\mathbb{T}^d) \hookrightarrow \begin{cases} H_{\text{mix}}^{\alpha + \beta/d}(\mathbb{T}^d) & : \beta \geq 0, \\ H_{\text{mix}}^{\alpha + \beta}(\mathbb{T}^d) & : \beta < 0. \end{cases}$$

This embedding immediately implies Theorem 2.9.

(ii) The restrictions in Theorem 2.9 are almost optimal. Indeed, let $g \in H^{\alpha + \beta}(\mathbb{T})$, then the function

$$f(x_1, \dots, x_d) := g(x_1), \quad x \in \mathbb{R}^d,$$

belongs to $H^{\alpha, \beta}(\mathbb{T}^d)$. Hence, from $H^{\alpha, \beta}(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$ we derive $H^{\alpha + \beta}(\mathbb{T}) \hookrightarrow C(\mathbb{T})$ which is known to be true if and only if $\alpha + \beta > 1/2$. In case $\alpha = 0$ we know $H^{\alpha, \beta}(\mathbb{T}^d) = H^\beta(\mathbb{T}^d)$. Hence, $H^{0, \beta} \hookrightarrow C(\mathbb{T}^d)$ if and only if $\beta/d > 1/2$.

We will need the following Bernstein type inequality.

Lemma 2.11. *Let $\min\{\alpha, \alpha + \beta - \gamma\} > 0$ and $\ell \in \mathbb{N}_0^d$. Then*

$$\|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)} \leq 2^{\alpha|\ell|_1 + (\beta - \gamma)|\ell|_\infty} \|f\|_{H^\gamma} \quad (2.9)$$

holds for all $f \in \mathcal{T}^\ell$.

Proof. Indeed, for $f \in \mathcal{T}^\ell$, we have

$$\begin{aligned} \|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}^2 &= \sum_{\substack{k_i \leq \ell_i \\ i=1, \dots, d}} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|\delta_k(f)\|_2^2 \leq \max_{\substack{k_i \leq \ell_i \\ i=1, \dots, d}} 2^{2(\alpha|k|_1 + (\beta - \gamma)|k|_\infty)} \sum_{\substack{k_i \leq \ell_i \\ i=1, \dots, d}} 2^{2\gamma|k|_\infty} \|\delta_k(f)\|_2^2 \\ &\leq 2^{2(\alpha|\ell|_1 + (\beta - \gamma)|\ell|_\infty)} \|f\|_{H^\gamma}^2. \end{aligned}$$

■

3 Sampling representations

Our main aim in this section consists in deriving a specific Nikol'skij-type representation for the spaces $H^{\alpha, \beta}(\mathbb{T}^d)$ in the spirit of Lemma 2.7. Specific in the sense, that the building blocks in the decomposition originate from associated sampling operators of type (1.12). First we need some technical lemmas.

Lemma 3.1. *Let $\alpha > 0$, $\beta \in \mathbb{R}$, $\min\{\alpha, \alpha + \beta\} > 0$ and*

$$\psi(k) := \alpha|k|_1 + \beta|k|_\infty \quad , \quad k \in \mathbb{N}_0^d.$$

Then there is an $\varepsilon > 0$ such that

$$\psi(k) \leq \psi(k') - \varepsilon(|k'|_1 - |k|_1)$$

holds for all $k', k \in \mathbb{N}_0^d$ with $k' \geq k$ component-wise.

Proof. Let $k' \geq k$. This implies

$$\psi(k) = \psi(k') - \alpha|k' - k|_1 - \beta(|k'|_\infty - |k|_\infty) \quad (3.1)$$

We need to distinguish two cases.

Case 1. If $\beta \geq 0$ we have as an immediate consequence of (3.1)

$$\psi(k) \leq \psi(k') - \alpha|k' - k|_1.$$

Case 2. Let $\beta < 0$. From (3.1) and

$$|k'|_\infty - |k|_\infty \leq |k' - k|_\infty \leq |k' - k|_1$$

we obtain

$$\psi(k) \leq \psi(k') - (\alpha + \beta)|k' - k|_1.$$

■

Recall the linear operator q_k has been defined in (1.12). Let us settle the following cancellation property.

Lemma 3.2. *Let $\ell, k \in \mathbb{N}_0^d$ with $k_n < \ell_n$ for some $n \in \{1, \dots, d\}$. Let further $f \in T^k$ and q_ℓ be the operator defined in (1.12). Then $q_\ell(f) = 0$.*

Proof. Since $f \in \mathcal{T}^k$ we have

$$f = \sum_{\substack{|m_j| \leq 2^{k_j} \\ j=1, \dots, d}} a_m e^{imx}$$

and

$$q_\ell(f)(x) = \sum_{\substack{|m_j| \leq 2^{k_j} \\ j=1, \dots, d}} a_m q_\ell(e^{im \cdot})(x) = \sum_{\substack{|m_j| \leq 2^{k_j} \\ j=1, \dots, d}} a_m \prod_{j=1}^d \eta_{\ell_j}(e^{im_j \cdot})(x_j).$$

Due to $2^{\ell_n-1} \geq 2^{k_n} \geq m_n$ we have

$$\eta_{\ell_n}(e^{im_n \cdot})(x_n) = (I_{2^{\ell_n}} - I_{2^{\ell_n-1}})(e^{im_n \cdot})(x_n) = 0$$

which implies $q_\ell(f) = 0$. ■

Now we are in the position to proof Nikol'skij's type representation theorems for the spaces $H^{\alpha, \beta}(\mathbb{T}^d)$.

Proposition 3.3. *Let $\min(\alpha, \alpha + \beta) > 1/2$. Then every function $f \in H^{\alpha, \beta}(\mathbb{T}^d)$ can be represented by the series*

$$f = \sum_{k \in \mathbb{N}_0^d} q_k(f) \tag{3.2}$$

converging unconditionally in $H^{\alpha, \beta}(\mathbb{T}^d)$, and satisfying the condition

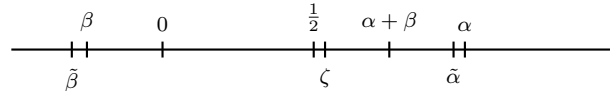
$$\sum_{k \in \mathbb{N}_0^d} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|q_k(f)\|_2^2 \leq C \|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}^2 \tag{3.3}$$

with a constant $C = C(\alpha, \beta, d) > 0$.

Proof. *Step 1.* We first prove (3.3) for $f \in H^{\alpha, \beta}(\mathbb{T}^d)$. Let us assume $\beta \neq 0$, otherwise set $\beta = \tilde{\beta} = 0$. For technical reasons we need to fix $\tilde{\alpha}, \zeta, \tilde{\beta} \in \mathbb{R}$ such that

$$\min\{\tilde{\alpha} - \zeta, \tilde{\alpha} - \zeta + \tilde{\beta}\} > 0, \quad \alpha - \tilde{\alpha} > 0, \quad \tilde{\beta} < \beta \text{ and } \zeta > \frac{1}{2} \tag{3.4}$$

holds. For $\beta > 0$ it is easy to find parameters $\tilde{\alpha}, \zeta, \tilde{\beta}$ fulfilling (3.4). Critical is the case $\beta < 0$. Here we choose the parameters in the following way:



The condition $\alpha + \beta > \frac{1}{2}$ implies that there is some $\varepsilon > 0$ such that $\alpha + \beta - \varepsilon > \frac{1}{2}$ holds. Choose now $\tilde{\alpha}, \tilde{\beta}, \zeta \in \mathbb{R}$ s.t. $\beta - \frac{\varepsilon}{2} < \tilde{\beta} < \beta$ and $\frac{1}{2} < \zeta < \tilde{\alpha} < \alpha$ with

$$0 < \alpha - \frac{1}{2} - \frac{\varepsilon}{2} < \tilde{\alpha} - \zeta < \alpha - \frac{1}{2}.$$

Obviously this is possible. It is easy to check that such a choice fulfills the properties in (3.4)

$$\begin{aligned}\tilde{\alpha} - \zeta + \tilde{\beta} &> \left(\alpha - \frac{1}{2} - \frac{\varepsilon}{2}\right) + \left(\beta - \frac{\varepsilon}{2}\right) \\ &= \left(\alpha + \beta - \varepsilon\right) - \frac{1}{2} \\ &> 0.\end{aligned}$$

We claim that there exists a constant c such that

$$2^{\tilde{\alpha}|\ell|_1 + \tilde{\beta}|\ell|_\infty} \|q_\ell(f)\|_2 \leq c \left(\sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{2(\tilde{\alpha}|k|_1 + \tilde{\beta}|k|_\infty)} \|\delta_k(f)\|_2^2 \right)^{\frac{1}{2}} \quad (3.5)$$

holds for all $\ell \in \mathbb{N}_0^d$. Because of $f = \sum_{k \in \mathbb{N}_0^d} \delta_k(f)$ and linearity of q_k we have

$$\|q_\ell(f)\|_2 = \left\| \sum_{k \in \mathbb{N}_0^d} q_\ell(\delta_k(f)) \right\|_2.$$

Using $\delta_k(f) \in \mathcal{T}^k$, Lemma 3.2, and the triangle inequality we find

$$\|q_\ell(f)\|_2 = \left\| \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} q_\ell(\delta_k(f)) \right\|_2 \leq \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} \|q_\ell(\delta_k(f))\|_2.$$

Using Lemma 5 in [30] and known results about the approximation power of the I_m , see [25], we obtain

$$\|q_\ell(f)\|_2 \lesssim 2^{-\zeta|\ell|_1} \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} \|\delta_k(f)\|_{H_{\text{mix}}^\zeta(\mathbb{T}^d)}.$$

Lemma 2.11 yields

$$\|q_\ell(f)\|_2 \lesssim 2^{-\zeta|\ell|_1} \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{\zeta|k|_1} \|\delta_k(f)\|_2.$$

We proceed by inserting an additional weight and apply Hölder's inequality

$$\|q_\ell(f)\|_2 \lesssim 2^{-\zeta|\ell|_1} \left(\sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{-2[(\tilde{\alpha}-\zeta)|k|_1 + \tilde{\beta}|k|_\infty]} \right)^{\frac{1}{2}} \left(\sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{2(\tilde{\alpha}|k|_1 + \tilde{\beta}|k|_\infty)} \|\delta_k(f)\|_2^2 \right)^{\frac{1}{2}}. \quad (3.6)$$

Lemma 3.1 with $\xi > 0$ chosen such that $\min\{\tilde{\alpha} - \zeta, \tilde{\alpha} - \zeta + \tilde{\beta}\} \geq \xi$ leads to

$$\begin{aligned}\sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{-2[(\tilde{\alpha}-\zeta)|k|_1 + \tilde{\beta}|k|_\infty]} &\leq 2^{-2[(\tilde{\alpha}-\zeta)|\ell|_1 + \tilde{\beta}|\ell|_\infty]} \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{-2\xi|k-\ell|_1} \\ &\lesssim 2^{-2[(\tilde{\alpha}-\zeta)|\ell|_1 + \tilde{\beta}|\ell|_\infty]}.\end{aligned}$$

Inserting this into (3.6) proves (3.5).

Taking squares and summing up with respect to ℓ in (3.5) we get

$$\sum_{\ell \in \mathbb{N}_0^d} 2^{2(\alpha|\ell|_1 + \beta|\ell|_\infty)} \|q_\ell(f)\|_2^2 \lesssim \sum_{\ell \in \mathbb{N}_0^d} 2^{2[(\alpha - \tilde{\alpha})|\ell|_1 + (\beta - \tilde{\beta})|\ell|_\infty]} \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{2(\tilde{\alpha}|k|_1 + \tilde{\beta}|k|_\infty)} \|\delta_k(f)\|_2^2.$$

Next, interchanging the order of summation yields

$$\sum_{\ell \in \mathbb{N}_0^d} 2^{2(\alpha|\ell|_1 + \beta|\ell|_\infty)} \|q_\ell(f)\|_2^2 \lesssim \sum_{k \in \mathbb{N}_0^d} 2^{2(\tilde{\alpha}|k|_1 + \tilde{\beta}|k|_\infty)} \|\delta_k(f)\|_2^2 \sum_{\substack{\ell_i \leq k_i \\ i=1, \dots, d}} 2^{2((\alpha - \tilde{\alpha})|\ell|_1 + (\beta - \tilde{\beta})|\ell|_\infty)}.$$

One more time we apply Lemma 3.1, this time with $0 < \xi \leq \alpha - \tilde{\alpha}$, which results in

$$\begin{aligned} & \sum_{\ell \in \mathbb{N}_0^d} 2^{2(\alpha|\ell|_1 + \beta|\ell|_\infty)} \|q_\ell(f)\|_2^2 \\ & \leq \sum_{k \in \mathbb{N}_0^d} 2^{2(\tilde{\alpha}|k|_1 + \tilde{\beta}|k|_\infty)} \|\delta_k(f)\|_2^2 2^{2((\alpha - \tilde{\alpha})|k|_1 + (\beta - \tilde{\beta})|k|_\infty)} \sum_{\substack{\ell_i \leq k_i \\ i=1, \dots, d}} 2^{-2\xi|k - \ell|_1} \\ & \lesssim \sum_{k \in \mathbb{N}_0^d} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|\delta_k(f)\|_2^2. \end{aligned}$$

This proves (3.3).

Step 2. Let $f \in H^{\alpha, \beta}(\mathbb{T}^d)$. We will show that f can be represented by the series (3.2) converging in the norm of $H^{\alpha, \beta}(\mathbb{T}^d)$. Applying Lemma 2.11, Hölder's inequality and (3.3) yields

$$\begin{aligned} \sum_{k \in \mathbb{N}_0^d} \|q_k(f)\|_{H^{\alpha, \beta}(\mathbb{T}^d)} & \leq \sum_{k \in \mathbb{N}_0^d} 2^{\alpha|k|_1 + \beta|k|_\infty} \|q_k(f)\|_2 \\ & \leq C \|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)} < \infty. \end{aligned} \tag{3.7}$$

Hence $\sum_{k \in \mathbb{N}_0^d} \|q_k(f)\|_{H^{\alpha, \beta}(\mathbb{T}^d)} < \infty$ and therefore $\sum_{k \in \mathbb{N}_0^d} q_k(f)$ converges unconditionally in $H^{\alpha, \beta}(\mathbb{T}^d)$ if $\min\{\alpha, \alpha + \beta\} > \frac{1}{2}$. We denote the limit as $F := \sum_{k \in \mathbb{N}_0^d} q_k(f)$. By the definition of the norm in $H^{\alpha, \beta}(\mathbb{T}^d)$

$$\|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}^2 = \sum_{\ell \in \mathbb{N}_0^d} 2^{2(\alpha|\ell|_1 + \beta|\ell|_\infty)} \|\delta_\ell(f)\|_2^2$$

we see that the trigonometric polynomials are dense in $H^{\alpha, \beta}(\mathbb{T}^d)$. Let now t be a trigonometric polynomial. We consider $\|F - t\|_{H^{\alpha, \beta}(\mathbb{T}^d)}$. Clearly, $t = \sum_{k \in \mathbb{N}_0^d} q_k(t)$ and, by definition, $F = \sum_{k \in \mathbb{N}_0^d} q_k(f)$ implying

$$F - t = \sum_{k \in \mathbb{N}_0^d} q_k(f - t) \tag{3.8}$$

with convergence in $H^{\alpha, \beta}(\mathbb{T}^d)$ for every trigonometric polynomial t . Now, for every trigonometric polynomial t we have

$$\|F - f\|_{H^{\alpha, \beta}(\mathbb{T}^d)} \leq \|F - t\|_{H^{\alpha, \beta}(\mathbb{T}^d)} + \|t - f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}. \tag{3.9}$$

By (3.7) and (3.8) we get

$$\|F - t\|_{H^{\alpha,\beta}(\mathbb{T}^d)} \leq C\|f - t\|_{H^{\alpha,\beta}(\mathbb{T}^d)}.$$

Putting this into 3.9 yields

$$\|F - f\|_{H^{\alpha,\beta}(\mathbb{T}^d)} \leq (C + 1)\|f - t\|_{H^{\alpha,\beta}(\mathbb{T}^d)}.$$

Choosing t close enough to f gives

$$\|F - f\|_{H^{\alpha,\beta}(\mathbb{T}^d)} < \varepsilon$$

for all $\varepsilon > 0$ and hence $\|F - f\|_{H^{\alpha,\beta}(\mathbb{T}^d)} = 0$ which is

$$f = \sum_{k \in \mathbb{N}_0^d} q_k(f)$$

in $H^{\alpha,\beta}(\mathbb{T}^d)$. ■

Proposition 3.4. *Let $\beta \in \mathbb{R}$, $\min\{\alpha, \alpha + \beta\} > 0$ and $(f_k)_{k \in \mathbb{N}_0^d}$ a sequence with $f_k \in \mathcal{T}^k$ satisfying*

$$\sum_{k \in \mathbb{N}_0^d} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|f_k\|_2^2 < \infty.$$

Assume that the series $\sum_{k \in \mathbb{N}_0^d} f_k$ converges in $L_2(\mathbb{T}^d)$ to a function f . Then $f \in H^{\alpha,\beta}(\mathbb{T}^d)$, and moreover, there is a constant $C = C(\alpha, \beta, d) > 0$ such that

$$\|f\|_{H^{\alpha,\beta}(\mathbb{T}^d)}^2 \leq C \sum_{k \in \mathbb{N}_0^d} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|f_k\|_2^2. \quad (3.10)$$

Proof. *Step 1.* Let $0 < \tilde{\alpha} < \alpha$ and $\tilde{\alpha} + \beta > 0$. We claim that there exists a constant c such that

$$2^{\tilde{\alpha}|\ell|_1 + \beta|k|_\infty} \|\delta_\ell(f)\|_2 \leq c \left(\sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|f_k\|_2^2 \right)^{\frac{1}{2}} \quad (3.11)$$

holds for all $\ell \in \mathbb{N}_0^d$. Clearly, $\delta_\ell : L_2(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ is an orthogonal projection. The projection properties of the operator δ_ℓ together with $f_k \in \mathcal{T}^k$ yields

$$\|\delta_\ell(f)\|_2 \leq \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} \|\delta_\ell(f_k)\|_2. \quad (3.12)$$

Thanks to $\|\delta_\ell|_{L_2(\mathbb{T}^d)} \rightarrow L_2(\mathbb{T}^d)\| = 1$ we conclude

$$\|\delta_\ell(f)\|_2 \leq \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} \|f_k\|_2. \quad (3.13)$$

Hölder's inequality yields

$$\|\delta_\ell(f)\|_2 \leq \left(\sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{-2(\tilde{\alpha}|k|_1 + \beta|k|_\infty)} \right)^{\frac{1}{2}} \left(\sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{2(\tilde{\alpha}|k|_1 + \beta|k|_\infty)} \|f_k\|_2^2 \right)^{\frac{1}{2}}. \quad (3.14)$$

Now we apply Lemma 3.1 and find

$$\sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{-2(\tilde{\alpha}|k|_1 + \beta|k|_\infty)} \leq 2^{-2(\tilde{\alpha}|\ell|_1 + \beta|\ell|_\infty)} \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{-2\varepsilon|k-\ell|_1} \lesssim 2^{-2(\tilde{\alpha}|\ell|_1 + \beta|\ell|_\infty)}.$$

This proves (3.11).

Step 2. Inequality (3.11) yields

$$\begin{aligned} \sum_{\ell \in \mathbb{N}_0^d} 2^{2(\alpha|\ell|_1 + \beta|\ell|_\infty)} \|\delta_\ell(f)\|_2^2 &\lesssim \sum_{\ell \in \mathbb{N}_0^d} 2^{2(\alpha - \tilde{\alpha})|\ell|_1} \sum_{\substack{k_i \geq \ell_i \\ i=1, \dots, d}} 2^{2(\tilde{\alpha}|k|_1 + \beta|k|_\infty)} \|f_k\|_2^2 \\ &= \sum_{k \in \mathbb{N}_0^d} 2^{2(\tilde{\alpha}|k|_1 + \beta|k|_\infty)} \|f_k\|_2^2 \sum_{\substack{\ell_i \leq k_i \\ i=1, \dots, d}} 2^{2(\alpha - \tilde{\alpha})|\ell|_1} \\ &\lesssim \sum_{k \in \mathbb{N}_0^d} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|f_k\|_2^2. \end{aligned}$$

Since the left-hand side coincides with $\|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}^2$ Proposition 3.4 is proved. \blacksquare

After one more notation we are ready for the main result of this section.

Definition 3.5. Let $\min\{\alpha, \alpha + \beta\} > \frac{1}{2}$. We define

$$\|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}^+ := \left(\sum_{k \in \mathbb{N}_0^d} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|q_k(f)\|_2^2 \right)^{\frac{1}{2}}$$

for all $f \in H^{\alpha, \beta}(\mathbb{T}^d)$.

Theorem 3.6. Let $\min\{\alpha, \alpha + \beta\} > \frac{1}{2}$. Then a function f on \mathbb{T}^d belongs to the space $H^{\alpha, \beta}(\mathbb{T}^d)$, if and only if f can be represented by the series (3.2) converging in $H^{\alpha, \beta}(\mathbb{T}^d)$ and satisfying the condition (3.3). Moreover, the norm $\|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}$ is equivalent to the norm $\|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}^+$.

Proof. This result is an easy consequence of Proposition 3.3 and Proposition 3.4, applied with $f_k = q_k(f)$. \blacksquare

Remark 3.7. (i) The restriction $\min\{\alpha, \alpha + \beta\} > \frac{1}{2}$ is essentially optimal, see Remark 2.10. (ii) The potential of sampling representations has been first recognized by Dũng [7, 8]. There the non-periodic situation in connection with tensor product B-spline series is treated in the unit cube.

4 Sampling on energy-norm based sparse grids

In this section we consider the quality of approximation by sampling operators using energy-norm based sparse grids. In fact, a suitable sampling operator Q_Δ uses a slightly larger set Δ_ε compared to Δ from (1.14) with the same combinatorial properties, see Lemma 6.4 below. We put

$$\Delta_\varepsilon(\xi) := \{k \in \mathbb{N}_0^d : (\alpha - \varepsilon)|k|_1 - (\gamma - \beta - \varepsilon)|k|_\infty \leq \xi\} \quad , \quad \xi > 0, \quad (4.1)$$

Theorem 4.1. *Let $\alpha > 0$, $\gamma \geq 0$ and $\beta < \gamma$ such that $\min\{\alpha, \alpha + \beta\} > \frac{1}{2}$. Let further $0 < \varepsilon < \gamma - \beta < \alpha$. Then there exists a constant $C = C(\alpha, \beta, \gamma, \varepsilon, d) > 0$ such that*

$$\|f - Q_{\Delta_\varepsilon(\xi)} f\|_{H^\gamma(\mathbb{T}^d)} \leq C 2^{-\xi} \|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)} \quad (4.2)$$

holds for all $f \in H^{\alpha, \beta}(\mathbb{T}^d)$ and all $\xi > 0$.

Proof. *Step 1.* The triangle inequality in $H^\gamma(\mathbb{T}^d)$, Lemma 2.11, and afterwards Hölder's inequality yield

$$\begin{aligned} \|f - Q_{\Delta_\varepsilon(\xi)} f\|_{H^\gamma(\mathbb{T}^d)} &= \left\| \sum_{k \notin \Delta_\varepsilon(\xi)} q_k(f) \right\|_{H^\gamma(\mathbb{T}^d)} \leq \sum_{k \notin \Delta_\varepsilon(\xi)} \|q_k(f)\|_{H^\gamma(\mathbb{T}^d)} \\ &\leq \sum_{k \notin \Delta_\varepsilon(\xi)} 2^{\gamma|k|_\infty} \|q_k(f)\|_2 \\ &= \sum_{k \notin \Delta_\varepsilon(\xi)} 2^{\alpha|k|_1 + \beta|k|_\infty} 2^{-(\alpha|k|_1 + \beta|k|_\infty)} 2^{\gamma|k|_\infty} \|q_k(f)\|_2 \\ &\leq \left(\sum_{k \notin \Delta_\varepsilon(\xi)} 2^{-2\alpha|k|_1 + 2(\gamma - \beta)|k|_\infty} \right)^{\frac{1}{2}} \left(\sum_{k \notin \Delta_\varepsilon(\xi)} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|q_k(f)\|_2^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Applying Theorem 3.6 we have

$$\left(\sum_{k \notin \Delta_\varepsilon(\xi)} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|q_k(f)\|_2^2 \right)^{\frac{1}{2}} \leq \|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}.$$

Consequently, we obtain the following inequality

$$\|f - Q_{\Delta_\varepsilon(\xi)} f\|_{H^\gamma(\mathbb{T}^d)} \leq \left(\sum_{k \notin \Delta_\varepsilon(\xi)} 2^{-2\alpha|k|_1 + 2(\gamma - \beta)|k|_\infty} \right)^{\frac{1}{2}} \|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}. \quad (4.3)$$

Step 2. Now we consider the sum

$$\sum_{k \notin \Delta_\varepsilon(\xi)} 2^{-2\alpha|k|_1 + 2(\gamma - \beta)|k|_\infty} \leq \sum_{i=1}^d \sum_{\substack{k \notin \Delta_\varepsilon(\xi) \\ k \in K_i}} 2^{-2\alpha|k|_1 + 2(\gamma - \beta)|k|_\infty},$$

where

$$K_i := \{k \in \mathbb{N}_0^d : k_i = |k|_\infty\} \quad i = 1, \dots, d. \quad (4.4)$$

We want to find a proper upper bound for

$$\sum_{\substack{k \notin \Delta_\varepsilon(\xi) \\ k \in K_i}} 2^{-2\alpha|k|_1 + 2(\gamma - \beta)|k|_\infty}.$$

For simplicity we restrict ourselves to the case $i = 1$ with $k_1 = |k|_\infty$ and set

$$\tilde{k} := (k_2, \dots, k_d). \quad (4.5)$$

Indeed, $|k|_1 = k_1 + |\tilde{k}|_1$ holds for all $k \in \mathbb{N}_0^d$. So the following equivalence is true

$$\begin{aligned} k \notin \Delta_\varepsilon(\xi) &\iff (\alpha - \varepsilon)|k|_1 - ((\gamma - \beta) - \varepsilon)k_1 > \xi \\ &\iff (\alpha - \varepsilon)|\tilde{k}|_1 + (\alpha - (\gamma - \beta))k_1 > \xi \\ &\iff k_1 > \frac{\xi - (\alpha - \varepsilon)|\tilde{k}|_1}{\alpha - (\gamma - \beta)}. \end{aligned}$$

Using this equivalence we can proceed with

$$\begin{aligned} \sum_{\substack{k \notin \Delta_\varepsilon(\xi) \\ k \in K_1}} 2^{-2\alpha|k|_1 + 2(\gamma-\beta)|k|_\infty} &= \sum_{\tilde{k} \in \mathbb{N}_0^{d-1}} 2^{-2\alpha|\tilde{k}|_1} \sum_{k_1 > \max \left\{ |\tilde{k}|_\infty - 1, \frac{\xi - (\alpha - \varepsilon)|\tilde{k}|_1}{\alpha - (\gamma - \beta)} \right\}} 2^{-2\alpha k_1 + 2(\gamma - \beta)k_1} \\ &= \sum_{\tilde{k} \in I_1} 2^{-2\alpha|\tilde{k}|_1} \sum_{k_1 \geq |\tilde{k}|_\infty} 2^{-2\alpha k_1 + 2(\gamma - \beta)k_1} \end{aligned} \quad (4.6)$$

$$+ \sum_{\tilde{k} \notin I_1} 2^{-2\alpha|\tilde{k}|_1} \sum_{k_1 > \frac{\xi - (\alpha - \varepsilon)|\tilde{k}|_1}{\alpha - (\gamma - \beta)}} 2^{-2\alpha k_1 + 2(\gamma - \beta)k_1}, \quad (4.7)$$

where

$$I_1 = \left\{ \tilde{k} \in \mathbb{N}_0^{d-1} : \frac{\xi - (\alpha - \varepsilon)|\tilde{k}|_1}{\alpha - (\gamma - \beta)} < |\tilde{k}|_\infty \right\}.$$

First we compute an upper bound for the sum in (4.6). Because of

$$2^{2((\gamma-\beta)-\alpha)|\tilde{k}|_\infty} \leq 2^{-2(\xi - [\alpha - \varepsilon])|\tilde{k}|_1} \quad \text{if} \quad \tilde{k} \in I_1,$$

we conclude

$$\begin{aligned} \sum_{\tilde{k} \in I_1} \sum_{k_1 \geq |\tilde{k}|_\infty} 2^{-2\alpha k_1 + 2(\gamma - \beta)k_1} &\leq C \sum_{\tilde{k} \in I_1} 2^{-2\alpha|\tilde{k}|_1} 2^{2((\gamma - \beta) - \alpha)|\tilde{k}|_\infty} \\ &\lesssim \sum_{\tilde{k} \in I_1} \sum_{k_1 \geq |\tilde{k}|_\infty} 2^{-2\alpha k_1 + 2(\gamma - \beta)k_1} \\ &\lesssim 2^{-2\xi} \sum_{\tilde{k} \in I_1} 2^{-2\varepsilon|\tilde{k}|_1} \\ &\lesssim 2^{-2\xi}. \end{aligned}$$

Here the constant behind \lesssim does not depend on ξ .

Step 2. Next, we estimate the sum in (4.7). Similarly as above we find

$$\begin{aligned} \sum_{\tilde{k} \notin I_1} 2^{-2\alpha|\tilde{k}|_1} \sum_{k_1 > \frac{\xi - (\alpha - \varepsilon)|\tilde{k}|_1}{\alpha - (\gamma - \beta)}} 2^{-2\alpha k_1 + 2(\gamma - \beta)k_1} &\lesssim \sum_{\tilde{k} \notin I_1} 2^{-2\alpha|\tilde{k}|_1} 2^{-2(\xi - (\alpha - \varepsilon))|\tilde{k}|_1} \\ &\lesssim 2^{-2\xi}. \end{aligned}$$

As a consequence we have

$$\sum_{k \notin \Delta_\varepsilon(\xi)} 2^{-2\alpha|k|_1 + 2(\gamma - \beta)|k|_\infty} \lesssim 2^{-2\xi}.$$

This together with (4.3) proves the claim. ■

The previous result includes the case $\gamma = 0$. Let us state this special case separately.

Corollary 4.2. *Let $\alpha > 0$, $\beta < 0$ such that $\alpha + \beta > \frac{1}{2}$ and $0 < \varepsilon < -\beta < \alpha$. Then there is a constant $C = C(\alpha, \beta, \varepsilon, d) > 0$ such that*

$$\|f - Q_{\Delta_\varepsilon(\xi)} f\|_2 \leq C 2^{-\xi} \|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)} \quad (4.8)$$

holds for all $f \in H^{\alpha, \beta}(\mathbb{T}^d)$ and $\xi > 0$.

Remark 4.3. (i) For the approximation of the embedding $I : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d)$, where $\alpha > \gamma > 0$, we could have used a simpler argument which does not require the sampling representation in Theorem 3.6 to estimate $\|f - Q_{\Delta_\varepsilon(\xi)} f\|_{H^\gamma(\mathbb{T}^d)}$. In fact, we estimate

$$\begin{aligned} \|f - Q_{\Delta_\varepsilon} f\|_{H^\gamma(\mathbb{T}^d)} &\leq \left\| \sum_{k \notin \Delta_\varepsilon(\xi)} q_k(f) \right\|_{H^\gamma(\mathbb{T}^d)} \leq \sum_{k \notin \Delta_\varepsilon(\xi)} \|q_k(f)\|_{H^\gamma(\mathbb{T}^d)} \\ &\leq \sum_{k \notin \Delta_\varepsilon(\xi)} 2^{\gamma|k|_\infty} \|q_k(f)\|_2. \end{aligned} \tag{4.9}$$

Due to the tensor product structure of the space $H_{\text{mix}}^\alpha(\mathbb{T}^d)$ we are allowed to use [30, Lemma 5] to estimate $\|q_k(f)\|_2$. Indeed, it holds

$$\begin{aligned} \|q_k(f)\|_2 &= \|(\eta_{k_1} \otimes \dots \otimes \eta_{k_d})f\|_2 \leq \left(\prod_{j=1}^d \|\eta_{k_j} : H^\alpha(\mathbb{T}) \rightarrow L_2(\mathbb{T})\| \right) \|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)} \\ &\lesssim 2^{-\alpha|k|_1} \|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)}. \end{aligned}$$

Putting this into (4.9) yields

$$\|f - Q_{\Delta_\varepsilon} f\|_{H^\gamma(\mathbb{T}^d)} \leq \|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)} \sum_{k \notin \Delta_\varepsilon(\xi)} 2^{-\alpha|k|_1 + \gamma|k|_\infty}$$

With exactly the same method as used in Step 2 of the proof of Theorem 4.1 we obtain that

$$\sum_{k \notin \Delta_\varepsilon(\xi)} 2^{-\alpha|k|_1 + \gamma|k|_\infty} \lesssim 2^{-\xi},$$

which yields

$$\|f - Q_{\Delta_\varepsilon} f\|_{H^\gamma(\mathbb{T}^d)} \lesssim 2^{-\xi} \|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)}.$$

(ii) The method from (i) is not suitable if $\gamma = 0$. In fact, it produces a worse bound compared to the one obtained in Theorem 5.4 below, namely

$$\|f - Q_{\Delta(\alpha m)} f\|_2 \lesssim 2^{-m\alpha} m^{d-1} \|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)}.$$

This is actually the strategy used in [33] to obtain (1.5), see also [2].

(iii) Estimates of sampling operators of Smolyak-type with respect to the embeddings $I : H_{\text{mix}}^\alpha([0, 1]^d) \rightarrow H^\gamma([0, 1]^d)$ may be found also in the papers [1, 2, 10, 24] and the recent one [13]. In particular, Bungartz and Griebel have used energy-norm based sparse grids in case $\alpha = 2$ and $\gamma = 1$. These authors have taken care of the dependence of all constants on the dimension d , an important problem in high-dimensional approximation, which we have ignored here.

5 Sampling on Smolyak grids

In this section we intend to apply our new method to situations where the classical Smolyak algorithm is used. On the one hand we give shorter proofs for existing results and extend some of them concerning the used approximating operators on the other hand.

5.1 The mixed-mixed case

We consider sampling operators for functions in $H_{\text{mix}}^\alpha(\mathbb{T}^d)$ measuring the error in $H_{\text{mix}}^\gamma(\mathbb{T}^d)$. The associated operator Q_Δ is this time given by

$$\Delta(\xi) = \Delta(\alpha, \gamma; \xi) := \{k \in \mathbb{N}_0^d : (\alpha - \gamma)|k|_1 \leq \xi\} \quad , \quad \xi > 0. \quad (5.1)$$

Theorem 5.1. *Let $\gamma > 0$ and $\alpha > \max\{\gamma, 1/2\}$. Then there is a constant $C = C(\alpha, \gamma, d) > 0$ such that*

$$\|f - Q_{\Delta(\xi)}f\|_{H_{\text{mix}}^\gamma(\mathbb{T}^d)} \leq C2^{-\xi}\|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)}$$

holds for all $f \in H_{\text{mix}}^\alpha(\mathbb{T}^d)$ and $\xi > 0$.

Proof. We employ Proposition 3.4 to $H_{\text{mix}}^\gamma(\mathbb{T}^d)$ with the sequence $(f_k)_{k \in \mathbb{N}_0^d}$ given by

$$f_k = \begin{cases} q_k(f) & : k \notin \Delta(\xi), \\ 0 & : k \in \Delta(\xi). \end{cases}$$

Note, that the only restriction for Proposition 3.4 is $\gamma > 0$. Clearly, $f - Q_{\Delta(\xi)}f = \sum_{k \in \mathbb{N}_0^d} f_k$ and

hence

$$\begin{aligned} \|f - Q_{\Delta(\xi)}f\|_{H_{\text{mix}}^\gamma(\mathbb{T}^d)}^2 &\lesssim \sum_{k \in \mathbb{N}_0^d} 2^{2\gamma|k|_1} \|f_k\|_2^2 \\ &= \sum_{k \notin \Delta(\xi)} 2^{2(\gamma-\alpha)|k|_1} 2^{2\alpha|k|_1} \|q_k(f)\|_2^2 \\ &\leq 2^{-2\xi} \sum_{k \in \mathbb{N}_0^d} 2^{2\alpha|k|_1} \|q_k(f)\|_2^2. \end{aligned}$$

Applying Theorem 3.6 (here we need $\alpha > 1/2$) completes the proof since

$$\sum_{k \in \mathbb{N}_0^d} 2^{2\alpha|k|_1} \|q_k(f)\|_2^2 \lesssim \|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)}^2.$$

■

As a direct consequence of Theorem 5.1, we obtain the following result for the weaker error norm $\|\cdot\|_{H^\gamma(\mathbb{T}^d)}$

Corollary 5.2. *Let $\alpha > \frac{1}{2}$ and $0 < \gamma < \alpha$. Then there is a constant $C = C(\alpha, \gamma, d) > 0$ such that*

$$\|f - Q_{\Delta(\xi)}f\|_{H^\gamma(\mathbb{T}^d)} \leq C2^{-\xi}\|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)} \quad (5.2)$$

holds for all $f \in H_{\text{mix}}^\alpha(\mathbb{T}^d)$ and $\xi > 0$.

Remark 5.3. Sampling with Smolyak operators has some history. Closest to us are Temlyakov [33, 34, 35] and Dũng [4]-[8], see also [26], [27] and [30]. In almost all contributions preference was given to situations where the target space was $L_q(\mathbb{T}^d)$. Let us also refer to the recent preprint [13].

5.2 The case $\alpha > \gamma - \beta = 0$

Now we are interested in the embedding

$$I : H^{\alpha, \beta}(\mathbb{T}^d) \rightarrow H^\beta(\mathbb{T}^d).$$

The sampling operator $Q_{\Delta(\xi)}$ is determined by $\Delta(\xi)$ from (1.16). Let us simplify the structure by considering the index sets $\Delta(\alpha m)$ for $m \in \mathbb{N}$ which consists of all $k \in \mathbb{N}_0^d$ satisfying $|k|_1 \leq m$.

Theorem 5.4. *Let $\beta = \gamma \geq 0$ and $\alpha > \frac{1}{2}$. Then there is a constant $C = C(\alpha, \beta, d) > 0$ such that*

$$\|f - Q_{\Delta(\alpha m)}f\|_{H^\beta(\mathbb{T}^d)} \leq C 2^{-m\alpha} m^{\frac{d-1}{2}} \|f\|_{H^{\alpha, \beta}(\mathbb{T}^d)}$$

holds for all $f \in H^{\alpha, \beta}(\mathbb{T}^d)$ and $m \in \mathbb{N}$.

Proof. We proceed as in proof of Theorem 4.1. The triangle inequality in $H^\beta(\mathbb{T}^d)$ yields

$$\|f - Q_{\Delta(\alpha m)}f\|_{H^\beta(\mathbb{T}^d)} = \left\| \sum_{k \notin \Delta(\alpha m)} q_k(f) \right\|_{H^\beta(\mathbb{T}^d)} \leq \sum_{k \notin \Delta(\alpha m)} \|q_k(f)\|_{H^\beta(\mathbb{T}^d)}.$$

Applying Lemma 2.11 gives

$$\|f - Q_{\Delta(\alpha m)}f\|_{H^\beta(\mathbb{T}^d)} \lesssim \sum_{k \notin \Delta(\alpha m)} 2^{\beta|k|_\infty} \|q_k(f)\|_2.$$

Proceeding with Hölder's inequality leads to

$$\|f - Q_{\Delta(\alpha m)}f\|_2 \leq \left(\sum_{|k|_1 > m} 2^{-2\alpha|k|_1} \right)^{\frac{1}{2}} \left(\sum_{|k|_1 > m} 2^{2(\alpha|k|_1 + \beta|k|_\infty)} \|q_k(f)\|_2^2 \right)^{\frac{1}{2}}.$$

Employing the upcoming lemma and Theorem 3.6 finishes the proof. ■

Lemma 5.5. *Let $\alpha > 0$. Then*

$$\sum_{|k|_1 > m} 2^{-2\alpha|k|_1} \lesssim m^{d-1} 2^{-2\alpha m} \tag{5.3}$$

holds for all $m > 0$.

Proof. This lemma is well known. Let us prove it for completeness. We decompose the sum in the following two parts

$$\sum_{|k|_1 > m} 2^{-2\alpha|k|_1} = \sum_{\substack{|k|_1 > m \\ |k|_\infty \leq m}} 2^{-2\alpha|k|_1} + \sum_{|k|_\infty > m} 2^{-2\alpha|k|_1}. \tag{5.4}$$

First we compute an upper bound for the second sum in (5.4). Again we use the convention for \tilde{k} of k from (4.5) and decompose as follows

$$\begin{aligned} \sum_{|k|_\infty > m} 2^{-2\alpha|k|_1} &\leq \sum_{i=1}^d \sum_{\substack{k_i > m \\ k \in \mathbb{N}_0^d}} 2^{-2\alpha|k|_1} = d \sum_{\tilde{k} \in \mathbb{N}_0^{d-1}} 2^{-2\alpha|\tilde{k}|_1} \sum_{k_1 > m} 2^{-\alpha k_1} \\ &\lesssim 2^{-\alpha m}. \end{aligned}$$

The first sum in (5.4) gives

$$\begin{aligned} \sum_{\substack{|k|_\infty \leq m \\ |k|_1 > m}} 2^{-2\alpha|k|_1} &\leq \sum_{k_2=0}^m \dots \sum_{k_d=0}^m \sum_{k_1=m-|\tilde{k}|_1}^{\infty} 2^{-2\alpha|k|_1} \\ &\lesssim (m+1)^{d-1} 2^{-2\alpha m}. \end{aligned}$$

Consequently,

$$\sum_{|k|_1 > m} 2^{-2\alpha|k|_1} \lesssim m^{d-1} 2^{-2\alpha m} \quad (5.5)$$

holds for all $m > 0$. ■

5.3 The case $\gamma = 0$

From Theorem 5.4 we immediately obtain the special case ($\gamma = \beta = 0$)

$$\|f - Q_{\Delta(\alpha m)} f\|_2 \leq C 2^{-m\alpha} m^{\frac{d-1}{2}} \|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)}, \quad m \in \mathbb{N},$$

compare with [26], [30]. With our methods we can additionally show an error bound for $L_\infty(\mathbb{T}^d)$ instead of $L_2(\mathbb{T}^d)$.

Theorem 5.6. *Let $\alpha > \frac{1}{2}$. Then there is a constant $C = C(\alpha, d) > 0$ such that*

$$\|f - Q_{\Delta(\alpha m)} f\|_\infty \leq C 2^{-m(\alpha - \frac{1}{2})} m^{\frac{d-1}{2}} \|f\|_{H_{\text{mix}}^\alpha(\mathbb{T}^d)}$$

holds for all $f \in H_{\text{mix}}^\alpha(\mathbb{T}^d)$ and $m \in \mathbb{N}$.

Proof. As above with Lemma 2.8 we conclude

$$\begin{aligned} \|f - Q_{\Delta(\alpha m)} f\|_\infty &= \left\| \sum_{k \notin \Delta(\alpha m)} q_k(f) \right\|_\infty \leq \sum_{k \notin \Delta(\alpha m)} \|q_k(f)\|_\infty \\ &\leq \sum_{|k|_1 > m} 2^{|k|_1/2} 2^{\alpha|k|_1} 2^{-\alpha|k|_1} \|q_k(f)\|_2 \\ &\leq \left(\sum_{|k|_1 > m} 2^{-2|k|_1(\alpha - \frac{1}{2})} \right)^{\frac{1}{2}} \left(\sum_{|k|_1 > m} 2^{2\alpha|k|_1} \|q_k(f)\|_2^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (5.6)$$

Applying Lemma 5.5 and Theorem 3.6 proves the claim. ■

Now we turn to the case $2 < q < \infty$. The following result allows for comparing the present situation with the results in Subsection 5.1.

Lemma 5.7. *Let $2 < q < \infty$. Then*

$$\|f\|_q \lesssim \left(\sum_{k \in \mathbb{N}_0^d} \|\delta_k(f)\|_q^2 \right)^{1/2} \lesssim \left(\sum_{k \in \mathbb{N}_0^d} 2^{2|k|_1(1/2 - 1/q)} \|\delta_k(f)\|_2^2 \right)^{1/2} = \|f\|_{H_{\text{mix}}^{\frac{1}{2} - \frac{1}{q}}(\mathbb{T}^d)}$$

holds true for any $f \in L_q(\mathbb{T}^d)$, where the right-hand side may be infinite.

Proof. The proof of the first relation in Lemma 5.7 is elementary using the Littlewood-Paley decomposition in $L_q(\mathbb{T}^d)$ together with $q/2 \geq 1$, see for instance [35, Theorem 0.3.2, Page 20]. The second relation follows by an application of Nikol'skij's inequality in Lemma 2.8. \blacksquare

Remark 5.8. *Let us mention that Lemma 5.7 can be refined to*

$$\|f\|_q \lesssim \left(\sum_{k \in \mathbb{N}_0^d} 2^{q|k|_1(1/2-1/q)} \|\delta_k(f)\|_2^q \right)^{1/q}.$$

For this deep result we refer to [35, Lemma II.2.1] and to [7, Lemma 5.3] as well as [23, Lemma 1] for non-periodic versions. In a more general context this embedding is a special case of a Jawerth/Franke type embedding, see [14].

6 Sampling numbers

In this section we will restate the approximation results from Sections 4 and 5 in terms of the number of degrees of freedom. We additionally show the asymptotic optimality with regard to sampling numbers of the sampling operators considered in Sections 4 and 5. This requires estimates of the rank of the corresponding sampling operators. A lower bound for the rank is deduced from the fact that the respective sampling operators reproduce trigonometric polynomials from modified hyperbolic crosses \mathcal{H}_Δ . Recall that our approximation scheme is based on the classical trigonometric interpolation. We have used several times the fact that the operator I_m defined in (1.11) reproduces univariate trigonometric polynomials of degree less than or equal to m . What concerns the operator Q_Δ in (1.13) we can prove the following general reproduction result.

Lemma 6.1. *Let $\Delta \subset \mathbb{N}_0^d$ be a solid finite set meaning that $k \in \Delta$ and $\ell \leq k$ implies $\ell \in \Delta$. Then Q_Δ reproduces trigonometric polynomials with frequencies in*

$$\mathcal{H}_\Delta := \bigcup_{k \in \Delta} \mathcal{P}_k, \tag{6.1}$$

where \mathcal{P}_k is defined in (1.18).

Proof. We follow the arguments in the proof of [30, Lemma 1]. By the fact that $|\Delta| < \infty$ we find a $m \geq 0$ such that

$$\Delta \subset \{0, \dots, m\}^d.$$

Let

$$T := \sum_{|k|_\infty \leq m} \bigotimes_{i=1}^d \eta_{k_i} \quad \text{and} \quad R := \sum_{\substack{k \notin \Delta \\ |k|_\infty \leq m}} \bigotimes_{i=1}^d \eta_{k_i}.$$

Of course, it holds

$$Q_\Delta = T - R.$$

Since

$$\sum_{k=0}^m \eta_k = I_{2^m}$$

we obtain

$$T = \bigotimes_{i=1}^d I_{2^m}.$$

Obviously, for $\ell \in \mathcal{H}_\Delta$ the univariate reproduction property yields

$$(Te^{i\ell})(x) = \prod_{j=1}^d (I_{2^m} e^{i\ell_j})(x_j) = e^{i\ell x}$$

for all $x \in \mathbb{T}^d$. It remains to prove $Re^{i\ell} \equiv 0$. Let $k = (k_1, \dots, k_d) \in \mathbb{N}_0^d$ such that $k \notin \Delta$. Due to $\ell \in H_\Delta$ there exists $u \in \Delta$ with $|\ell_i| \leq 2^{u_i}$ for all $i = 1, \dots, d$. The solidity property of Δ yields the existence of $j \in \{1, \dots, d\}$ with

$$u_j < k_j.$$

This gives

$$|\ell_j| \leq 2^{u_j} \leq 2^{k_j-1} < 2^{k_j}.$$

Finally, by the univariate reproduction property, we obtain

$$\eta_{k_j}(e^{i\ell_j}) = (I_{2^{k_j}} - I_{2^{k_j-1}})e^{i\ell_j} = 0.$$

■

The previous result immediately implies the relation

$$\text{rank } Q_\Delta \geq \sum_{k \in \Delta} 2^{|k|_1}$$

if $\Delta \subset \mathbb{N}_0^d$ is solid.

Lemma 6.2. *Let $\alpha > 0$, $\gamma \geq 0$ and $\beta < \gamma$ such that $0 < \gamma - \beta \leq \alpha$*

- (i) *The index sets $\Delta(\alpha, \beta, \gamma; \xi)$ defined in (1.14) and $\Delta(\varepsilon, \alpha, \beta, \gamma; \xi)$ defined in (1.15) are solid sets in the sense of Lemma 6.1 for every $\xi > 0$.*
- (ii) *The index set $\Delta(\alpha; \xi)$ defined in (1.16) is a solid set for every $\xi > 0$.*

Proof. The second result is trivial. We prove the first one. Let

$$\psi(k) := \alpha|k|_1 - (\gamma - \beta)|k|_\infty.$$

The set $\Delta(\xi)$ consists of all $k \in \mathbb{N}_0^d$ with $\psi(k) \leq \xi$. Applying Lemma 3.1 yields

$$\psi(k') \leq \psi(k) \leq \xi$$

for all $k' \leq k \in \Delta(\xi)$. That means all the k' also belong to $\Delta(\xi)$. ■

Remark 6.3. *Hyperbolic crosses $\mathcal{H}_{\Delta(\xi)}$ (with $\Delta(\xi)$ from (1.14) and (1.16)) in the 2-plane:*

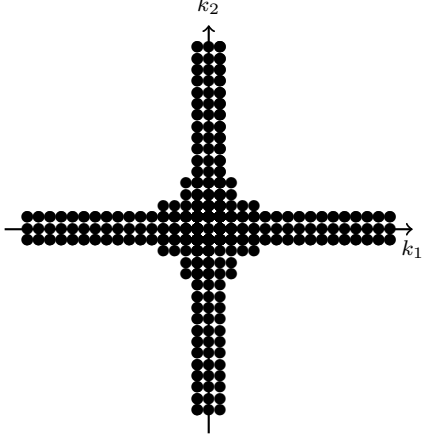


Figure 3: $\alpha = 2, \beta = 0, \gamma = 1, \xi = 4$

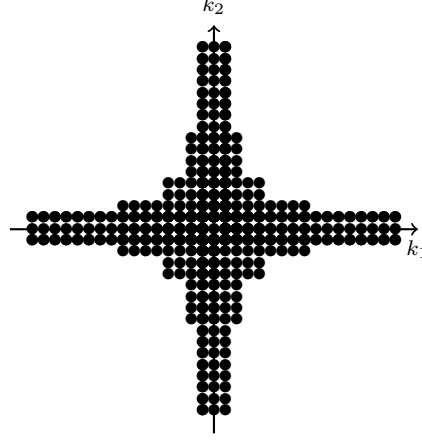


Figure 4: $\alpha = 1, \xi = 4$

Comparing Figure 3 and 4 shows that energy norm based hyperbolic crosses contain “mostly” anisotropic building blocks than its classical (Smolyak) counterpart.

In the next lemma we give sharp estimates for $\sum_{k \in \Delta(\xi)} 2^{|k|_1}$ with $\Delta(\xi)$ from (1.14).

Lemma 6.4. *Let $\alpha > 0, \gamma \geq 0, \beta \in \mathbb{R}$ such that $\gamma > \beta$ and $\alpha > \gamma - \beta$. Then*

$$\sum_{k \in \Delta(\xi)} 2^{|k|_1} \asymp 2^{\frac{\xi}{\alpha - (\gamma - \beta)}}$$

holds for all $\xi \geq \alpha - (\gamma - \beta)$, where the constants behind “ \asymp ” only depend on $\alpha, \gamma - \beta$, and d .

Proof. *Step 1.* First we deal with the upper bound. We are going to use the same notation as in (4.4) and (4.5). We obtain the following inequality

$$\sum_{k \in \Delta(\xi)} 2^{|k|_1} \leq \sum_{i=1}^d \sum_{k \in K_i \cap \Delta(\xi)} 2^{|k|_1}.$$

By symmetry it will be enough to deal with $i = 1$. Hence

$$\sum_{k \in \Delta(\xi)} 2^{|k|_1} \leq d \sum_{k \in K_1 \cap \Delta(\xi)} 2^{|k|_1}.$$

Now we want to decompose the summation over k . Since $k_1 \geq |k|_\infty$ we find

$$\begin{aligned} k \in \Delta(\xi) &\iff \alpha|k|_1 - (\gamma - \beta)k_1 \leq \xi \\ &\iff \alpha(|\tilde{k}|_1 + k_1) - (\gamma - \beta)k_1 \leq \xi \\ &\iff k_1 \leq \frac{\xi - \alpha|\tilde{k}|_1}{\alpha - (\gamma - \beta)}. \end{aligned}$$

This implies

$$|\tilde{k}|_\infty \leq \frac{\xi - \alpha|\tilde{k}|_1}{\alpha - (\gamma - \beta)} \iff \alpha|\tilde{k}|_1 + (\alpha - (\gamma - \beta))|\tilde{k}|_\infty \leq \xi.$$

We shall use these inequalities to produce an appropriate decomposition of $K_1 \cap \Delta(\xi)$ which results in

$$\begin{aligned}
\sum_{k \in \Delta(\xi)} 2^{|k|_1} &\leq d \sum_{\substack{\tilde{k} \in \mathbb{N}_0^{d-1} \\ \alpha|\tilde{k}|_1 + (\alpha - (\gamma - \beta))|\tilde{k}|_\infty \leq \xi}} 2^{|\tilde{k}|_1} \sum_{k_1 = |\tilde{k}|_\infty}^{\frac{\xi - \alpha|\tilde{k}|_1}{\alpha - (\gamma - \beta)}} 2^{k_1} \\
&\lesssim 2^{\frac{\xi}{\alpha - (\gamma - \beta)}} \sum_{\alpha|\tilde{k}|_1 + (\alpha - (\gamma - \beta))|\tilde{k}|_\infty \leq \xi} 2^{\frac{-\tilde{\alpha}}{\alpha - (\gamma - \beta)}|\tilde{k}|_1} \\
&\lesssim 2^{\frac{\xi}{\alpha - (\gamma - \beta)}},
\end{aligned}$$

since $\alpha/(\alpha - (\gamma - \beta)) > 0$.

Step 2. We prove the lower bound. First we claim that

$$k^* := \left\lfloor \frac{\xi}{\alpha - (\gamma - \beta)} \right\rfloor (1, 0, \dots, 0) \in \Delta(\xi).$$

Indeed,

$$\begin{aligned}
k^* \in \Delta(\xi) &\iff \alpha|k^*|_1 - (\gamma - \beta)|k^*|_\infty \leq \xi \\
&\iff (\alpha - \varepsilon) \left\lfloor \frac{\xi}{\alpha - (\gamma - \beta)} \right\rfloor - ((\gamma - \beta) - \varepsilon) \left\lfloor \frac{\xi}{\alpha - (\gamma - \beta)} \right\rfloor \leq \xi \\
&\iff (\alpha - (\gamma - \beta)) \left\lfloor \frac{\xi}{\alpha - (\gamma - \beta)} \right\rfloor \leq \xi.
\end{aligned}$$

Obviously, the last inequality is true. Consequently

$$\sum_{k \in \Delta(\xi)} 2^{|k|_1} \geq 2^{|k^*|_1} = 2^{\left\lfloor \frac{\xi}{\alpha - (\gamma - \beta)} \right\rfloor} \geq 2^{\frac{\xi}{\alpha - (\gamma - \beta)} - 1}.$$

The proof is complete. ■

Corollary 6.5. *Let $\alpha > 0$, $\gamma \geq 0$, $\beta \in \mathbb{R}$ such that $\gamma > \beta$ and $\alpha > \gamma - \beta$. Let further $\Delta(\xi)$ as in (1.14).*

(i) *The sampling operator $Q_{\Delta(\xi)}$ uses at most $C2^{\frac{\xi}{\alpha - (\gamma - \beta)}}$ function values, where the constant $C > 0$ only depends on α , $\gamma - \beta$ and d .*

(ii) *The rank of the linear operator $Q_{\Delta(\xi)}$ satisfies*

$$\text{rank } Q_{\Delta(\xi)} \asymp 2^{\frac{\xi}{\alpha - (\gamma - \beta)}}, \quad \xi \geq \alpha - (\gamma - \beta),$$

where the constants behind “ \asymp ” only depend on α , $\gamma - \beta$, and d .

Proof. Clearly, $I_m f$ uses $2m + 1$ values of function f , hence $\eta_m f$ is using $\leq 2^{m+2}$ function values. This implies that $q_k f$ applies $\leq 2^{2d} 2^{|k|_1}$ function values. As a consequence of Lemma 6.4 we find that $Q_{\Delta(\xi)} f$ is using

$$\lesssim \sum_{k \in \Delta(\xi)} 2^{|k|_1} \asymp 2^{\frac{\xi}{\alpha - (\gamma - \beta)}}$$

function values of f . Part (ii) follows from Lemma 6.1 and the lower bound in Lemma 6.4. ■

Let us now count the degree of freedom for a classical Smolyak grid.

Lemma 6.6. *For any $d \in \mathbb{N}$ and $m \in \mathbb{N}_0^d$, we have the inequality*

$$\left(\frac{m+d-1}{d-1}\right)^{d-1} 2^m \leq \sum_{|k|_1 \leq m} 2^{|k|_1} \leq \left[\frac{e(m+d-1)}{d-1}\right]^{d-1} 2^{m+1}.$$

Proof. This assertion is a direct consequence of [9, Lemma 3.10] together with the well-known relation

$$\left(\frac{N}{n}\right)^n \leq \binom{N}{n} \leq \left(\frac{eN}{n}\right)^n.$$

■

Corollary 6.7. *Let $m \in \mathbb{N}$ and*

$$\Delta = \{k \in \mathbb{N}_0^d : |k|_1 \leq m\}.$$

(i) *The sampling operator Q_Δ is using at most $Cm^{d-1}2^m$ function values, where C decays super-exponentially in d .*

(ii) *The rank of the linear operator Q_Δ satisfies*

$$\text{rank } Q_\Delta \asymp m^{d-1} 2^m, \quad m \in \mathbb{N}.$$

Proof. Part (i) follows from the fact that $q_k(f)$ uses $2^{2d|k|_1}$ function values for any k together with the upper bound in Lemma 6.6. The second assertion can be derived by using the reproduction properties of Q_Δ , see Lemma 6.1, and the lower bound in Lemma 6.6. ■

Remark 6.8. *For $d = 2$ the sampling grids of $Q_{\Delta(\xi)}$ as in (1.14) and (1.16) look like:*

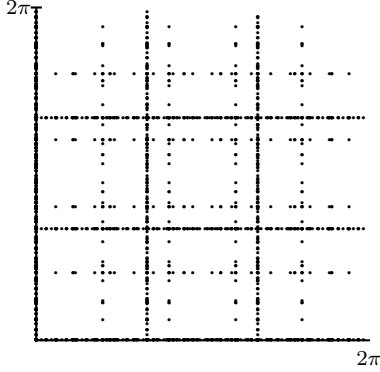


Figure 5: $\alpha = 2, \beta = 0, \gamma = 1, \xi = 5$

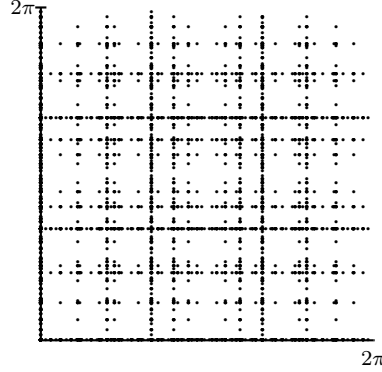


Figure 6: $\alpha = 1, \xi = 5$

These figures show that the point sets of $Q_{\Delta(\xi)}$ have a lot of internal structure. However, they are far from being uniformly distributed within \mathbb{T}^d .

Now we are in position to formulate our results in terms of sampling numbers.

Theorem 6.9. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\min\{\alpha, \alpha + \beta\} > 1/2$, $\gamma \geq 0$ and $0 < \gamma - \beta < \alpha$. Then it holds*

$$g_m(I_1 : H^{\alpha, \beta}(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d)) \asymp a_m(I_1 : H^{\alpha, \beta}(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d)) \asymp m^{-(\alpha - \gamma + \beta)}, \quad m \geq 1. \quad (6.2)$$

Proof. Proposition 7.1 below shows

$$m^{-(\alpha-(\gamma-\beta))} \lesssim a_m(I_1 : H^{\alpha,\beta}(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d)) \leq g_m(I_1 : H^{\alpha,\beta}(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d)), \quad m \in \mathbb{N}.$$

Suppose $0 < \varepsilon < \gamma - \beta$. Let $D_\varepsilon(\xi)$ be the number of function values the operator $\Delta_\varepsilon(\xi)$ is using. Then Theorem 4.1 yields

$$\|f - Q_{\Delta_\varepsilon(\xi)} f\|_{H^\gamma(\mathbb{T}^d)} \lesssim \left(\frac{D_\varepsilon(\xi)}{2^{\xi/(\alpha-(\gamma-\beta))}} \right)^{\alpha-(\gamma-\beta)} D_\varepsilon(\xi)^{-(\alpha-(\gamma-\beta))} \|f\|_{H^{\alpha,\beta}(\mathbb{T}^d)}.$$

Applying Corollary 6.5, (i) with $\alpha - \varepsilon$ and $\gamma - \beta - \varepsilon$ shows that

$$\frac{D_\varepsilon(\xi)}{2^{\xi/(\alpha-(\gamma-\beta))}} \leq C_1(\varepsilon, \alpha, \gamma - \beta, d).$$

This proves the estimate from above in case $m = D_\varepsilon$. The corresponding estimate for all m follows by a simple monotonicity argument. \blacksquare

Remark 6.10. In case $\beta = 0$ Griebel and Hamaekers recently proved a similar upper bound for $g_m(I_1)$ (see [13, Lemma 9]). Under the conditions of Theorem 6.9 the family of sampling operators $Q_{\Delta_\varepsilon(\xi)}$ for $0 < \varepsilon < \gamma - \beta$ is optimal in order.

The next theorem collects sharp results for sampling numbers which are based on Smolyak's algorithm.

Theorem 6.11. *Let $\alpha > 1/2$ and suppose $0 < \gamma < \alpha$.*

(i) *We have for $m \geq 2$*

$$g_m(I_5 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H_{\text{mix}}^\gamma(\mathbb{T}^d)) \asymp a_m(I_5 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H_{\text{mix}}^\gamma(\mathbb{T}^d)) \asymp m^{-(\alpha-\gamma)} (\log m)^{(d-1)(\alpha-\gamma)}. \quad (6.3)$$

(ii) *Let $2 < q < \infty$. Then we have for $m \geq 2$*

$$\begin{aligned} g_m(I_4 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow L_q(\mathbb{T}^d)) &\asymp a_m(I_4 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow L_q(\mathbb{T}^d)) \\ &\asymp m^{-\alpha+\frac{1}{2}-\frac{1}{q}} (\log m)^{(d-1)(\alpha-\frac{1}{2}+\frac{1}{q})}. \end{aligned} \quad (6.4)$$

(iii) *In case $q = \infty$ it holds for all $m \geq 2$*

$$g_m(I_4 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow L_\infty(\mathbb{T}^d)) \asymp a_m(I_4 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow L_\infty(\mathbb{T}^d)) \asymp m^{-\alpha+\frac{1}{2}} (\log m)^{(d-1)\alpha}. \quad (6.5)$$

Proof. *Proof of (i).*

Proposition 7.1, (iii) below shows for $m \geq 2$

$$m^{-(\alpha-\gamma)} (\log m)^{(d-1)(\alpha-\gamma)} \lesssim a_m(I_5 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H_{\text{mix}}^\gamma(\mathbb{T}^d)) \leq g_m(I_5 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d)).$$

Concerning the estimate from above we apply Theorem 5.1 with $\xi = (\alpha - \gamma)m$ for $m \in \mathbb{N}$. This gives

$$\|f - Q_{\Delta((\alpha-\gamma)m)} f\|_{H_{\text{mix}}^\gamma(\mathbb{T}^d)} \lesssim 2^{-(\alpha-\gamma)m}, \quad m \in \mathbb{N}. \quad (6.6)$$

Let $D(m)$ be the number of function values used by $Q_{\Delta((\alpha-\gamma)m)} f$. By Theorem 6.7,(i),(ii) we know that

$$D(m) \asymp m^{d-1} 2^m \quad \text{and} \quad \log D(m) \asymp \log m.$$

Rewriting (6.6) gives

$$\|f - Q_{\Delta((\alpha-\gamma)m)} f\|_{H_{\text{mix}}^\gamma(\mathbb{T}^d)} \lesssim D(m)^{-(\alpha-\gamma)} (\log D(m))^{(d-1)(\alpha-\gamma)}.$$

Obvious monotonicity arguments complete the proof.

Proof of (ii).

The estimate from below for the approximation numbers is due to Romanyuk [21]. The corresponding estimate from above for the sampling numbers is an immediate consequence of Lemma 5.7 together with (i), where $\gamma = 1/2 - 1/q$.

Proof of (iii).

The estimate from below for the approximation numbers is due to Temlyakov [34]. Let us mention that this lower bound is also applied by a recent general result by Cobos, Kühn and Sickel [3]. For the details we refer to Proposition 7.1 below. The estimate from above for sampling numbers follows from Theorem 5.6 combined with Corollary 6.7,(i),(ii) in the same way as in (i). \blacksquare

Remark 6.12. As we have mentioned before, not all the results in Theorem 6.11 are new. Part (iii) reproduces a result due to Temlyakov [34]. Note, that our methods allow for proving this result in the framework of classical trigonometric interpolation, see Theorem 5.6, whereas Temlyakov had to use de la Vallée-Poussin sampling operators. In any case, it is remarkable that Smolyak's algorithm yields optimal bounds here. A non-periodic version of (ii) has been proved recently in Dũng [7].

7 Appendix: approximation numbers

Corresponding estimates for the approximation numbers serve as a natural benchmark for the sampling problem we are interested in. In the sequel we mainly collect the relevant results from [9].

Proposition 7.1. (i) *Let $\alpha > \gamma - \beta > 0$. Then*

$$a_n(I_1 : H^{\alpha,\beta}(\mathbb{T}^d) \rightarrow H^\gamma(\mathbb{T}^d)) \asymp n^{-\alpha+\gamma-\beta}, \quad n \in \mathbb{N}.$$

(ii) *Let $\alpha > \gamma - \beta = 0$. Then*

$$a_n(I_2 : H^{\alpha,\beta}(\mathbb{T}^d) \rightarrow H^\beta(\mathbb{T}^d)) \asymp n^{-\alpha} (\log n)^{(d-1)\alpha}, \quad 2 \leq n \in \mathbb{N}.$$

(iii) *Let $\alpha > \gamma \geq 0$. Then*

$$a_n(I_5 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H_{\text{mix}}^\gamma(\mathbb{T}^d)) \asymp n^{-(\alpha-\gamma)} (\log n)^{(d-1)(\alpha-\gamma)}, \quad 2 \leq n \in \mathbb{N}.$$

In particular,

$$a_n(I_3 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \asymp n^{-\alpha} (\log n)^{\alpha(d-1)}, \quad 2 \leq n \in \mathbb{N}. \quad (7.1)$$

(iv) *Let $\alpha > \frac{1}{2}$. Then*

$$a_n(I_4 : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow L_\infty(\mathbb{T}^d)) \asymp n^{-\alpha+\frac{1}{2}} (\log n)^{\alpha(d-1)}, \quad n \in \mathbb{N}.$$

Proof. Let us consider (iii) first. The relation in (7.1) is due to Temlyakov [34, Theorem III.4.4]. For $\gamma > 0$ we use the commutative diagram

$$\begin{array}{ccc} H_{\text{mix}}^\alpha(\mathbb{T}^d) & \xrightarrow{I} & H_{\text{mix}}^\gamma(\mathbb{T}^d) \\ A \downarrow & & \uparrow B \\ H_{\text{mix}}^{\alpha-\gamma}(\mathbb{T}^d) & \xrightarrow{I^*} & L_2(\mathbb{T}^d), \end{array}$$

where

$$\begin{aligned} Af(x) &:= \sum_{k \in \mathbb{Z}^d} c_k(f) \prod_{j=1}^d (1 + |k_j|^2)^{\gamma/2} e^{ikx}, \\ Bf(x) &:= \sum_{k \in \mathbb{Z}^d} c_k(f) \prod_{j=1}^d (1 + |k_j|^2)^{-\gamma/2} e^{ikx}. \end{aligned}$$

Clearly, $A : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H_{\text{mix}}^{\alpha-\gamma}(\mathbb{T}^d)$ and $B : L_2(\mathbb{T}^d) \rightarrow H_{\text{mix}}^\gamma(\mathbb{T}^d)$ are isomorphisms. In addition, we have $I = B \circ I^* \circ A$. The multiplicativity of the approximation numbers implies

$$a_n(I : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H_{\text{mix}}^\gamma(\mathbb{T}^d)) \leq \|A\| \|B\| a_n(I^* : H_{\text{mix}}^{\alpha-\gamma}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)).$$

Taking (7.1) into account yields the estimate from above. For the lower bound we use the commutative diagram the other way around to see $I^* = B^{-1} \circ I \circ A^{-1}$. We obtain

$$a_n(I^* : H_{\text{mix}}^{\alpha-\gamma}(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \leq \|A^{-1}\| \|B^{-1}\| a_n(I : H_{\text{mix}}^\alpha(\mathbb{T}^d) \rightarrow H_{\text{mix}}^\gamma(\mathbb{T}^d)).$$

Again (7.1) yields (iii). To prove (ii) we use the commutative diagram

$$\begin{array}{ccc} H^{\alpha,\beta}(\mathbb{T}^d) & \xrightarrow{I} & H^\beta(\mathbb{T}^d) \\ A \downarrow & & \uparrow B \\ H_{\text{mix}}^\alpha(\mathbb{T}^d) & \xrightarrow{I^*} & L_2(\mathbb{T}^d) \end{array}$$

with A, B modified accordingly. The result follows by (7.1).

The proof of (i) can be found in [9, Theorem 4.7], however, with the additional restriction that $2(\gamma - \beta) > \alpha > \gamma - \beta$. For the convenience of the reader we give a proof without this restriction. The lower bound in (i) is a consequence of a well-known abstract result (see [36, Theorem 1] or [17, Theorem 1.4, p. 405]) on lower bounds for linear n -widths, namely

Lemma 7.2. *Let L_{n+1} be an $n + 1$ -dimensional subspace in a Banach space X , and $B_{n+1}(r) := \{f \in L_{n+1} : \|f\|_X \leq r\}$. Then*

$$\lambda_n(B_{n+1}(r), X) \geq r.$$

Here $\lambda_n(B_{n+1}(r), X)$ denotes the linear n -width of the set $B_{n+1}(r)$ in X .

We apply this Lemma with $X = H^\gamma$ and L_{n+1} to be the subspace of all trigonometric polynomials with frequencies in $\mathcal{H}_{\Delta(\xi)}$ from (6.1) with $\Delta(\xi) = \Delta(\alpha, \beta, \gamma; \xi)$ and ξ chosen accordingly. From Lemma 6.4 we get $n \asymp 2^{\xi/(\alpha - (\gamma - \beta))}$. We immediately see the Bernstein type inequality

$$\|f\|_{H^{\alpha,\beta}} \lesssim 2^\xi \|f\|_{H^\gamma} \quad , \quad f \in L_{n+1}. \quad (7.2)$$

Hence, by choosing $r := 2^{-\xi}$ we get from (7.2) that $B_{n+1}(r)$ is contained in the unit ball of $H^{\alpha,\beta}$. Finally, by Lemma 7.2 we conclude

$$a_n(I_1) \geq \lambda_n(B_{n+1}(2^{-\xi}), H^\gamma) = 2^{-\xi} \asymp n^{-(\alpha - (\gamma - \beta))}.$$

For the proof of (iv) we apply a lemma that goes back to the work of Osipenko and Parfenov (see [20]). For more details we refer to the recent preprint by Cobos, Kühn and Sickel [3].

Plugging (7.1) into [3, Lemma 3.3] yields

$$\begin{aligned} a_n(I_4 : H_{mix}^\alpha(\mathbb{T}^d) \rightarrow L_\infty(\mathbb{T}^d)) &\geq \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\sum_{j=n}^{\infty} a_j^2(I_3 : H_{mix}^\alpha(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)) \right)^{\frac{1}{2}} \\ &\gtrsim \left(\sum_{j=n}^{\infty} j^{-2\alpha} \log(j)^{2(d-1)\alpha} \right)^{\frac{1}{2}}. \end{aligned} \quad (7.3)$$

Estimating the sum by an integral gives

$$\begin{aligned} \sum_n^{\infty} j^{-2\alpha} (\log j)^{2(d-1)\alpha} &\asymp \int_n^{\infty} y^{-2\alpha} (\log y)^{(d-1)2\alpha} dy \\ &\geq (\log n)^{2(d-1)\alpha} \int_n^{\infty} y^{-2\alpha} dy \\ &\asymp (\log n)^{2(d-1)\alpha} n^{-2\alpha+1}. \end{aligned} \quad (7.4)$$

Inserting (7.4) into (7.3) yields the lower bound in (iv). ■

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