Spline Interpolation on Sparse Grids

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We investigate the rate of convergence of interpolating splines with respect to sparse grids for Besov spaces of dominating mixed smoothness (tensor product Besov spaces). Main emphasis is given to the approximation by piecewise linear functions.

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1. Introduction

Many problems in real world applications involve functions depending on a large number d of variables. It is well-known that numerical treatments of such problems often suffer from the fact that the cost of an algorithm grows fast in d. See for instance the recent monographs of Novak and Woźniakowski [1, 2]. One is interested in finding suitable models and frameworks in order to restrict the bad influence of the dimension d. A prominent model is to impose specific smoothness conditions on the functions to be approximated. From the work of Korobov, Bakhvalov, Babenko and Smolyak starting in the late 1950s it is known that the boundedness of certain mixed derivatives of order d is a suitable assumption. We refer, e.g., to the monographs of Tikhomirov [3], Temlyakov [4] and Nikol'skij [5], to the recent survey of Bungartz and Griebel [6], and to papers by Bazarkhanov [7–9], Dinh Zung [10], Griebel [11], Oswald [12], Romanyuk [13–16] and [17–21] just to mention a few.
This paper continues the research in this direction. More precisely, we study the approximation of functions belonging to a certain Besov or Sobolev space of mixed smoothness by tensor product splines using exclusively function values on a grid which is generated by a Smolyak algorithm (see Paragraph 4.1).
For $m \in \mathbb{N}$ we define

$$\mathcal{G}(m,d)^{\square} := \Big\{(2^{-j_1}k_1, \dots, 2^{-j_d}k_d) \subset [0,1]^d : \; j_1 + ... + j_d \le m\Big\}. \qquad (1)$$

This set represents a sparse grid in $[0,1]^d$, i.e., a grid, consisting of considerably fewer sampling points than the full Cartesian product of the interval grids. Let $\mathcal{N}$

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denote the normalized cardinal B-spline of order two, i.e.,

$$\mathcal{N}(t) := \begin{cases} t & \text{if} \quad 0 \le t \le 1\,, \\ 2-t & \text{if} \quad 1 < t \le 2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Our main result is the following. For any $m \in \mathbb{N}$ there is a linear operator A_m, defined on the class $C([0,1]^d)$, that only involves function values on $\mathcal{G}(m,d)^\square$ of its argument f. The approximant $A_m f$ is a linear combination of functions of the form

$$\mathcal{N}(2^{j_1}x_1 - k_1) \cdot \ldots \cdot \mathcal{N}(2^{j_d}x_d - k_d)\,,$$

where $0 \le k_i \le 2^{j_i}$, $i = 1, \ldots, d$, and $j_1 + \ldots + j_d \le m$. The function $A_m f$ interpolates f, i.e.,

$$A_m f(x) = f(x) \qquad \text{for all} \quad x \in \mathcal{G}(m,d)^\square\,.$$

For $1 \le p \le \infty$ and $1/p < r < 2$, the error norm behaves like

$$\| \, I - A_m \, | S^r_{p,p}B((0,1)^d) \to L_p((0,1)^d)\| \asymp m^{(d-1)(1-1/p)}\, 2^{-rm}\,. \tag{2}$$

Here I denotes the identity operator

$$I : S^r_{p,p}B((0,1)^d) \to L_p((0,1)^d)$$

and $S^r_{p,p}B((0,1)^d)$ is a Besov space of dominating mixed smoothness. The restriction $r > 1/p$ guarantees the embedding $S^r_{p,p}B((0,1)^d) \hookrightarrow C(\mathbb{R}^d)$, which ensures that our approximation procedure is well defined. This class of functions can also be interpreted as a tensor product of univariate Besov spaces

$$S^r_{p,p}B((0,1)^d) = B^r_{p,p}(0,1) \otimes_p \ldots \otimes_p B^r_{p,p}(0,1)\,, \qquad 1 \le p < \infty\,,$$

in the sense of equivalent norms (see Paragraph A.2). Additionally, we give some extensions to higher order splines in order to treat spaces with smoothness $r \ge 2$. Furthermore, we consider interpolation on the entire $\mathbb{R}^d$.

Let $M = M(m) \asymp 2^m m^{d-1}$ be the size of the grid $\mathcal{G}(m,d)^\square$, i.e., the number of function values used by A_m. Then (2) can be rewritten as

$$\| \, I - A_m \, | S^r_{p,p}B((0,1)^d) \to L_p((0,1)^d)\| \asymp M^{-r} (\log M)^{(d-1)(r+1-1/p)}\,.$$

Comparing our sequence of linear operators A_m to in order optimal linear approximation, it turns out that

$$\frac{\| \, I - A_m \, | S^r_{p,p}B((0,1)^d) \to L_p((0,1)^d)\|}{a_M(S^r_{p,p}B((0,1)^d), L_p((0,1)^d))} \asymp \begin{cases} (\log M)^{(d-1)/2} & : 2 \le p < \infty \\ (\log M)^{(d-1)(1-1/p)} & : 1 < p \le 2 \end{cases}\,.$$

Here a_M denotes the M-th approximation number (see Paragraph 6.3.1) of the respective identity. For $1 \le p \le \infty$, it is an open question whether A_m realizes the optimal error if we restrict the class of rank M linear operators to those which exclusively use M functions values. The associated widths are sometimes called sampling numbers, see Paragraph 6.3.2. If the answer were positive, then $p = 2$

and $r > 1/2$ would yield an example of a pair of Hilbert spaces for which the asymptotic behavior of the approximation numbers and the sampling numbers do not coincide, a problem discussed recently in [22].
The paper is organized as follows. Section 2 is devoted to the study of interpolation and sampling of univariate functions on $\mathbb{R}$ in a rather general framework. We continue in Section 3 by collecting the consequences for the two examples we are most interested in, namely spline interpolation and Whittaker's cardinal series. In Section 4, we discuss the associated Smolyak algorithms on $\mathbb{R}^d$, which are applied to the examples in Section 5. Section 6 switches to the same problem for functions on the cube $(0,1)^d$. We point out the consequences for the problem of optimal recovery from function values. Finally, the Appendices A and B provide all the necessary facts about tensor products of function spaces and spaces of dominating mixed smoothness on $\mathbb{R}^d$ and $(0,1)^d$, respectively.
On the one hand, the paper represents a partial non-periodic analog of [19], on the other hand, it is a continuation of [21]. In the periodic context the related problems are part of classical approximation theory, see, e.g., the monograph [4]. It is connected to the problem of optimal recovery of functions and approximation by harmonics with frequencies in the hyperbolic cross.

1.1. *Notation*

The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ denote the real numbers, complex numbers, natural numbers, natural numbers including 0, and the integers. The natural number d is reserved for the dimension of the Euclidean space $\mathbb{R}^d$ under consideration. The Euclidean distance of $x \in \mathbb{R}^d$ to the origin is denoted by $|x|_2$, whereas the ℓ_1^d-norm is denoted by $|x|_1$. We often need further vector-type quantities like multi-indices. They are denoted by $\bar{\ell}, \bar{k}, \bar{j}$ and $\bar{\alpha}$ with numbered components. For a multi-index $\bar{\alpha}$ we define the differential operator $D^{\bar{\alpha}}$ by

$$D^{\bar{\alpha}} = \frac{\partial^{|\bar{\alpha}|_1}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \,.$$

The symbol $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse. We normalize these transformations by

$$\mathcal{F}f(\xi) := \frac{1}{(\sqrt{2\pi})^d} \int\limits_{\mathbb{R}^d} e^{-ix\cdot\xi} \, f(x) \, dx \,, \qquad \xi \in \mathbb{R}^d \,,$$

where $f \in L_1(\mathbb{R}^d)$. We further use the notation $L_p(\mathbb{R}^d)$ and $L_p(\Omega)$, $1 \le p \le \infty$, for the usual Lebesgue spaces, whereas uniformly continuous, bounded functions are collected in $C(\mathbb{R}^d)$. Sequence spaces of p-summable sequences are denoted by ℓ_p, where $1 \le p \le \infty$.
If $(X, \|\cdot|X\|)$ and $(Y, \|\cdot|Y\|)$ are Banach spaces, then we shall write $\mathcal{L}(X,Y)$ for the class of all linear and bounded operators $P: X \to Y$ equipped with the usual operator norm denoted by $\|\,P\,|X \to Y\|$. If $X = Y$ we simply write $\mathcal{L}(X)$. Recall, that for a subset M of a vector space X, $\operatorname{span} M$ denotes the set of all finite linear combinations.
The notation $a \asymp b$ is used if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1}\, a \le b \le c\, a \,.$$

Constants will change their value from line to line, sometimes indicated by adding subscripts.

2. Interpolation and sampling on uniform grids on $\mathbb{R}$

To prepare our estimates for the d-dimensional case we first recall some results from the one-dimensional context. We shall discuss two different approaches. The first one goes back to Oswald [23] and concentrates on a situation where the interpolant of each function belongs to the space where f is taken from. The second approach allows sampling instead of interpolation. It uses the Strang-Fix conditions of the basic function Λ. Here we employ some ideas from Ries and Stens [24], see also Jetter and Zhou [25], and [26].

2.1. *Preparations*

We start with a uniform continuous square integrable function $\Lambda : \mathbb{R} \to \mathbb{R}$ and the following assumptions:

(C) The series

$$\sum_{k=-\infty}^{\infty} |\Lambda(t-k)|$$

converges uniformly on $[0,1]$.

(I) (Interpolation property) For $k \in \mathbb{Z}$ it holds

$$\Lambda(k) = \delta_{0,k} \, .$$

(R) (Refinement condition) There exists a sequence $\{m_k\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ such that

$$\Lambda(t) = \sum_{k\in\mathbb{Z}} m_k \Lambda(2t-k) \, , \tag{3}$$

where the convergence is considered in $L_2(\mathbb{R})$.

For $j \geq 0$ we define

$$V_j(\Lambda) := \operatorname{span}\{\Lambda(2^j \cdot -k) \ : \ k \in \mathbb{Z}\} \, .$$

If there is no danger of confusion we will write simply V_j. For $1 \leq p \leq \infty$ and $\Lambda \in L_p(\mathbb{R})$ we shall use the notation $\overline{V_j}^{(p)}$ for the closure of V_j with respect to $\| \cdot |L_p(\mathbb{R})\|$. As a consequence of (C) and $\ell_2 \hookrightarrow \ell_\infty$ the expansion (3) converges as well in $C(\mathbb{R})$ and the limits can be identified. Hence, $V_{j-1} \subset \overline{V_j}^{(\infty)}$. This results in a multiresolution analysis, i.e.

$$\overline{V_0}^{(\infty)}(\Lambda) \subset \overline{V_1}^{(\infty)}(\Lambda) \subset \ \ldots \ \subset \overline{V_j}^{(\infty)}(\Lambda) \subset \ldots \subset C(\mathbb{R}) \, , \quad j \in \mathbb{N}_0 \, . \tag{4}$$

Now we define a family of linear operators by

$$Q_j f(t) := \sum_{k=-\infty}^{\infty} f(2^{-j}k)\, \Lambda(2^j t - k)\,, \qquad t \in \mathbb{R}\,, \qquad j \in \mathbb{N}_0\,. \tag{5}$$

From property (C) we derive that Q_j is well-defined on $C(\mathbb{R})$.

Lemma 2.1: *Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be a uniform continuous square integrable function satisfying (C).*

(i) For $j \in \mathbb{N}_0$ we have $Q_j \in \mathcal{L}(C(\mathbb{R}))$ and

$$\| \, Q_j \, | \mathcal{L}(C(\mathbb{R})) \| = \sup_{0 \le t \le 1} \sum_{k=-\infty}^{\infty} |\Lambda(t-k)|\,.$$

(ii) If Λ additionally satisfies (I) and (R) then

$$Q_j f = f\,, \quad f \in \overline{V_\ell}^{(\infty)}\,,$$

for all $0 \le \ell \le j$.

Proof: Part (i) is proved in [24]. Now we turn to (ii). By assumption (R) and (4) it suffices to consider the case $\ell = j$. Any $f \in V_j$ can be written as

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \, \Lambda(2^j t - k)\,, \qquad t \in \mathbb{R}\,,$$

for some finite sequence $(c_k)_k$. By (I) we conclude that

$$f(k2^{-j}) = c_k \quad , \quad k \in \mathbb{Z}.$$

Hence, $Q_j f = f$ on the dense subset V_j and (i) concludes the proof. □

2.2. *On the norm of the operators $I - Q_j$*

We shall work with Besov spaces $B^s_{p,p}(\mathbb{R})$. Formally, its elements are equivalence classes of functions. Since we want to access function samples, recall that

$$B^s_{p,p}(\mathbb{R}) \hookrightarrow C(\mathbb{R}) \qquad \text{if} \quad s > 1/p\,. \tag{6}$$

The correct interpretation is that the equivalence class $f \in B^s_{p,p}(\mathbb{R})$ contains one representative which is continuous. Furthermore, it is also well-known that $B^{1/p}_{p,p}(\mathbb{R})$, $1 < p \le \infty$ contains unbounded functions.
Let us fix $1 \le p \le \infty$. We need some additional assumptions.

(S) (Stability) There exist positive constants A, B s.t.

$$A \, \| \, (c_k)_k \, | \ell_p \| \le \| \sum_k c_k \, \Lambda(\,\cdot\, - k) \, | L_p(\mathbb{R}) \| \le B \, \| \, (c_k)_k \, | \ell_p \|$$

holds for all finite sequences $(c_k)_k$.

($\mathbf{D_s}$) (Decomposition property) There exist projections

$$P_j : B^s_{p,p}(\mathbb{R}) \to \overline{V}_j^{(\infty)}$$

such that $f = \sum_{j=0}^\infty P_j f$ (convergence in $\| \cdot |B^s_{p,p}(\mathbb{R})\|$) and

$$\|f|B^s_{p,p}(\mathbb{R})\| \asymp \Big(\sum_{j=0}^\infty 2^{jsp} \| P_j f \,|L_p(\mathbb{R})\|^p\Big)^{1/p} .$$

Moreover, we assume $P_i P_j = \delta_{i,j} P_i$.

Our main tool in deriving the estimate for interpolation on sparse grids will be the following inequality for $d = 1$.

Proposition 2.2: *Let $1 \le p \le \infty$ and $s > 1/p$. Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be a function satisfying conditions (C), (I), (R), (S) and (D_s). Then there exists a constant $c > 0$ such that*

$$\Big(\sum_{j\in\mathbb{N}_0} 2^{jsp}\|Q_j f - Q_{j-1} f|L_p(\mathbb{R})\|^p\Big)^{1/p} \le c\,\|f|B^s_{p,p}(\mathbb{R})\|$$

holds for all $f \in B^s_{p,p}(\mathbb{R})$.

Proof: We follow Oswald [23, Page 60]. Let us assume $1 \le p < \infty$. The case $p = \infty$ follows by obvious modifications. For brevity we put

$$g_\ell := P_\ell f\,, \quad \ell \in \mathbb{N}_0\,,$$

according to assumption (D_s). By Lemma 2.1 we know $Q_j g_\ell = Q_{j-1} g_\ell$ if $\ell < j$. Hence

$$\begin{aligned} &\| Q_j f - Q_{j-1} f \,|L_p(\mathbb{R})\|^p \\ &\quad\lesssim \Big\| Q_j\Big(\sum_{\ell\ge j} g_\ell\Big) \Big|L_p(\mathbb{R})\Big\|^p + \Big\| Q_{j-1}\Big(\sum_{\ell\ge j} g_\ell\Big) \Big|L_p(\mathbb{R})\Big\|^p . \end{aligned} \tag{7}$$

Let us concentrate on the first summand on the right-hand side. By condition (S) we obtain

$$\begin{aligned} 2^{jsp}\Big\|Q_j\Big(\sum_{\ell\ge j} g_\ell\Big)|L_p\Big\|^p &\lesssim 2^{j(sp-1)} \sum_{k=-\infty}^\infty \Big|\sum_{\ell=j}^\infty g_\ell(k2^{-j})\Big|^p \\ &= 2^{j(sp-1)} \sum_{k=-\infty}^\infty \Big|\sum_{\ell=j}^\infty 2^{-\ell\varepsilon}2^{\ell\varepsilon} g_\ell(k2^{-j})\Big|^p \\ &\lesssim 2^{j(sp-1)} \sum_{k=-\infty}^\infty 2^{-jp\varepsilon} \sum_{\ell=j}^\infty |2^{\ell\varepsilon} g_\ell(k2^{-j})|^p \\ &\lesssim 2^{j(s-1/p-\varepsilon)p} \sum_{\ell=j}^\infty 2^{\ell\varepsilon p} \sum_{k=-\infty}^\infty |g_\ell(k2^{-j})|^p\,, \end{aligned}$$

where $\varepsilon > 0$ will be chosen later. Because of the inclusion

$$\{k2^{-j} \ : \ k \in \mathbb{Z}\} \subset \{k2^{-\ell} \ : \ k \in \mathbb{Z}\}$$

for $\ell \geq j$ and property (S) we can continue estimating

$$\begin{aligned} 2^{jsp} \Big\| \, Q_j \Big(\sum_{\ell \geq j} g_\ell \Big) \, \Big| L_p(\mathbb{R}) \Big\|^p & \lesssim 2^{j(s-1/p-\varepsilon)p} \sum_{\ell=j}^{\infty} 2^{\ell\varepsilon p} \sum_{k=-\infty}^{\infty} |g_\ell(k2^{-\ell})|^p \\ & \lesssim 2^{j(s-1/p-\varepsilon)p} \sum_{\ell=j}^{\infty} 2^{\ell(\varepsilon+1/p)p} \, \| \, g_\ell \, | L_p(\mathbb{R}) \|^p \, . \end{aligned}$$

Hence, choosing $\varepsilon > 0$ such that $s > 1/p + \varepsilon$, we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} 2^{jsp} \Big\| \, Q_j \Big(\sum_{\ell \geq j} g_\ell \Big) \, \Big| L_p(\mathbb{R}) \Big\|^p & \lesssim \sum_{\ell=0}^{\infty} 2^{\ell(\varepsilon+1/p)p} \, \| \, g_\ell \, | L_p(\mathbb{R}) \|^p \sum_{j=0}^{\ell} 2^{j(s-1/p-\varepsilon)p} \\ & \lesssim \sum_{\ell=0}^{\infty} 2^{\ell sp} \, \| \, g_\ell \, | L_p(\mathbb{R}) \|^p \\ & \asymp \| f | B^s_{p,p}(\mathbb{R}) \|^p \, . \end{aligned}$$

In the same way we may argue for the second summand on the right-hand side of (7). Let us complete the proof by a comment on some technicalities in connection with the calculations done above. Indeed, because of (D_s) we have $g_\ell \in C(\mathbb{R})$ for every $\ell \in \mathbb{N}_0$. Hence, $Q_j g_\ell$ makes sense. Furthermore, the series $\sum_\ell g_\ell$ converges in $B^s_{p,p}(\mathbb{R})$ which implies the convergence in $C(\mathbb{R})$ as a consequence of $s > 1/p$, see (6). Lemma 2.1/(i) yields

$$\sum_{k \in \mathbb{Z}} \Big(\sum_{\ell} g_\ell(k2^{-j}) \Big) \Lambda(2^j \cdot -k) = Q_j (\sum_\ell g_\ell) = \sum_\ell Q_j g_\ell \, .$$

The proof is complete. □

Let us put now and in the sequel $\Delta_j := Q_j - Q_{j-1}$, $j \in \mathbb{N}$, and $\Delta_0 := Q_0$.

Corollary 2.3: *Under the same restrictions as in Proposition 2.2 we have*

$$\sup_{j \in \mathbb{N}_0} 2^{js} \, \| \, \Delta_j \ : B^s_{p,p}(\mathbb{R}) \to L_p(\mathbb{R}) \| < \infty \, .$$

It further holds

$$\sup_{j \in \mathbb{N}_0} 2^{js} \, \| \, I - Q_j \ : B^s_{p,p}(\mathbb{R}) \to L_p(\mathbb{R}) \| < \infty \, .$$

Proof: The first statement is a consequence of $\ell_p \hookrightarrow \ell_\infty$ and Proposition 2.2. To prove the second one we have to modify the proof of Proposition 2.2 slightly. Starting with

$$\begin{aligned} & \| f - Q_j f | L_p(\mathbb{R}) \| \\ & \quad \leq \sum_{\ell > j} \| g_\ell | L_p(\mathbb{R}) \| + \Big\| \, Q_j \Big(\sum_{\ell > j} g_\ell \Big) \, \Big| L_p(\mathbb{R}) \Big\| \end{aligned} \qquad (8)$$

instead of (7) and using

$$2^{js}\sum_{\ell>j}\|g_\ell|L_p\| = \sum_{\ell>j} 2^{(j-\ell)s}2^{\ell s}\|g_\ell|L_p\| \lesssim \|f|B^s_{p,p}(\mathbb{R})\| \, .$$

The second summand in (8) is estimated as above and we obtain

$$\sup_{j\in\mathbb{N}_0} 2^{js}\| \, I - Q_j \, : B^s_{p,p}(\mathbb{R}) \to L_p(\mathbb{R})\| < \infty \, .$$

□

Remark 1 : (i) Interpolation on uniform grids on $\mathbb{R}$ is a classical topic. We refer to Jetter's survey [27] and the references given there.
(ii) In the following subsection we shall deal with some more general results, at least partly. However, Proposition 2.2 and its proof is very similar to the sparse grid generalization which we are going to discuss in Theorem 4.5.

2.3. *Wavelets and Besov spaces*

Let ϕ denote a univariate scaling function associated with the wavelet ψ, i.e.,

$$\psi(t) = \sum_{k=-\infty}^{\infty} h_k \, \phi(2t-k) \, , \qquad t\in\mathbb{R} \, , \tag{9}$$

for an appropriate sequence $(h_k)_k \in \ell_2$. We define

$$\psi_{0,k}(t) := \phi(t-k) \, , \qquad k\in\mathbb{Z},$$
$$\psi_{j+1,k}(t) := 2^{j/2}\, \psi(2^j t - k) \, , \qquad k\in\mathbb{Z}, \quad j\in\mathbb{N}_0$$

and require that the functions $\psi_{j,k}$ form an orthonormal basis of $L_2(\mathbb{R})$. Furthermore we assume some regularity, namely $\phi,\psi \subset C^N(\mathbb{R}^d)$, and some decay

$$\sup_{t\in\mathbb{R}} (1+|x|)^M \, |\psi^{(\ell)}(x)| < \infty \, , \quad |\ell| \le N \, ,$$

as well as a moment condition

$$\int_{-\infty}^{\infty} t^\ell \, \psi(t) \, dt = 0 \, , \quad |\ell| < N \, ,$$

for some $M > 1$ and some $N > 0$. For the following proposition we refer to [28], [29] and [30].

Proposition 2.4: *Let $s\in\mathbb{R}$, $1\le p\le\infty$ and $1\le q\le\infty$. Further we suppose*

$$N > |s| \qquad \text{and} \qquad M > 1+N \, .$$

Then, for every $f\in B^s_{p,q}(\mathbb{R})$, we have

$$f = \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} \langle f, \, \psi_{j,k}\rangle \, \psi_{j,k} \, ,$$

convergence in $\mathcal{S}'(\mathbb{R})$ *(and in* $B^s_{p,q}(\mathbb{R})$ *if* $\max(p,q) < \infty$*), and*

$$\| f \,|B^s_{p,q}(\mathbb{R})\| \asymp \Big(\sum_{j=0}^{\infty} 2^{j(s+\frac{1}{2}-\frac{1}{p})q} \Big(\sum_{k\in\mathbb{Z}} |\langle f,\, \psi_{j,k}\rangle|^p \Big)^{q/p} \Big)^{1/q} . \tag{10}$$

Under the conditions of Proposition 2.4 property (D_s) is satisfied.

Lemma 2.5: *Let* $1 \le p \le \infty$ *and* $s > 1/p$*. We define*

$$P_j f := \sum_{k=-\infty}^{\infty} \langle f,\, \psi_{j,k}\rangle \, \psi_{j,k} \,, \qquad j \in \mathbb{N}_0 \,.$$

Then, under the conditions of Proposition 2.4 and with $(h_k)_k \in \ell_1$*, see (9), property* (D_s) *is satisfied with* $V_j = V_j(\phi)$*.*

Proof: The required orthonormality $P_i\, P_j = \delta_{i,j}\, P_j$ follows from the fact that the wavelet system is orthonormal. $(h_k)_k \in \ell_1$ implies that the range of P_j is contained in $\overline{V_j}^{(p)}(\phi)$ since span $\{\psi_{j,k} : \, k \in \mathbb{Z}\}$ is contained in $\overline{V_j}^{(p)}(\phi)$. Because of $B^s_{p,p}(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, see (6), we also get $P_j(B^s_{p,p}(\mathbb{R})) \hookrightarrow \overline{V_j}^{(\infty)}(\phi)$. Employing Proposition 2.4 with $s = 0$, $q = 1, \infty$, and taking into account the chain of continuous embeddings

$$B^0_{p,1}(\mathbb{R}) \hookrightarrow L_p(\mathbb{R}) \hookrightarrow B^0_{p,\infty}(\mathbb{R})$$

we obtain

$$\| \sum_{k=-\infty}^{\infty} a_k \, \psi_{j,k} \,|L_p(\mathbb{R})\| \asymp 2^{j(\frac{1}{2}-\frac{1}{p})} \Big(\sum_{k\in\mathbb{Z}} |a_k|^p \Big)^{1/p} , \tag{11}$$

where the constants behind $\asymp$ do not depend on the sequence $(a_k)_k$ and on $j \in \mathbb{N}_0$. By orthonormality of our wavelet system we know $\langle f,\, \psi_{j,k}\rangle = \langle P_j f,\, \psi_{j,k}\rangle$. Hence, as a consequence of (10), we obtain

$$\| f \,|B^s_{p,q}(\mathbb{R})\| \asymp \Big(\sum_{j=0}^{\infty} 2^{jsp} \, \| \sum_{k\in\mathbb{Z}} \langle f,\, \psi_{j,k}\rangle \, \psi_{j,k} \,|L_p(\mathbb{R})\|^p \Big)^{1/p} .$$

This proves the claim. □

Remark 2: Given a wavelet characterization as above there is not always an associated fundamental interpolant Λ s.t. $\overline{V_0}^{(\infty)}(\Lambda) = \overline{V_0}^{(\infty)}(\phi)$. One example is obtained by considering the multiresolution analysis associated to cardinal B-splines of odd order.

2.4. *Sampling on uniform grids on* $\mathbb{R}$

Here we want to avoid assumption (I). It will be convenient for us to work with the following family of operators

$$I_W f(t) := \sum_{k=-\infty}^{\infty} f(k/W) \, \Lambda(Wt - k) \,, \qquad t \in \mathbb{R} \,,$$

where $W \in \mathbb{R}$, $W \geq 1$. We will not restrict to the dyadic subsequence Q_j, see (5). Let us start with the remarkable result of Jetter and Zhou [25].

Proposition 2.6: *Let $\Lambda \in L_2(\mathbb{R})$ and $s > 1/2$. Then the following two assertions are equivalent.*

- *There exists a constant c such that for all $f \in B^s_{2,2}(\mathbb{R})$ and all $W \geq 1$ we have the inequality*

$$\| f - I_W f \,|L_2(\mathbb{R})\| \leq c\, W^{-s} \, \| \, |\xi|^s \, \mathcal{F}f(\xi) \,|L_2(\mathbb{R})\| \,. \tag{12}$$

- *The function*

$$|\xi|^{-2s} \Big(1 - \frac{\mathcal{F}\Lambda(\xi)}{\sqrt{2\pi}} \Big)^2 + \sum_{k \neq 0} |\mathcal{F}\Lambda(\xi + 2\pi k)|^2$$

belongs to $L_\infty(-\pi, \pi)$.

The expression $\| \, |\xi|^s \, \mathcal{F}f(\xi) \,|L_2(\mathbb{R})\|$, see the right-hand side in (12), represents the homogeneous part of the norm of the Besov space $B^s_{2,2}(\mathbb{R})$. There are many ways to split the norm in a way s.t.

$$\| f \,|B^s_{p,q}(\mathbb{R})\| \asymp \| f \,|L_p(\mathbb{R})\| + \| f \,|\dot{B}^s_{p,q}(\mathbb{R})\| \,.$$

Whenever we have an estimate of the norm $\| I - I_W \,|B^s_{p,\infty}(\mathbb{R}) \to L_p(\mathbb{R})\|$, then we get a somewhat stronger estimate by working with $\| f \,|\dot{B}^s_{p,q}(\mathbb{R})\|$ instead of $\| f \,|B^s_{p,q}(\mathbb{R})\|$. This is probably well-known, but we did not find a reference. For that reason we give a precise formulation and a proof.

Lemma 2.7: *Let $1 \leq p, q \leq \infty$, $s > 0$ and suppose*

$$\| I - I_W \,|B^s_{p,q}(\mathbb{R}) \to L_p(\mathbb{R})\| \lesssim W^{-s} \,, \qquad W \geq 1 \,. \tag{13}$$

Then there exists a constant $c > 0$ s.t.

$$\| f - I_W f \,|L_p(\mathbb{R})\| \leq c\, W^{-s} \, \| f \,|\dot{B}^s_{p,q}(\mathbb{R})\| \tag{14}$$

holds for all $W \geq 1$ and all $f \in B^s_{p,q}(\mathbb{R})$.

Proof: We assume that (14) is not true. Then there exists a sequences $(W_N)_N$ and $(f_N)_N$ such that

$$\| f_N \,|B^s_{p,q}(\mathbb{R})\| = 1 \qquad \text{and} \qquad \| f_N - I_{W_N} f_N \,|L_p(\mathbb{R})\| \geq N\, W_N^{-s} \, \| f_N \,|\dot{B}^s_{p,q}(\mathbb{R})\| \,.$$

Estimate (13) in combination with this last inequality yields

$$\begin{aligned} N\, W_N^{-s} \, \| f_N \,|\dot{B}^s_{p,q}(\mathbb{R})\| &\leq \| f_N - I_{W_N} f_N \,|L_p(\mathbb{R})\| \\ &\leq c\, W_N^{-s} \| f_N \,|B^s_{p,q}(\mathbb{R})\| \end{aligned}$$

This yields

$$\lim_{N \to \infty} \| f_N \,|\dot{B}^s_{p,q}(\mathbb{R})\| = 0 \,. \tag{15}$$

By assumption this also implies

$$\lim_{N\to\infty} \| f_N \,|L_p(\mathbb{R})\| = 1\,. \tag{16}$$

But this is impossible. To see this we switch to homogeneous Besov spaces for a moment, also denoted by $\dot{B}^s_{p,q}(\mathbb{R})$, see e.g. Peetre [31] or Triebel [32, Chapt. 5] for the definition and some properties. The elements of these classes are equivalence classes modulo polynomials of degree $\le [s]$ (integer part). From (15) we derive the existence of a sequence $(p_N)_N$ of polynomials of degree $\le [s]$ s.t.

$$\lim_{N\to\infty} f_N + p_N = 0 \qquad (\text{convergence in} \quad S'(\mathbb{R}))\,.$$

But this is in contradiction with (16). □

We continue with results based on Strang-Fix and discrete moment conditions. Let us start by recalling the Strang-Fix conditions.

Definition 2.8: (i) We say that Λ satisfies a discrete moment condition of order $\alpha > 0$ if

$$m_\alpha(\Lambda) := \sup_{t\in\mathbb{R}} \sum_{k=-\infty}^{\infty} |t-k|^\alpha \, |\Lambda(t-k)| < \infty\,. \tag{17}$$

(ii) We say that the function $\Lambda \in L_1(\mathbb{R})$ satisfies the Strang-Fix condition of order 1 if

$$\mathcal{F}\Lambda(2\pi k) = \delta_{0,k}\, \frac{1}{\sqrt{2\pi}}\,, \qquad k \in \mathbb{Z}\,. \tag{18}$$

(iii) We say that the function Λ satisfies the Strang-Fix condition of order $r \in \mathbb{N}$, $r > 1$, if Λ satisfies the Strang-Fix condition of order 1 and in addition

$$(\mathcal{F}\Lambda)^{(\ell)}(2\pi k) = 0\,, \qquad k \in \mathbb{Z}\,, \quad \ell = 1, 2, \ldots, r-1\,. \tag{19}$$

Remark 3: (i) Observe, that if $\Lambda \in C(\mathbb{R})$, then $m_\alpha(\Lambda) < \infty$ implies $m_\beta(\Lambda) < \infty$ for all $0 < \beta < \alpha$.
(ii) If Λ satisfies (C), then the Strang-Fix condition of order 1 and

$$\sum_{k=-\infty}^{\infty} \Lambda(t-k) = 1\,, \qquad t \in \mathbb{R}\,,$$

are equivalent, see [24].

Now we are ready to recall the main results of [24]. As usual, if $0 < s \le 1$,

$$\|f \,|\mathrm{Lip}\, s\| := \sup_{u\neq v} \frac{|f(u)-f(v)|}{|u-v|^s}\,.$$

Proposition 2.9: (i) *Let* $0 < s \le 1$*. Let* Λ *be a function satisfying* $m_s(\Lambda) < \infty$ *and (18). Then*

$$\| I - I_W \,|\mathrm{Lip}\, s(\mathbb{R}) \to C(\mathbb{R})\| \lesssim W^{-s}\,, \qquad W \ge 1\,.$$

(ii) *Let Λ be a function satisfying $m_r(\Lambda) < \infty$ for some $r \in \mathbb{N}$, $r > 1$. Then the following assertions are equivalent:*
(a) *We have*

$$\| \, I - I_W \, | C^r(\mathbb{R}) \to C(\mathbb{R}) \| \lesssim W^{-r} \, , \qquad W \geq 1 \, .$$

(b) *Λ satisfies the Strang-Fix conditions of order r.*

Now we turn to a p-version of this result.

Proposition 2.10: *Let Λ be a continuous function satisfying $m_{r+1}(\Lambda) < \infty$, (18), and (19) for some $r \in \mathbb{N}$, $r > 1$. Let $1 \leq p \leq \infty$ and $1/p < s < r$. Then*

$$\| \, I - I_W \, | B^s_{p,\infty}(\mathbb{R}) \to L_p(\mathbb{R}) \| \lesssim W^{-s} \, , \qquad W \geq 1 \, . \tag{20}$$

Proof: We proceed as in [26] (proof of Proposition 9). Only a few modifications are necessary.

Step 1. Since $m_r(\Lambda) < \infty$, we have the equivalence of (19) and the discrete moment condition

$$\sum_{k=-\infty}^{\infty} (t-k)^j \, \Lambda(t-k) = 0 \, , \qquad j = 1, 2, \ldots , r-1 \, , \; t \in \mathbb{R} \, , \tag{21}$$

see [24]. Temporarily we assume that f is smooth. We employ Taylor's formula and obtain

$$f(x) = \sum_{j=0}^{r-2} \frac{f^{(j)}(x_0)}{j!} \, (x - x_0)^j + \int_{x_0}^{x} f^{(r-1)}(u) \, \frac{(x-u)^{r-2}}{(r-2)!} \, du \, . \tag{22}$$

Let t and $W \geq 1$ be fixed. We use $\sum_{k=-\infty}^{\infty} \Lambda(t-k) = 1$ for all $t \in \mathbb{R}$, see Remark **3**, and (22) with $k/W = x$, $x_0 = t$ and see that

$$\begin{aligned} f - I_W f(t) &= \sum_{k=-\infty}^{\infty} \big[f(t) - f(\frac{k}{W}) \big] \; \Lambda(Wt-k) \\ &= \sum_{j=1}^{r-2} \frac{f^{(j)}(t)}{j!} \sum_{k=-\infty}^{\infty} (\frac{k}{W} - t)^j \Lambda(Wt-k) \\ &\quad + \sum_{k=-\infty}^{\infty} \int_t^{k/W} f^{(r-1)}(u) \, \frac{(\frac{k}{W} - u)^{r-2}}{(r-2)!} \, du \; \Lambda(Wt-k) \, . \end{aligned} \tag{23}$$

Observe

$$f^{(r-1)}(t) \, \frac{(\frac{k}{W} - t)^{r-1}}{(r-1)!} = \int_t^{k/W} f^{(r-1)}(t) \, \frac{(\frac{k}{W} - u)^{r-2}}{(r-2)!} \, du \, .$$

Substituting into (23) and using the discrete moment conditions (21) gives

$$\begin{aligned} & f(t) - I_W f(t) \\ & \quad = \sum_{k=-\infty}^{\infty} \Big[\int_t^{k/W} (f^{(r-1)}(u) - f^{(r-1)}(t)) \, \frac{(\frac{k}{W} - u)^{r-2}}{(r-2)!} \, du \Big] \, \Lambda(Wt-k) \\ & \quad = \sum_{k=-\infty}^{\infty} \Big[\int_0^{k/W-t} (\Delta_h f^{(r-1)}(t)) \, \frac{(\frac{k}{W} - t - h)^{r-2}}{(r-2)!} \, dh \Big] \, \Lambda(Wt-k) \, . \end{aligned}$$

Let $0 < s < 1$. Since $|h| \le |k/W - t|$ the triangle and the Minkowski inequality yield

$$\begin{aligned} & \| \, f - I_W f \, | L_p(\mathbb{R}) \| \\ & \quad \le \Big\| \sum_{k=-\infty}^{\infty} |\frac{k}{W} - t|^{r-2} \, |\Lambda(Wt-k)| \int_{|h| \le |k/W-t|} |\Delta_h f^{(r-1)}(t)| \, dh \, \Big| L_p(\mathbb{R}) \Big\| \\ & \quad \le \Big\| \sum_{k=-\infty}^{\infty} |\frac{k}{W} - t|^{r+s-1} \, |\Lambda(Wt-k)| \int_{|h| \le |k/W-t|} |h|^{-s-1} |\Delta_h f^{(r-1)}(t)| \, dh \, \Big| L_p(\mathbb{R}) \Big\| \\ & \quad \le m_{r+1}(\Lambda) \, W^{-r-s+1} \int_0^{\infty} |h|^{-s} \| \Delta_h f^{(r-1)} \, | L_p(\mathbb{R}) \| \, \frac{dh}{|h|} \\ & \quad \lesssim W^{-r-s+1} \, \| \, f \, | B^{s+r-1}_{p,1}(\mathbb{R}) \| \, . \end{aligned} \tag{24}$$

Since $C_0^\infty(\mathbb{R})$ is dense in $B^{s+r-1}_{p,1}(\mathbb{R})$, this inequality extends from $C_0^\infty(\mathbb{R})$ to $B^{s+r-1}_{p,1}(\mathbb{R})$. Real interpolation with $0 < s_0 < s_1 < 1$ and $0 < \theta < 1$ yields

$$\begin{aligned} & \| \, I - I_W \, | B^s_{p,\infty}(\mathbb{R}) \to L_p(\mathbb{R}) \| \\ & \quad \lesssim \| \, I - I_W \, | B^{s_0}_{p,1}(\mathbb{R}) \to L_p(\mathbb{R}) \|^{1-\theta} \| \, I - I_W \, | B^{s_1}_{p,1}(\mathbb{R}) \to L_p(\mathbb{R}) \|^{\theta} \, , \end{aligned}$$

where $s := (1-\theta)\, s_0 + \theta\, s_1$, see e.g. [32, 2.4.2]. This proves the claim for all values of the smoothness parameter within the interval $(r-1, r)$.

Step 2. Let $1 < s < r$ and $1 \le p < \infty$. Since the discrete moment condition and the Strang-Fix conditions are monotone in r we immediately obtain (20) for all such s and p by applying the arguments from Step 1.

Step 3. Let $p = \infty$. Let $s > 0$, $s \notin \mathbb{N}$. We shall use $\operatorname{Lip} s(\mathbb{R}) \cap C(\mathbb{R}) = B^s_{\infty,\infty}(\mathbb{R})$, $0 < s < 1$ (in the sense of equivalent norms) in combination with

$$B^s_{\infty,\infty}(\mathbb{R}) = (B^{s_0}_{\infty,\infty}(\mathbb{R}), B^{s_1}_{\infty,\infty}(\mathbb{R}))_{\theta,\infty} \, ,$$

and

$$B^s_{\infty,\infty}(\mathbb{R}) = (B^{s_0}_{\infty,\infty}(\mathbb{R}), C^{s_1}(\mathbb{R}))_{\theta,\infty} \, ,$$

also valid in the sense of equivalent norms, and $s := (1-\theta)\, s_0 + \theta\, s_1$, $s_0 \ne s_1$ (with $s_1 \in \mathbb{N}$ in the second formula), see e.g. [32, 2.4.2]. Hence, as a consequence of Proposition 2.9, we obtain (20) with $p = \infty$ and $0 < s < r$.

Step 4. Let $1 \le p < \infty$. This time we use complex interpolation, see [33]. We have

$$B^s_{p,p}(\mathbb{R}) = [B^{s_0}_{1,1}(\mathbb{R}), B^{s_1}_{\infty,\infty}(\mathbb{R})]_\theta\,, \qquad \frac{1}{p} := 1-\theta\,, \quad s := (1-\theta)\, s_0 + \theta\, s_1\,.$$

For appropriate combinations of $s_0 > 1$ and $s_1 > 0$ we obtain

$$\|\, I - I_W\, |B^s_{p,p}(\mathbb{R}) \to L_p(\mathbb{R})\| \lesssim W^{-s}\,, \qquad W \ge 1\,,$$

for all s, $1/p < s < r$. Another application of real interpolation in case $1 < p < \infty$, namely

$$B^s_{p,\infty}(\mathbb{R}) = (B^{s_0}_{p,\infty}(\mathbb{R}), B^{s_1}_{p,p}(\mathbb{R}))_{\theta,\infty}\,, \qquad s := (1-\theta)\, s_0 + \theta\, s_1\,,$$

yields (20) for the full range of s. □

Remark 4 : Proposition 2.10 with the additional assumption $\operatorname{supp}\Lambda$ compact has been proved in [26]. However, the proofs in both situations used ideas from [24].

Sampling kernels with compact support – limiting situations

With Propositions 2.9 and 2.6 we have limiting situations with $p = \infty$ and $p = 2$, respectively. It is natural to look at $p = 1$ now. Following [34, pp. 52] we denote by $\mathrm{Lip}\,(1, L_1(\mathbb{R}))$ the set of all locally integrable functions f s.t. the semi-norm

$$\|\, f\, |\mathrm{Lip}\,(1, L_1(\mathbb{R}))\| := \sup_{t>0} \frac{1}{t} \sup_{|h|<t} \int_{-\infty}^{\infty} |\Delta_h f(x)|\, dx \tag{25}$$

is finite. In what follows we shall need the inequality

$$\sup_{t>0} \frac{1}{t} \sup_{|h|<t} \int_{-\infty}^{\infty} |\Delta_h f(x)|\, dx \le \|\, f'\, |L_1(\mathbb{R})\|\,. \tag{26}$$

This can be proved either directly, see e.g. Ziemer [35, pp. 46] or by using the connection with functions of bounded variation, see e.g. [34, Thm. 2.9.3].

Proposition 2.11: *Let Λ be a continuous compactly supported function satisfying (17), (18), and (19) for some $r \in \mathbb{N}$, $r > 1$. Then*

$$\|\, f - I_W f\, |W^r_1(\mathbb{R}) \to L_1(\mathbb{R})\| \lesssim W^{-r}\,, \qquad W \ge 1\,,$$

and

$$\|\, f - I_W f\, |W^r_\infty(\mathbb{R}) \to C(\mathbb{R})\| \lesssim W^{-r}\,, \qquad W \ge 1\,,$$

follow.

Proof: *Step 1.* Let $p = 1$. It will be enough to deal with $f \in C_0^\infty(\mathbb{R})$. Let $\operatorname{supp}\Lambda \subset [-a, a]$ for some $a > 0$. We proceed as in proof of Proposition 2.6 and obtain, see

(24),

$$
\begin{aligned}
&\| f - I_W f \,|L_1(\mathbb{R})\| \\
&\qquad \le \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \Big|\frac{k}{W} - t\Big|^{r-2} |\Lambda(Wt-k)| \int_{|h|\le |k/W-t|} |\Delta_h f^{(r-1)}(t)| \, dh \, dt \,.
\end{aligned}
$$

Here we apply $m_r(\Lambda) < \infty$. Using the compact support of Λ this yields

$$
\begin{aligned}
&\| f - I_W f \,|L_1(\mathbb{R})\| \\
&\quad \le \Big(\sup_{t\in\mathbb{R}} \sum_{k=-\infty}^{\infty} \Big|\frac{k}{W} - t\Big|^{r-2} |\Lambda(Wt-k)| \Big) \int_{|h|\le a/W} \int_{-\infty}^{\infty} |\Delta_h f^{(r-1)}(t)| \, dt \, dh \\
&\quad \le 2a \, m_{r-2}(\Lambda) \, W^{-r+1} \sup_{|h|\le a/W} \int_{-\infty}^{\infty} |\Delta_h f^{(r-1)}(t)| \, dt \,.
\end{aligned}
$$

Hence, by using (26) we conclude

$$
\| f - I_W f \,|L_1(\mathbb{R})\| \lesssim W^{-r} \, \| f^{(r-1)} \,|\mathrm{Lip}\,(1, L_1(\mathbb{R}))\| \lesssim W^{-r} \, \| f^{(r)} \,|L_1(\mathbb{R})\| \,.
$$

This proves the claim for $p = 1$.

Step 2. Let $p = \infty$. Again we use (24) as a starting point. Because of the Lipschitz continuity of $f^{(r-1)}$ we do not need a density argument. Consequently

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}} | f(t) - I_W f(t) | \\
&\qquad \le \Big(\sup_{t\in\mathbb{R}} \sum_{k=-\infty}^{\infty} \Big|\frac{k}{W} - t\Big|^{r-2} |\Lambda(Wt-k)| \Big) \int_{|h|\le a/W} \sup_{t\in\mathbb{R}} |\Delta_h f^{(r-1)}(t)| \, dh \\
&\qquad \le m_{r-2}(\Lambda) \, W^{-r+1} \sup_{|h|\le a/W} \| \Delta_h f^{(r-1)} \,|L_\infty(\mathbb{R})\| \,.
\end{aligned}
$$

The Lipschitz continuity of $f^{(r-1)}$ yields

$$
\| f - I_W f \,|C(\mathbb{R})\| \lesssim W^{-r} \, \| f^{(r)} \,|L_\infty(\mathbb{R})\| \,.
$$

The proof is complete. □

For Λ being compactly supported we can summarize our results as follows.

Corollary 2.12: *Let Λ be a continuous compactly supported function satisfying (17), (18), and (19) for some $r \in \mathbb{N}$, $r > 1$. Let $1 \le p \le \infty$. Then*

$$
\| I - I_W \,|W_p^r(\mathbb{R}) \to L_p(\mathbb{R})\| \lesssim W^{-r} \,, \qquad W \ge 1 \,. \tag{27}
$$

Proof: We shall use an interpolation formula established by DeVore and Scherer, namely

$$
W_p^r(\mathbb{R}) = (W_1^r(\mathbb{R}), W_\infty^r(\mathbb{R}))_{\theta,p} \,, \qquad \frac{1}{p} = 1 - \theta \,,
$$

see [36, Cor. 5.5.13]. This combined with Proposition 2.11 yields (27). □

Remark 5: By means of $W_2^s(\mathbb{R}) = B^s_{2,2}(\mathbb{R})$ (in the sense of equivalent norms) we can compare Corollary 2.12 with Proposition 2.6. Clearly, Proposition 2.6 is a stronger result than Corollary 2.12 in case $p = 2$.

3. Examples I.

We deal with two types of examples only. The first one is related to splines of even order and the second one is generated by the sinc-function. For more examples we refer to Jetter's survey [27].

3.1. *Spline interpolation*

Our general reference for B-splines and spline interpolation is Chui's monograph [37, Chapt. 4]. A few properties of B-splines and related wavelet bases are given in the appendix, Paragraph A.3.
Let $n \in \mathbb{N}$, $n \ge 2$. Beside the case $n = 2$ we can not work with the B-spline $\mathcal{N}_n$ itself. Recall $\mathcal{N}_2 = \mathcal{N}$. The function

$$S_n(\xi) := \sum_{k=-\infty}^{\infty} \mathcal{F}\mathcal{N}_n(\xi + 2\pi k)\,, \qquad \xi \in \mathbb{R}\,,$$

is called the symbol of $\mathcal{N}_n$. Poisson's summation formula yields

$$S_n(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{\ell=1}^{n-1} \mathcal{N}_n(\ell)\, e^{-i\xi\ell}\,,$$

hence the symbol is a trigonometric polynomial. It is well-known that this symbol does not have zeros if and only if n is even. For simplicity we concentrate on n even in the sequel. We define

$$\Lambda_{2n}(t) := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\Big[\frac{\mathcal{F}\mathcal{N}_{2n}(\xi)}{S_{2n}(\xi)}\Big](t)\,, \qquad t \in \mathbb{R}\,.$$

Observe $\Lambda_2 = \mathcal{N}_2(\cdot + 1)$. The function Λ_{2n} is a fundamental interpolant, i.e., property (I) is satisfied. Expanding $1/S_{2n}$ into a Fourier series

$$\frac{1}{S_{2n}(\xi)} = \sum_{k=-\infty}^{\infty} a_{n,k}\, e^{-ik\xi}$$

one obtains, using arguments from complex analysis, that the sequence $(|a_{n,k}|)_k$ is exponentially decaying. Hence

$$\Lambda_{2n}(t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} a_{n,k}\, \mathcal{N}_{2n}(t-k)\,. \tag{28}$$

Let us collect the important properties for our purpose.

- Λ_{2n} belongs to $\overline{V_0}^{(\infty)}(\mathcal{N}_{2n})$.
- Λ_{2n} is compactly supported if, and only if, $n = 1$.
- $\Lambda_{2n} \in B^{2n-1+1/p}_{p,\infty}(\mathbb{R})$, $1 \le p \le \infty$.

- Λ_{2n} is Lipschitz continuous of order $2n-1$.
- $\overline{V_0}^{(p)}(\Lambda_{2n}) = \overline{V_0}^{(p)}(\mathcal{N}_{2n})$ for all p, $1 \le p \le \infty$.

Proposition 3.1: *Let $n \in \mathbb{N}$ and $1 \le p \le \infty$.* (i) *The fundamental interpolant Λ_{2n} provides (C), (I), (R), (S), and (D_r) if $1/p < r < 2n-1+1/p$.*
(ii) *The function Λ_{2n} has finite discrete moments of any order. Furthermore, it satisfies the Strang-Fix conditions of order $2n$.*

Proof: *Step 1.* Proof of (i). *Step 1.1.* Property (C) follows from the exponential decay of the function Λ_{2n}. Properties (I) and (R) have been explained above. By (28) we have $\Lambda_{2n}(t) = \sum_{k=-\infty}^{\infty} c_k \mathcal{N}_{2n}(t-k)$ with $(c_k)_k \in \ell_1(\mathbb{Z})$. On the other hand it holds

$$\mathcal{N}_{2n}(t) = \sum_{\ell=0}^{2n-1} \mathcal{N}_{2n}(\ell)\, \Lambda_{2n}(t-\ell)\,.$$

These two identities, combined with the fact that $(\mathcal{N}_{2n}(\,\cdot\,-k)_{k\in\mathbb{Z}})$ is a Riesz basis in $\overline{V_0}^{(p)}(\mathcal{N}_{2n})$, yields (S) for Λ_{2n}.

Step 1.2. Choosing

$$P_j f(t) := \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k}^{2n}\rangle\, \psi_{j,k}^{2n}(t)\,, \qquad t \in \mathbb{R}, \quad j \in \mathbb{N}_0\,,$$

then the norm equivalence required in property (D_r) follows from Proposition A.9 in the Appendix below and Property (S) for the system $\{\psi_{j,k}^{2n}\}_{k\in\mathbb{Z}}$ in $L_p(\mathbb{R})$ or alternatively from Proposition A.9 in combination with Lemma 2.5. This lemma also implies $P_j : \ B^r_{p,p}(\mathbb{R}) \to \overline{V_j}^{(\infty)}$.

Step 2. Proof of (ii). Based on

$$\mathcal{F}\Lambda_2(\xi) = \frac{1}{\sqrt{2\pi}}\Big(\frac{\sin(\xi/2)}{(\xi/2)}\Big)^2\,, \quad \xi \in \mathbb{R}\,,$$

the claim is easy to check for $n=1$, see Subsection A.3 in the appendix. Let now $n>1$ and define

$$I_W^n f(t) := \sum_{k=-\infty}^{\infty} f(k/W)\, \Lambda_{2n}(Wt-k)\,, \quad t \in \mathbb{R}\,, \quad W \ge 1\,. \tag{29}$$

By Step 1 and Proposition 2.2 we know

$$\|\, f - I_W^n f\,|L_2(\mathbb{R})\| \le c\, W^{-s}\, \|\, f\,|B^s_{2,2}(\mathbb{R})\|$$

as long as $1/2 < s < 2n-1/2$. Here the switch from the dyadic subsequence to the sequence I_W^n is not a big deal. It requires only some more or less obvious modifications of the original proof. Lemma 2.7 yields that this inequality remains true with $\|\, f\,|\dot{B}^s_{2,2}(\mathbb{R})\|$. In view of Proposition 2.6 this implies that

$$|\xi|^{-s}\Big(1 - \frac{\mathcal{F}\Lambda(\xi)}{\sqrt{2\pi}}\Big)$$

belongs to $L_\infty(-\pi,\pi)$. Since $\mathcal{F}\Lambda(\xi)$ is the quotient of two smooth functions and the denominator is not vanishing we conclude that

$$1 - \mathcal{F}\Lambda(\xi)/\sqrt{2\pi} = O(|\xi|^{2n})\,, \qquad \xi \to 0\,.$$

Hence we have found

$$(\mathcal{F}\Lambda)^{(\ell)}(0) = \delta_{\ell,0}\,\frac{1}{\sqrt{2\pi}}\,, \qquad \ell = 0,1,\ldots,2n-1\,.$$

To prove $(\mathcal{F}\Lambda)^{(\ell)}(2\pi k) = 0$, $k \in \mathbb{Z}\setminus\{0\}$, $\ell = 0,1,\ldots,2n-1$, it is enough to observe that $\mathcal{F}\mathcal{N}_{2n}$ has a zero of order $2n-1$ in these points $2\pi k$. □

The following assertions are direct consequences of Propositions 2.9, 2.10, 2.6, 3.1 and Corollary 2.12.

Corollary 3.2: *Let $1 \le p \le \infty$ and $n \in \mathbb{N}$.*
(i) *Let $1/p < r < 2n$. Then we have*

$$\|\, I - I_W^n \,|B^r_{p,\infty}(\mathbb{R}) \to L_p(\mathbb{R})\| \lesssim W^{-r}\,, \qquad W \ge 1\,.$$

(ii) *In the limiting situation $r = 2n$ it holds*

$$\|\, I - I_W^n \,|B^{2n}_{2,2}(\mathbb{R}) \to L_2(\mathbb{R})\| \lesssim W^{-2n}\,, \qquad W \ge 1\,,$$

as well as

$$\|\, I - I_W^n \,|C^{2n}(\mathbb{R}) \to C(\mathbb{R})\| \lesssim W^{-2n}\,, \qquad W \ge 1\,.$$

(iii) *In the particular case $n = 1$ and $r = 2$ we even know*

$$\|\, I - I_W^1 \,|W^2_p(\mathbb{R}) \to L_p(\mathbb{R})\| \lesssim W^{-2}\,, \qquad W \ge 1\,.$$

Remark 1 : In [38], [39] and [26] also the following sampling operators, based on the functions

$$\begin{aligned}\psi_1(t) &:= 4\,\mathcal{N}_3(t+3/2) - 3\,\mathcal{N}_4(t+2)\,,\\ \psi_2(t) &:= 21\,\mathcal{N}_5(t+5/2) - 35\,\mathcal{N}_6(t+3) + 15\,\mathcal{N}_7(t+7/2)\,,\end{aligned}$$

are discussed. These compactly supported spline functions satisfy the Strang-Fix conditions of order $r = 3$ and $r = 4$, respectively. Let us denote the associated sampling operators by $I_W^{\psi_1}$ and $I_W^{\psi_2}$. Hence

$$\|\, I - I_W^{\psi_1} \,|B^s_{p,\infty}(\mathbb{R}) \to L_p(\mathbb{R})\| \lesssim W^{-s}\,, \qquad W \ge 1\,, \tag{30}$$

if $1/p < s < 3$ and

$$\|\, I - I_W^{\psi_2} \,|B^s_{p,\infty}(\mathbb{R}) \to L_p(\mathbb{R})\| \lesssim W^{-s}\,, \qquad W \ge 1\,, \tag{31}$$

if $1/p < s < 4$. For the limiting situations $s = 3$ and $s = 4$ we may apply Corollary 2.12. The corresponding operators $I_W^{\psi_i}$ are no longer projections, $I_W^{\psi_i} f$ is not interpolating the function f in general.

3.2. *Sinus Cardinalis*

We consider the function

$$\Lambda(t) = \mathrm{sinc}\,(t) = \frac{\sin(\pi t)}{\pi t}\,, \qquad t \in \mathbb{R}\,.$$

The associated family of operators Q_j, this time denoted by S_j,

$$S_j f(t) := \sum_{k=-\infty}^{\infty} f(2^{-j}k)\,\mathrm{sinc}\,(2^j t - k)\,, \qquad t \in \mathbb{R}\,,$$

are the classical Whittaker cardinal series. It is obvious that (I) is satisfied and (C) is not. However, the counterparts of Proposition 2.2 and Corollary 2.3 are known in this case, see [40]. Observe further that (S) is nothing than the classical Plancherel-Polya inequalities, see e.g. [26]. There is also an associated wavelet, namely the Shannon wavelet. We omit details and recall a result from [40].

Corollary 3.3: *Let $1 < p < \infty$, $1 \le q \le \infty$ and $1/p < r < \infty$. Then*

$$\| I - S_j \,|B^r_{p,q}(\mathbb{R}) \to L_p(\mathbb{R})\| \asymp 2^{-jr}\,, \quad j \in \mathbb{N}_0\,,$$

holds.

4. Interpolation on sparse grids on $\mathbb{R}^d$

Now we turn to the d-dimensional situation. To begin with we recall some properties of the Smolyak algorithm.

4.1. *The Smolyak Algorithm*

For a sequence of linear operators $L = (L_j)_j$ we put

$$\Delta_j(L) := \begin{cases} L_j - L_{j-1} & \text{if} \quad j \in \mathbb{N}\,, \\ L_0 & \text{if} \quad j = 0\,. \end{cases}$$

Definition 4.1: Let $m \in \mathbb{N}_0$. The Smolyak-Algorithm $A(m, d, \vec{L})$ of order m relative to the d sequences $L^1 := (L^1_j)_{j=0}^\infty, \ldots, L^d := (L^d_j)_{j=0}^\infty$, is the linear operator

$$A(m, d, \vec{L}) := \sum_{j_1+\ldots+j_d \le m} \Delta_{j_1}(L^1) \otimes \ldots \otimes \Delta_{j_d}(L^d)\,.$$

Remark 1: Originally introduced in [41] there are now hundreds of references dealing with this construction. A few basics and some references can be found in [42], [2] and [43]. In particular the following formula is proved in [43]:

$$A(m, d, \vec{L}) = \sum_{m-d+1 \le |\bar{j}|_1 \le m} (-1)^{m-|\bar{j}|_1} \binom{d-1}{m-|\bar{j}|_1} L^1_{j_1} \otimes \ldots \otimes L^d_{j_d}\,. \qquad (32)$$

For later use we need to define the following subspaces

$$V_{\bar{\ell}}(\Lambda) := \mathrm{span}\{\Lambda(2^{\ell_1} \cdot - k_1) \otimes \cdots \otimes \Lambda(2^{\ell_d} \cdot - k_d) \;:\; \bar{k} \in \mathbb{Z}^d\} \quad , \quad \bar{\ell} \in \mathbb{N}_0^d\,.$$

Moreover, we need the so-called sparse grid ansatz spaces $\mathcal{V}_m(\Lambda)$ for $m \in \mathbb{N}$. They are defined as follows

$$\mathcal{V}_m(\Lambda) := \mathrm{span}\{\Lambda(2^{\ell_1} \cdot -k_1) \otimes \cdots \otimes \Lambda(2^{\ell_d} \cdot -k_d) \ : \ |\bar{\ell}|_1 = m, \ \bar{k} \in \mathbb{Z}^d\} . \tag{33}$$

Again, we denote by $\overline{\mathcal{V}}_m^{(p)}$ the closure of $\mathcal{V}_m$ and by $\overline{V_{\bar{\ell}}}^{(p)}$ the closure of $V_{\bar{\ell}}$ in $L_p(\mathbb{R})$. Putting $\vec{Q} = ((Q_j)_j, \ldots, (Q_j)_j)$ we consider the related Smolyak algorithm $A(m, d, \vec{Q})$. The crucial observation is the following.

Lemma 4.2: *Let the sequence $(Q_j)_j$ satisfy*

$$Q_j f = f \quad , \quad f \in V_\ell , \tag{34}$$

for all $0 \le \ell \le j$. Then

$$A(m, d, \vec{Q}) f = f \tag{35}$$

holds for all $f \in \overline{\mathcal{V}}_m^{(\infty)}$ and all $m \in \mathbb{N}$.

Proof: Obviously, it is enough to prove (35) for

$$f(x_1, ..., x_d) = \Lambda(2^{u_1} x_1 - k_1) \otimes \cdots \otimes \Lambda(2^{u_d} x_d - k_d) ,$$

where $|\bar{u}|_1 \le m$ and $\bar{k} \in \mathbb{Z}^d$. Let

$$T := \sum_{0 \le \ell_1 \le m} \cdots \sum_{0 \le \ell_d \le m} \bigotimes_{i=1}^{d} \Delta_{\ell_i} \quad \text{and} \quad R := \sum_{|\bar{\ell}|_1 > m} \bigotimes_{i=1}^{d} \Delta_{\ell_i}$$

Hence, we get $A(m, d, \vec{Q}) = T - R$. Since $\sum_{j=0}^{m} \Delta_j = Q_m$, we obtain

$$T = \bigotimes_{i=1}^{d} Q_m .$$

Assumption (34) yields then $Tf = f$. Therefore, it suffices to prove that $Rf = 0$. We prove a bit more, namely that

$$\Big(\bigotimes_{i=1}^{d} \Delta_{\ell_i} \Big) f = 0 \tag{36}$$

for any $|\bar{\ell}|_1 > m$. Indeed, since $|\bar{\ell}|_1 > m \ge |\bar{u}|_1$ we find at least one $i \in \{1, ..., d\}$ such that $\ell_i - 1 \ge u_i$. A consequence of (34) is

$$\Delta_{\ell_i}[\Lambda(2^{u_i} \cdot -k_i)](t) = Q_{\ell_i}[\Lambda(2^{u_i} \cdot -k_i)](t) - Q_{\ell_i - 1}[\Lambda(2^{u_i} \cdot -k_i)](t) = 0 ,$$

which yields (36). □

The set of sampling points used by Q_j will be denoted by $\mathcal{T}_j$. Then we put

$$\mathcal{G}(m, d) := \bigcup_{m-d+1 \le |\bar{j}|_1 \le m} \mathcal{T}_{j_1} \times \ldots \times \mathcal{T}_{j_d} .$$

By (32), the operator $A(m,d,\vec{Q})$ uses only samples from the grid $\mathcal{G}(m,d)$. The nestedness of the grids $\mathcal{T}_j$ implies

$$\begin{aligned}\mathcal{G}(m,d) &= \bigcup_{|\bar{j}|_1=m} \mathcal{T}_{j_1} \times \ldots \times \mathcal{T}_{j_d} \\ &= \Big\{(2^{-j_1}k_1, \ldots, 2^{-j_d}k_d) : \quad |\bar{j}|_1 = m\,,\, k \in \mathbb{Z}^d\Big\}\,.\end{aligned}$$

Since the operators Q_j interpolate every $f \in C(\mathbb{R})$ on $\mathcal{T}_j$ we have the following counterpart for the operator $A(m,d,\vec{Q})$.

Lemma 4.3: *Let the function Λ satisfy (C), (I) and (R). Then*

$$A(m,d,\vec{Q})f(x) = f(x)\,, \qquad x \in \mathcal{G}(m,d)\,.$$

holds for all $x \in \mathcal{G}(m,d)$ and all $f \in C(\mathbb{R}^d)$.

Proof: The proof is similar to the proof of Lemma 4.2. We employ the same notation and decomposition of $A(m,d,\vec{Q}) = T - R$ as in the proof of Lemma 4.2. Since Q_m interpolates on $\mathcal{T}_m$ the operator T interpolates on $\mathcal{T}_m \times \ldots \times \mathcal{T}_m$. Hence, it is enough to prove

$$Rf(x) = 0 \qquad \text{for all} \quad x \in \mathcal{T}_{k_1} \times \ldots \times \mathcal{T}_{k_d}\,, \qquad |\bar{k}|_1 = m\,,$$

and all $f \in C(\mathbb{R}^d)$. We shall prove even more, namely

$$\Big(\Delta_{j_1} \otimes \ldots \otimes \Delta_{j_d}\Big)f(x) = 0\,, \qquad x \in \mathcal{T}_{k_1} \times \ldots \times \mathcal{T}_{k_d}\,, \quad |\bar{k}|_1 = m$$

and $|\bar{j}|_1 > m$.
Let $\bar{j}$, $|\bar{j}|_1 > m$, $\bar{k}$, $|\bar{k}|_1 = m$ and $x \in \mathcal{G}(m,d,\vec{L})$ be given. For $1 \le u \le d$ we put $g_u(t) := f(x_1, \ldots, x_{u-1}, t\,, x_{u+1}, \ldots, x_d)$, $t \in \mathbb{R}$. Furthermore, there exists at least one component u such that $k_u < j_u$. This implies $Q_{j_u} g_u(x_u) = Q_{j_u-1} g_u(x_u)$ which proves the claim. □

The next subsection is devoted to upper estimates of

$$\|I - A(m,d,\vec{Q}) : S^r_{p,q}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\|\,.$$

4.2. *Estimates from above for interpolation on sparse grids*

We will present two different approaches to error estimates for the Smolyak algorithm. The first one is rather close to what has been done in the univariate situation in Subsection 2.2. In particular, it is applicable only under the assumption (I). The second one is less complicated, can be applied even to sampling operators. However, the outcome is an estimate which is not as good as in the first approach.
We need a d-dimensional counterpart of the decomposition property (D_s). Let $1 \le p, q \le \infty$.

($\bar{\mathbf{D}}_{\mathbf{r}}$) (Multivariate decomposition) There exist projections $P_{\bar{\ell}}$,

$$P_{\bar{\ell}} :\ S^r_{p,q}B(\mathbb{R}^d) \to \overline{V_{\bar{\ell}}}^{(\infty)}$$

such that $f = \sum_{\bar{\ell}\in\mathbb{N}_0^d} P_{\bar{\ell}} f$ (convergence in $\| \cdot |S^r_{p,q}B(\mathbb{R}^d)\|$) and

$$\| f \,|S^r_{p,q}B(\mathbb{R}^d)\| \asymp \Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{|\bar{\ell}|_1 rq} \, \| P_{\bar{\ell}} f \,|L_p(\mathbb{R}^d)\|^q \Big)^{1/q} .$$

Additionally, we assume $P_{\bar{k}} P_{\bar{\ell}} = \delta_{\bar{k},\bar{\ell}} P_{\bar{\ell}}$.

If necessary we shall indicate the dependence on p and q by writing $(\bar{D}_r)(p,q)$ instead of $(\bar{D}_r)$. If our starting point is a wavelet system $(\psi_{j,k})_{j,k}$ which allows a characterization of $B^r_{p,p}(\mathbb{R})$, then we get $(\bar{D}_r)(p,p)$ for free.

Lemma 4.4: *Let $1 \le p \le \infty$. Let the univariate wavelet system $(\psi_{j,k})_{j,k}$ satisfy the assumptions in Proposition 2.4 for some $N > 0$ and $M > 1$. Furthermore we assume (9) with $(h_k)_k \in \ell_1$. Let $V_{\bar{\ell}} := V_{\bar{\ell}}(\phi)$, $\ell \in \mathbb{N}_0^d$. Then*

$$P_{\bar{\ell}} f := \sum_{\bar{k}\in\mathbb{Z}^d} \langle f, \psi_{\bar{\ell},\bar{k}} \rangle \, \psi_{\bar{\ell},\bar{k}} \, ,$$

where

$$\psi_{\bar{\ell},\bar{k}}(x) := \psi_{\ell_1,k_1}(x_1) \cdot \ldots \cdot \psi_{\ell_d,k_d}(x_d) \, ,$$

satisfies $(\bar{D}_r)(p,p)$ for all $1/p < r < N$.

Proof: We consider the mapping

$$J : \; f \mapsto (\langle f, \psi_{j,k} \rangle)_{j,k} \, .$$

By Proposition 2.4 J is an isomorphism mapping the Besov space $B^r_{p,p}(\mathbb{R})$ onto the sequence space b^r_p, see Def. A.8 in Appendix A. This implies that the tensor product operator $J \otimes \ldots \otimes J$ maps the tensor product of the Besov spaces isomorphically onto the tensor product of the sequence spaces. But

$$B^r_{p,p}(\mathbb{R}) \otimes_{\delta_p} \, \ldots \, \otimes_{\delta_p} B^r_{p,p}(\mathbb{R}) = S^r_{p,p}B(\mathbb{R}^d)$$

(see (A2)) and

$$b^r_p \otimes_{\delta_p} \, \ldots \, \otimes_{\delta_p} b^r_p = s^r_p b \, ,$$

where $s^r_p b$ is defined in Def. A.8. For the correct interpretation of these formulas and more details we refer to [21]. Hence

$$\| f \,|S^r_{p,p}B(\mathbb{R}^d)\| \asymp \Big(\sum_{\bar{j}\in\mathbb{N}_0^d} \sum_{\bar{k}\in\mathbb{Z}^d} 2^{|\bar{j}|_1 (r + \frac{1}{2} - \frac{1}{p})p} \, |\langle f, \psi_{\bar{\ell},\bar{k}} \rangle|^p \Big)^{1/p}$$

for all $|r| < N$. Next we consider the mapping J restricted to the range $\overline{W}^{(p)}_j \subset \overline{V}^{(p)}_j$ of P_j. To avoid misunderstandings we denote this restriction by T_j. Since T_j is an isomorphism of $\overline{W}^{(p)}_j$ onto $\ell_p(\mathbb{Z})$, see (11), the associated tensor product operator $T_{\bar{\ell}}$,

$$T_{\bar{\ell}} := T_{\ell_1} \otimes \, \ldots \, \otimes T_{\ell_d}$$

becomes an isomorphism of

$$\overline{W}^{(p)}_{\ell_1} \otimes_{\delta_p} \ldots \otimes_{\delta_p} \overline{W}^{(p)}_{\ell_d} \qquad \text{onto} \qquad \ell_p(\mathbb{Z}^d)\,,$$

since

$$\ell_p(\mathbb{Z}) \otimes_{\delta_p} \ldots \otimes_{\delta_p} \ell_p(\mathbb{Z}) = \ell_p(\mathbb{Z}^d)\,,$$

see [44]. Let $d = 2$ for a moment. By $X \otimes Y$ we denote the algebraic tensor product of X and Y. By $\overline{W}^{(p)}_{\bar\ell}$ we denote the L_p-closure of the set

$$\operatorname{span}\{\psi_{\ell_1,k} \otimes \ldots \otimes \psi_{\ell_d,k} :\ k \in \mathbb{Z}^d\}\,.$$

Then it is easy to see that

$$\Big(\operatorname{span}\{\psi_{\ell_1,k} : k \in \mathbb{Z}\} \otimes \operatorname{span}\{\psi_{\ell_2,k} : k \in \mathbb{Z}\}\Big) \subset \overline{W}^{(p)}_{\ell_1} \otimes \overline{W}^{(p)}_{\ell_2} \subset \overline{W}^{(p)}_{\bar\ell}\,, \qquad \bar\ell = (\ell_1, \ell_2)\,.$$

Taking the closure with respect to the tensor norm δ_p, then, by using this sandwich type argument, we conclude

$$\overline{W}^{(p)}_{\ell_1} \otimes_{\delta_p} \overline{W}^{(p)}_{\ell_2} = \overline{W}^{(p)}_{\bar\ell}\,.$$

Iteration yields

$$\overline{W}^{(p)}_{\ell_1} \otimes_{\delta_p} \ldots \otimes_{\delta_p} \overline{W}^{(p)}_{\ell_d} = \overline{W}^{(p)}_{\bar\ell}\,.$$

Employing this identity and taking (11) into account we obtain

$$\| \sum_{k \in \mathbb{Z}^d}^{\infty} a_k\, \psi_{j,k}\, |L_p(\mathbb{R})\| \asymp 2^{|\bar j|_1(\frac{1}{2}-\frac{1}{p})} \Big(\sum_{k \in \mathbb{Z}^d} |a_k|^p \Big)^{1/p}\,, \tag{37}$$

where the constants behind $\asymp$ do not depend on $j \in \mathbb{Z}^d$ and $(a_k)_k \in \ell_p(\mathbb{Z}^d)$. Here we used

$$\| T_{\bar\ell}\, |\overline{W}^{(p)}_{\bar\ell} \to \ell_p(\mathbb{Z}^d)\| = \prod_{i=1}^{d} \| T_{\ell_i}\, |\overline{W}^{(p)}_{\ell_i} \to \ell_p(\mathbb{Z})\|\,.$$

The inequality $\le$ follows from the general theory on tensor product operators and the fact, that δ_p is a uniform cross norm, see [44]. The reverse inequality follows by the same argument but applied to $T^{-1}_{\bar\ell} = T^{-1}_{\ell_1} \otimes \ldots \otimes T^{-1}_{\ell_d}$. Hence

$$\| f\, |S^r_{p,p}B(\mathbb{R}^d)\| \asymp \Big(\sum_{\bar j \in \mathbb{N}_0^d} 2^{|\bar j|_1 rp}\, \| \sum_{\bar k \in \mathbb{Z}^d} \langle f, \psi_{\bar\ell,\bar k}\rangle\, \psi_{\bar\ell,\bar k}\, |L_p(\mathbb{R}^d)\|^p \Big)^{1/p}\,.$$

To prove $P_{\bar\ell} :\ S^r_{p,q}B(\mathbb{R}^d) \to \overline{V}^{(\infty)}_{\bar\ell}$ it is enough to observe that $S^r_{p,q}B(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$ if $r > 1/p$ and

$$\overline{W}^{(p)}_{\bar\ell} \subset \overline{V}^{(p)}_{\bar\ell} \subset \overline{V}^{(\infty)}_{\bar\ell}\,,$$

see (37). □

Remark 2: Let $(\,\cdot\,,\,\cdot\,)_{\theta,q}$ denote the real interpolation. Let $1 \le p,q \le \infty$, $r > 0$ and $0 < \theta < 1$. Then, in contrast to the isotropic situation, we have

$$(S^r_{p,p}B(\mathbb{R}^d), L_p(\mathbb{R}^d))_{\theta,q} \neq S^{r(1-\theta)}_{p,q}B(\mathbb{R}^d)$$

in general, see [18]. Hence, at least at this moment, we do not have a simple argument to prove that $(\bar{D}_r)(p,p)$ implies $(\bar{D}_s)(p,q)$ for all $0 < s < r$ and $1 \le q \le \infty$.

Now we continue with the main assertion in this section.

Theorem 4.5: *Let $1 \le p < \infty$, $1 \le q \le \infty$, $1/p < r_0 < r$ and the sequence $(Q_j)_j$ be given by (5). Suppose further, that*

$$\sup_{j\in\mathbb{N}_0} 2^{jr_0} \,\|\, \Delta_j \,|B^r_{p,p}(\mathbb{R}) \to L_p(\mathbb{R})\| < \infty$$

and

$$Q_j f = f \quad , \quad f \in V_\ell\,,$$

for all $0 \le \ell \le j$. Additionally, we need $(\bar{D}_r(p,p))$, $(\bar{D}_{r_0}(p,p))$ and with coincidence of the projections $P_{\bar{\ell}}$ in $(\bar{D}_{r_0})$ and $(\bar{D}_r)$. Then

$$\|\, I - A(m,d,\vec{Q})\,|S^r_{p,q}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \lesssim 2^{-mr} m^{(d-1)(1-1/q)}\,, \qquad m \in \mathbb{N}\,. \quad (38)$$

Proof: Using $(\bar{D}_r)$ we can decompose $f \in S^r_{p,q}B(\mathbb{R}^d)$ in

$$f = \sum_{\bar{u}\in\mathbb{N}_0^d} g_{\bar{u}}\,. \quad (39)$$

We want to rearrange this decomposition in a proper way. Let $\bar{b} = (b_1,\dots,b_d)$, $b_i \in \{0,1\}$, $i = 1,\dots,d$. For fixed m we define the index sets $I^m_0 := \{0,...,m\}$, $I^m_1 := \{m+1, m+2, ...\}$ and

$$P^m_{\bar{b}} := \{\bar{u} \in \mathbb{N}_0^d \;:\; |\bar{u}|_1 > m,\; u_i \in I^m_{b_i},\; i = 1,\dots,d\}\,.$$

Rearranging (39) gives

$$f = h + \sum_{b\in\{0,1\}^d} f^b\,,$$

where

$$h := \sum_{|\bar{u}|_1\le m} g_{\bar{u}} \in \overline{\mathcal{V}}^{(\infty)}_m \qquad \text{and} \qquad f^b := \sum_{\bar{u}\in P^m_{\bar{b}}} g_{\bar{u}}\,. \quad (40)$$

Then, by Lemma 4.2, we obtain $A(m,d,Q)h = h$. It remains to estimate the 2^d terms $\|f^b - A(m,d,\vec{Q})f^b|L_p(\mathbb{R}^d)\|$. By triangle inequality we obtain

$$\|f^b - A(m,d,\vec{Q})f^b|L_p(\mathbb{R}^d)\| \le \sum_{\bar{u}\in P^m_{\bar{b}}} \|g_{\bar{u}} - A(m,d,\vec{Q})g_{\bar{u}}|L_p(\mathbb{R}^d)\|$$

Clearly, by means of Lemma 4.2 we know that

$$A(|\bar{u}|_1, d, \vec{Q}) g_{\bar{u}} = g_{\bar{u}} \,.$$

Next we claim

$$\big(\Delta_{\ell_1} \otimes \ldots \otimes \Delta_{\ell_d}\big) g_{\bar{u}} = 0$$

if there exists some index i_0 s.t. $\ell_{i_0} > u_{i_0}$. Since $g_{\bar{u}} \in \bar{V}_{\bar{u}}^{(\infty)}$ it is enough to test the operator on functions of the type $\Lambda(2^{u_1}x_1 - k_1) \cdot \ldots \cdot \Lambda(2^{u_d}x_d - k_d)$. But $\Delta_{\ell_{i_0}} \Lambda(2^{u_{i_0}} t - n) = 0$, see Lemma 2.1. Both identities, combined with Corollary A.7 in the Appendix, yield

$$\begin{aligned}
&\| \, g_{\bar{u}} - A(m, d, \vec{Q}) g_{\bar{u}} \, | L_p(\mathbb{R}^d) \| \\
&\quad = \| \sum_{m < |\bar{\ell}|_1 \le |\bar{u}|_1} \big(\Delta_{\ell_1} \otimes \ldots \otimes \Delta_{\ell_d}\big) g_{\bar{u}} \, | L_p(\mathbb{R}^d) \| \\
&\quad = \| \sum_{\bar{\ell} \in W_{\bar{u}}^m} \big(\Delta_{\ell_1} \otimes \ldots \otimes \Delta_{\ell_d}\big) g_{\bar{u}} \, | L_p(\mathbb{R}^d) \| \\
&\quad \le \sum_{\bar{\ell} \in W_{\bar{u}}^m} \| \big(\Delta_{\ell_1} \otimes \ldots \otimes \Delta_{\ell_d}\big) \, g_{\bar{u}} \, | L_p(\mathbb{R}^d) \| \\
&\quad \le \| \, g_{\bar{u}} \, | S^{r_0}_{p,p} B(\mathbb{R}^d) \| \sum_{\bar{\ell} \in W_{\bar{u}}^m} \| \, \Delta_{\ell_1} \otimes \ldots \otimes \Delta_{\ell_d} \, | S^{r_0}_{p,p} B(\mathbb{R}^d) \to L_p(\mathbb{R}^d) \| \\
&\quad \le \| \, g_{\bar{u}} \, | S^{r_0}_{p,p} B(\mathbb{R}^d) \| \sum_{\bar{\ell} \in W_{\bar{u}}^m} \prod_{i=1}^{d} \| \, \Delta_{\ell_i} \, | B^{r_0}_{p,p}(\mathbb{R}) \to L_p(\mathbb{R}) \|
\end{aligned}$$

where

$$W_{\bar{u}}^m := \{ \bar{\ell} \in \mathbb{N}_0^d : \quad |\bar{\ell}|_1 > m \,, \quad \ell_i \le u_i \,, \; i = 1, \ldots d \} \,.$$

By Corollary 2.3

$$\| \, \Delta^i_{\ell_i} \, | B^{r_0}_{p,p}(\mathbb{R}) \to L_p(\mathbb{R}) \| \; \lesssim 2^{-\ell_i r_0} \,.$$

Hence

$$\| \, g_{\bar{u}} - A(m, d, \vec{Q}) g_{\bar{u}} \, | L_p(\mathbb{R}^d) \| \lesssim \; \| \, g_{\bar{u}} \, | S^{r_0}_{p,p} B(\mathbb{R}^d) \| \sum_{\bar{\ell} \in W_{\bar{u}}^m} 2^{-r_0 |\bar{\ell}|_1} \,. \tag{41}$$

Using $(\bar{D}_{r_0})$ we obtain

$$\| g_{\bar{u}} | S^{\bar{r_0}}_{p,p} B(\mathbb{R}^d) \| \asymp 2^{|\bar{u}|_1 r_0} \| g_{\bar{u}} | L_p(\mathbb{R}^d) \| \,.$$

Plugging this into (41) and using (40) in combination with $(\bar{D}_r)$ we arrive at

$$\begin{aligned}
&\| f^b - A(m,d,\vec{Q}) f^b \,|L_p(\mathbb{R}^d)\| \\
&\lesssim \sum_{\bar{u}\in P_b^m} 2^{r_0|\bar{u}|_1} \,\| g_{\bar{u}} \,|L_p(\mathbb{R}^d)\| \, 2^{-r_0 m}\, |W_{\bar{u}}^m| \\
&\lesssim 2^{-r_0 m} \sum_{\bar{u}\in P_b^m} 2^{(r_0-r)|\bar{u}|_1} \,|W_{\bar{u}}^m|\, 2^{r|\bar{u}|_1} \,\| g_{\bar{u}} \,|L_p(\mathbb{R}^d)\| \\
&\lesssim \| f \,|S^r_{p,q}B(\mathbb{R}^d)\| \, 2^{-r_0 m} \Big(\sum_{\bar{u}\in P_b^m} 2^{(r_0-r)|\bar{u}|_1 q'} \,|W_{\bar{u}}^m|^{q'} \Big)^{1/q'}
\end{aligned}$$

where we used Hölder's inequality with $1/q + 1/q' = 1$ in the last step. Observe, that

$$W_{\bar{u}}^m \subset \Big[m - \sum_{\substack{i=1\\ i\neq 1}}^d u_i, u_1 \Big] \times \cdots \times \Big[m - \sum_{\substack{i=1\\ i\neq d}}^d u_i, u_d \Big] \,.$$

This implies

$$|W_{\bar{u}}^m| \le \min \Big((|\bar{u}|_1 - m)^d \ , \ \prod_{i=1}^d u_i \Big) \,.$$

Now we have to distinguish two cases depending on the size of $|b|_1$.

Step 1. Let $|b|_1 \le 1$. Using $|W_{\bar{u}}^m| \le (|\bar{u}|_1 - m)^d$ yields

$$2^{-r_0 m} \Big(\sum_{u\in P_b^m} 2^{(r_0-r)|\bar{u}|_1 q'} \,|W_{\bar{u}}^m|^{q'} \Big)^{1/q'} \le c\, 2^{-mr}\, m^{(d-1)(1-1/q)}$$

for some c independent of m.

Step 2. Let $|b|_1 > 1$. In this case we use $|W_{\bar{u}}^m| \le \prod_{i=1}^d u_i$ and obtain

$$\begin{aligned}
2^{-r_0 m} \Big(\sum_{\bar{u}\in P_b^m} 2^{(r_0-r)|\bar{u}|_1 q'} \,|W_{\bar{u}}^m|^{q'} \Big)^{1/q'} &\le c\, 2^{-mr}\, m^d\, 2^{m(r_0-r)|b|_1} \,, \\
&\le c 2^{-mr} \,,
\end{aligned}$$

where again c does not depend on m. Summarizing we have proved

$$\| f^b - A(m,d,\vec{Q}) f^b \,|L_p(\mathbb{R}^d)\| \lesssim 2^{-mr}\, m^{(d-1)(1-1/q)} \,\| f \,|S^r_{p,q}B(\mathbb{R}^d)\| \,.$$

Using triangle inequality we finally obtain

$$\| f - A(m,d,\vec{Q}) f \,|L_p(\mathbb{R}^d)\| \lesssim 2^{-mr}\, m^{(d-1)(1-1/q)} \,\| f \,|S^r_{p,q}B(\mathbb{R}^d)\| \,.$$

This completes the proof. □

Remark 3 : A periodic counterpart of (38) has been proved in [19]. A few ideas of the proof are taken from there.

4.3. *Estimates from below for interpolation on sparse grids*

We call a function $g : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous of order $k \in \mathbb{N}$ if $g \in C^{k-1}(\mathbb{R})$ and

$$\sup_{x \neq y} \frac{|g^{(k-1)}(x) - g^{(k-1)}(y)|}{|x-y|} < \infty \, .$$

Theorem 4.6: *Let $1 \le p \le \infty$, $s > 1$, and $\Lambda \in B^s_{1,1}(\mathbb{R})$ be Lipschitz continuous of order $k \in \mathbb{N}$. Furthermore, we assume that Λ is compactly supported, satisfies (I), (R), (C) and*

$$\Phi(x) = \Lambda(x/2) - \Lambda(x) \ge 0 \quad , \quad x \in \mathbb{R} \, . \tag{42}$$

Then, for all r with

$$\frac{1}{p} < r < \frac{s}{p} + k\Big(1 - \frac{1}{p}\Big) \, ,$$

we have for the corresponding operators $A(m,d,\vec{Q})$

$$\| I - A(m,d,\vec{Q}) : S^r_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d) \| \gtrsim 2^{-mr}\, m^{(d-1)(1-1/p)} \, , \qquad m \in \mathbb{N} \, . \tag{43}$$

Proof: For $m \in \mathbb{N}$ we introduce the following index sets

$$I_m := \Big\{ (\bar{u},\bar{k}) \in \mathbb{N}_0^d \times \mathbb{Z}^d : \quad |\bar{u}|_1 = m+1 \, , \min_{1 \le i \le d} u_i \ge 2 \, , 1 \le k_\ell \le 2^{u_\ell - 2} \, , \ell = 1, \ldots, d \Big\} \, .$$

To prove (43) we shall use the following sequence of test functions

$$f_m(x_1, ..., x_d) := \sum_{(\bar{u},\bar{k}) \in I_m} \Lambda(2^{u_1} x_1 - 2k_1) \cdot \, \ldots \, \cdot \Lambda(2^{u_d} x_d - 2k_d) \, .$$

Step 1. Estimate of $\| f_m - A(m,d,\vec{Q}) f_m \,| L_p(\mathbb{R}^d) \|$ from below. Clearly, every f_m belongs to the space $\mathcal{V}_{m+1}$, which implies, see Lemma 4.2, that

$$\begin{aligned} & f_m - A(m,d,\vec{Q}) f_m \\ & \quad = A(m+1,d,\vec{Q}) f_m - A(m,d,\vec{Q}) f_m \\ & \quad = \sum_{|\bar{j}|_1 = m+1} \sum_{(\bar{u},\bar{k}) \in I_m} \Delta_{j_1}[\Lambda(2^{u_1} \cdot -2k_1)](x_1) \cdot \, \ldots \, \cdot \Delta_{j_d}[\Lambda(2^{u_d} \cdot -2k_d)](x_d) \, . \end{aligned}$$

Using Lemma 2.1, we observe, that $\Delta_{j_i}[\Lambda(2^{u_i} \cdot -2k_i)](x_i) \neq 0$ implies that $u_i \ge j_i$. Taking $|\bar{u}|_1 = |\bar{j}|_1$ into account this leads to $u_i = j_i$ for all i. Therefore, using property (I), we conclude

$$\Delta_{j_i}[\Lambda(2^{j_i} \cdot -2k_i)](x_i) = -\Phi(2^{j_i} x_i - 2k_i) \le 0 \quad , \quad i = 1, ..., d.$$

This yields

$$f_m - A(m,d,\vec{Q}) f_m = (-1)^d \sum_{(\bar{j},\bar{k}) \in I_m} \Phi(2^{j_1} x_1 - 2k_1) \cdot \ldots \cdot \Phi(2^{j_d} x_d - 2k_d) \, .$$

Observe, if $\operatorname{supp}\Phi \subset [-N,N]$, then $\operatorname{supp} f_m \subset [-N-1,N+1]$ follows. Hence

$$\begin{aligned}
&\| f_m - A(m,d,\vec{Q})f_m \,|L_p(\mathbb{R}^d)\| \qquad (44)\\
&\quad = \Big\| \sum_{(\bar{j},\bar{k})\in I_m} \Phi(2^{j_1}x_1-2k_1)\cdot\ \ldots\ \cdot\Phi(2^{j_d}x_d-2k_d)\,\Big|L_p([-N-1,N+1]^d)\Big\|\\
&\quad \geq C_N\Big\| \sum_{(\bar{j},\bar{k})\in I_m} \Phi(2^{j_1}x_1-2k_1)\cdot\ \ldots\ \cdot\Phi(2^{j_d}x_d-2k_d)|L_1([-N-1,N+1]^d)\Big\|
\end{aligned}$$

Since we sum over nonnegative functions inside the L_1-norm we finally conclude

$$\begin{aligned}
&\|f_m - A(m,d,\vec{Q})f_m|L_p(\mathbb{R}^d)\|\\
&\quad \geq \sum_{(\bar{j},\bar{k})\in I_m} \|\Phi(2^{j_1}\cdot-2k_1)|L_1(\mathbb{R})\|\cdot\ldots\cdot\|\Phi(2^{j_d}\cdot-2k_d)|L_1(\mathbb{R})\|\\
&\quad = \sum_{(\bar{j},\bar{k})\in I_m} 2^{-|\bar{j}|_1}\,\|\,\Phi\,|L_1(\mathbb{R})\|^d \qquad (45)\\
&\quad \gtrsim m^{d-1}\,.
\end{aligned}$$

Step 2. Estimate of $\|\, f_m\,|S^r_{p,p}(\mathbb{R}^d)\|$ from above.

Substep 2.1. Let $p=1$. Since $\Lambda\in B^s_{1,1}(\mathbb{R})$, Theorem A.6, see the Appendix below, yields $\Lambda(2^{u_1}x_1-2k_1)\cdot\ \ldots\ \cdot\Lambda(2^{u_d}x_d-2k_d)\in S^s_{1,1}B(\mathbb{R}^d)$. Applying triangle inequality and a homogeneity argument, see e.g. [32, Prop. 3.4.1], we obtain

$$\begin{aligned}
\|\, f_m\,|S^s_{1,1}B(\mathbb{R}^d)\| &\leq \sum_{(\bar{j},\bar{k})\in I_m} \|\,\Phi(2^{j_1}x_1-2k_1)\cdot\ldots\cdot\Phi(2^{j_d}x_d-2k_d)\,|S^s_{1,1}B(\mathbb{R}^d)\|\\
&\lesssim \sum_{(\bar{j},\bar{k})\in I_m} \|\,\Phi(2^{j_1}\ \cdot\ -2k_1)\,|B^s_{1,1}(\mathbb{R})\|\cdot\ \ldots\ \cdot\|\,\Phi(2^{j_d}\ \cdot\ -2k_d)\,|B^s_{1,1}(\mathbb{R})\|\\
&\lesssim \sum_{(\bar{j},\bar{k})\in I_m} 2^{|\bar{j}|_1(s-1)}\,\|\,\Phi\,|B^s_{1,1}(\mathbb{R})\|^d\\
&\lesssim m^{d-1}\,2^{ms}\,. \qquad (46)
\end{aligned}$$

The above estimates can be repeated for any r, $0<r<s$, since $B^s_{1,1}(\mathbb{R})\hookrightarrow B^r_{1,1}(\mathbb{R})$.

Substep 2.2. Let $p=\infty$. Since Λ is Lipschitz continuous of order $k\in\mathbb{N}$ and compactly supported the functions $\Phi(2^{j_1}x_1-2k_1)\cdot\ldots\cdot\Phi(2^{j_d}x_d-2k_d)$ are k-atoms centered at $Q_{\bar{j},2\bar{k}}$ (up to a universal constant), see Def. A.11 and Remark *A5* below. Hence, if $0<t<k$, Theorem A.12 yields

$$\|\, f_m\,|S^t_{\infty,\infty}B(\mathbb{R}^d)\| \lesssim \sup_{|\bar{u}|_1=m+1} 2^{|\bar{u}|_1 t} \lesssim 2^{tm}\,. \qquad (47)$$

Substep 2.3. Let $1<p<\infty$. We shall use complex interpolation, see [45, Thm. 4.6]. Together with (46), (47) it follows

$$\begin{aligned}
\|\, f_m\,|S^r_{p,p}B(\mathbb{R}^d)\| &\lesssim \|\, f_m\,|S^s_{1,1}B(\mathbb{R}^d)\|^{1-\theta}\,\|\, f_m\,|S^t_{\infty,\infty}B(\mathbb{R}^d)\|^{\theta}\\
&\lesssim (m^{d-1}\,2^{ms})^{1-\theta}\,(2^{tm})^{\theta}\\
&\lesssim m^{(d-1)/p}\,2^{mr}\,, \qquad (48)
\end{aligned}$$

where $0 < \theta < 1$,

$$\frac{1}{p} = 1 - \theta \qquad \text{and} \qquad r := s\,(1-\theta) + t\,\theta\,.$$

Combining (45) with (48) the claim follows. □

4.4. *Estimates from above for sampling on sparse grids*

This time our analysis is based on the following decomposition of the error $I - A(m,d,\vec{L})$.

Lemma 4.7: *The following identity holds for all $d \geq 2$ and all $m \in \mathbb{N}$:*

$$\begin{aligned} I \otimes \cdots \otimes I - A(m,d,\vec{L}) \\ &= \sum_{j_1+\cdots+j_d=m} \Delta_{j_1}(L^1) \otimes \Delta_{j_2}(L^2) \otimes \cdots \otimes \Delta_{j_{d-1}}(L^{d-1}) \otimes (I - L^d_{j_d}) \\ &+ \sum_{j_1+\cdots+j_{d-1}=m} \Delta_{j_1}(L^1) \otimes \Delta_{j_2}(L^2) \otimes \cdots \otimes \Delta_{j_{d-2}}(L^{d-2}) \otimes (I - L^{d-1}_{j_{d-1}}) \otimes I \\ &\ \vdots \\ &+ \sum_{j_1+j_2=m} \Delta_{j_1}(L^1) \otimes (I - L^2_{j_2}) \otimes I \otimes \cdots \otimes I \\ &+ (I - L^1_m) \otimes I \otimes \cdots \otimes I\,. \end{aligned}$$

Proof: The formula follows by induction with respect to d and with fixed m. Obviously, the following recursion

$$\begin{aligned} I \otimes \cdots \otimes I - A(m,d,\vec{L}) &= (I - L^1_m) \otimes (I \otimes \cdots \otimes I) + L^1_m \otimes I \otimes \cdots \otimes I \\ &\quad - \sum_{j_1=0}^{m} \Delta_{j_1}(L^1) \otimes A(m-j_1, d-1, \vec{L} \setminus L^1) \end{aligned}$$

holds. Using $\sum_{j=0}^m \Delta_j(L^1) = L^1_m$ we get

$$\begin{aligned} I \otimes \cdots \otimes I - A(m,d,\vec{L}) &= (I - L^1_m) \otimes (I \otimes \cdots \otimes I) \\ &\quad + \sum_{j_1=0}^{m} \Delta_{j_1}(L^1) \otimes \big(\underbrace{I \otimes \cdots \otimes I}_{d-1} - A(m-j_1, d-1, \vec{L} \setminus L^1)\big)\,. \end{aligned}$$

Applying the induction hypothesis the claim follows. □

Remark 4: Similar ideas have been used by Wasilkowski and Woźniakowski [43]. For $d = 2$ such identities can be found also in Delvos and Schempp [46, Prop. 1.4/2].

We are going to use these identities with respect to the sequence $\vec{Q}$.

Theorem 4.8: *Let $1 \leq p \leq \infty$ and $r > 0$. Let $\vec{Q} = ((Q_j)_j, \ldots, (Q_j)_j)$ be given. We suppose*

$$\| \, I - Q_j \, | B^r_{p,p}(\mathbb{R}) \to L_p(\mathbb{R}) \| \lesssim 2^{-jr}\,, \qquad j \in \mathbb{N}_0\,.$$

Then

$$\| I - A(m,d,\vec{Q}) \,| S^r_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \lesssim 2^{-mr}\, m^{d-1}\,, \qquad m \in \mathbb{N}\,,$$

follows.

Proof: *Step 1.* Let $1 \le p < \infty$. It is enough to combine Corollary A.7 with Lemma 4.7.

Step 2. Let $p = \infty$. By means of Lemma 4.7 we conclude

$$\begin{aligned}\|I - A(m,d,\vec{Q})\| \le & \sum_{j_1+\cdots+j_d=m} \|\Delta_{j_1}\otimes\Delta_{j_2}\otimes\cdots\otimes\Delta_{j_{d-1}}\otimes(I-Q_{j_d})\| \\ & + \sum_{j_1+\cdots+j_{d-1}=m} \|\Delta_{j_1}\otimes\Delta_{j_2}\otimes\cdots\otimes\Delta_{j_{d-2}}\otimes(I-Q_{j_{d-1}})\otimes I\| \\ & \vdots \\ & + \sum_{j_1+j_2=m} \|\Delta_{j_1}\otimes(I-Q_{j_2})\otimes I\otimes\cdots\otimes I\| \\ & + \|(I-Q_m)\otimes I\otimes\cdots\otimes I\|\,. \end{aligned} \tag{49}$$

Let $f \in S^r_{\infty,\infty}B(\mathbb{R}^d)$. We proceed estimating $\|(P_1\otimes\cdots\otimes P_d)f|L_\infty(\mathbb{R}^d\|$, where the operators P_i are chosen accordingly to the summands in (49) . Furthermore, let $\Delta^n_{h,i}$ denote the finite difference operator of order n in direction of the i-th coordinate, see (B1) in the appendix. We find, by taking into account the characterization of $B^r_{\infty,\infty}(\mathbb{R})$ in terms of differences,

$$\begin{aligned}&\|(P_1\otimes\cdots\otimes P_d)f|L_\infty(\mathbb{R}^d\| = \sup_{x_2,\ldots,x_d}\sup_{x_1} |P_1(I\otimes P_2\otimes ...\otimes P_d)f(\cdot,x_2,...,x_d)(x_1)| \\ &\lesssim \|P_1 : B^r_{\infty,\infty}(\mathbb{R}) \to L_\infty(\mathbb{R})\| \sup_{x_2,\ldots,x_d} \|(I\otimes P_2\otimes\cdots\otimes P_d)f(\cdot,x_2,...,x_d)|B^r_{\infty,\infty}(\mathbb{R})\| \\ &\asymp \|P_1\| \Big(\sup_{x_1,\ldots,x_d} |(I\otimes P_2\otimes\cdots\otimes P_d)f(x_1,x_2,...,x_d)| \\ &\qquad + \sup_{t_1>0} {t_1}^{-r} \sup_{|h_1|\le t_1}\sup_{x_1,\ldots,x_d} |(\Delta^n_{h_1}\otimes P_2\otimes\cdots\otimes P_d)f(x_1,...,x_d)|\Big)\,. \end{aligned}$$

Iterating this procedure yields

$$\begin{aligned}&\|(P_1\otimes\cdots\otimes P_d)f|L_\infty(\mathbb{R}^d)\| \\ &\lesssim \|P_1\|\cdot ...\cdot\|P_d\| \Big(\|f|L_\infty(\mathbb{R}^d)\| + \sum_{\substack{A\subset\{1,\ldots,d\}\\ A\neq\emptyset}} \sup_{\substack{t_i>0\\ i=1,\ldots,n}} t_1^{-r}\cdots t_n^{-r} \|(D_1\otimes ...\otimes D_d)f|L_\infty\|\Big) \\ &\asymp \|P_1\|\cdot ...\cdot\|P_d\|\cdot\|f|S^r_{\infty,\infty}B(\mathbb{R}^d)\|\,,\end{aligned}$$

where

$$D_i := \begin{cases} \Delta^n_{h,i} & \text{if} \quad i\in A \\ I & \text{if} \quad i \notin A \end{cases}\,.$$

For the corresponding characterization of $S^r_{\infty,\infty}B(\mathbb{R}^d)$ in terms of differences we refer to [47] and Appendix B. Therefore,

$$\| P_1 \otimes \cdots \otimes P_d \,|S^r_{\infty,\infty}B(\mathbb{R}^d) \to L_\infty(\mathbb{R}^d)\| \lesssim \prod_{i=1}^{d} \| P_i \,|B^r_{\infty,\infty}(\mathbb{R}) \to L_\infty(\mathbb{R})\| \,.,$$

analogous to the situation with $p < \infty$. Returning to (49) yields

$$\begin{aligned} \| I - A(m,d,\vec{Q}) \,|S^r_{\infty,\infty}B(\mathbb{R}^d) \to L_\infty(\mathbb{R}^d)\| &\lesssim (1 + m + \cdots + m^{d-1})\, 2^{-mr} \\ &\lesssim m^{d-1}\, 2^{-mr} \,. \end{aligned}$$

The proof is complete. □

Remark 5 : We compare advantages and disadvantages of the Theorems 4.5 and 4.8. Of course, we have to concentrate to $p = q$. Under more restrictive conditions on Λ we have found in Theorem 4.5 an estimate with error $\lesssim 2^{-mr}\, m^{(d-1)(1-1/p)}$, whereas in Theorem 4.8 we produced an error $\lesssim 2^{-mr}\, m^{d-1}$. Hence, there is a difference of order $m^{(d-1)/p}$ in case $1 \le p < \infty$.

As a supplement to Theorem 4.8 we formulate the counterpart for Sobolev spaces $H^r_p(\mathbb{R})$ of fractional order.

Theorem 4.9 : *Let $1 < p < \infty$ and $r > 0$. Let $\vec{Q} = ((Q_j)_j, \ldots , (Q_j)_j)$ be given. We suppose*

$$\| I - Q_j \,|H^r_p(\mathbb{R}) \to L_p(\mathbb{R})\| \lesssim 2^{-jr} \,, \qquad j \in \mathbb{N}_0 \,.$$

Then

$$\| I - A(m,d,\vec{Q}) \,|S^r_pW(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \lesssim 2^{-mr}\, m^{d-1} \,, \qquad m \in \mathbb{N} \,,$$

follows.

Proof : Again it will be enough to combine Corollary A.7 with Lemma 4.7. □

5. Examples II.

We summarize the outcome with respect to the examples already treated in the univariate case, see Section 3.

5.1. *Spline interpolation*

Proposition 5.1 : *Let $n \in \mathbb{N}$ and $1 \le p < \infty$. Then Λ_{2n} provides $(\bar{D}_r(p,p))$ if $1/p < r < 2n - 1 + 1/p$.*

Proof : We simply refer to Lemma 4.4 and Proposition 5.1. □

Let us define $\vec{I}_n := ((I^n_{2^j})_j, \ldots , (I^n_{2^j})_j)$, see (29). Then the following assertions are consequences of Theorems 4.5 and 4.8.

Corollary 5.2: *Let $n \in \mathbb{N}$ and $1 \le p \le \infty$.*
(i) *Let $1/p < r < 2n$. Then we have*

$$\| I - A(m,d,\vec{I}_n) \,|S^r_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \lesssim 2^{-mr}\, m^{(d-1)(1-1/p)}\,, \quad m \in \mathbb{N}_0\,. \qquad (50)$$

(ii) *In the limiting situation $r = 2n$ we have*

$$\| I - A(m,d,\vec{I}_n) \,|S^{2n}_{2,2}B(\mathbb{R}^d) \to L_2(\mathbb{R}^d)\| \lesssim 2^{-mr}\, m^{(d-1)}\,, \quad m \in \mathbb{N}_0\,.$$

Proof: *Step 1.* Part (i) with $1 \le p < \infty$ and $1/p < r < 2n - 1 + 1/p$ follows from Theorem 4.5 and Proposition 5.1. Theorem 4.8 and Corollary 3.2/(i) yield

$$\| I - A(m,d,\vec{I}_n) \,|S^r_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \lesssim 2^{-mr}\, m^{(d-1)}\,, \quad m \in \mathbb{N}_0\,,$$

if $1 \le p \le \infty$ and $1/p < r < 2n$. For $1 < p < \infty$ we interpolate with respect to the pair $(S^r_{1,1}B(\mathbb{R}^d), S^r_{\infty,\infty}B(\mathbb{R}^d))$. Since

$$S^r_{p,p}B(\mathbb{R}^d) = [S^r_{1,1}B(\mathbb{R}^d), S^r_{\infty,\infty}B(\mathbb{R}^d)]_\theta \qquad \text{and} \qquad L_p(\mathbb{R}^d) = [L_1(\mathbb{R}^d), L_\infty(\mathbb{R}^d)]_\theta\,,$$

$\theta = 1 - 1/p$, see [45], the estimate (50) follows from the interpolation property of the complex method.

Step 2. Proof of (ii). We employ Theorem 4.8 in connection with Corollary 3.2. □

Remark 1:

(i) Corollary 5.2 is an improvement of earlier results on spline interpolation on sparse grids obtained by Sprengel and one of the authors, see [17].
(ii) In his recent book [48] Triebel proved the validity of $\bar{D}_r(p,q)$ in case $1 \le p,q < \infty$ and $1/p < r < 2n - 1 + 1/p$, but restricted to $d = 2$. Hence, there is some hope to extend the Corollary 5.2(i) also to values of q different from p.

In the particular case $n = 1$ Corollary 5.2 can be supplemented in two directions, namely by some more limiting cases and by an estimate from below. Obviously, $\Lambda_2 = \mathcal{N}_2(\cdot + 1)$ satisfies (42).

Corollary 5.3: (i) *Let $1 \le p \le \infty$ and $1/p < r < 2$. Then we have*

$$\| I - A(m,d,\vec{I}_1) \,|S^r_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \asymp 2^{-mr}\, m^{(d-1)(1-1/p)}\,, \quad m \in \mathbb{N}_0\,.$$

(ii) *Let $1 < p < \infty$ and $r = 2$. Then we have*

$$\| I - A(m,d,\vec{I}_1) \,|S^2_pH(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \lesssim 2^{-2m}\, m^{d-1}\,, \quad m \in \mathbb{N}_0\,.$$

Proof: *Step 1.* Proof of (i). The estimate from above is contained in (50).

Step 1.1. Let $1/p < r < 1 + 1/p$. For the estimate from below we shall apply Theorem 4.6. Since $\mathcal{N}_2$ is Lipschitz continuous we may choose $k = 1$. The parameter s can be chosen to be any positive number strictly less than 2. This leads to the restrictions

$$\frac{1}{p} < r < \frac{2}{p} + 1 - \frac{1}{p} = 1 + \frac{1}{p}$$

for r.

Step 1.2. Now let $1+1/p \le r < 2$. We argue by contradiction. We assume

$$\lim_{m\to\infty} 2^{mr}\, m^{-(d-1)(1-1/p)}\, \| \, I - A(m,d,\vec{I}_1)\, |S^r_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| = 0\,.$$

More exactly, we had to argue with a subsequence of $\mathbb{N}$ in this formula. But the arguments would be the same. So we omit this keeping notations simpler in that way. Now we interpolate with respect to the pair $(S^r_{p,p}B(\mathbb{R}^d), S^{r_0}_{p,p}B(\mathbb{R}^d))$, where $1/p < r_0 < 1+1/p$. Let $r_1 := (1-\theta)\, r + \theta\, r_0$. For $\theta \downarrow 0$ and $r_0 \downarrow 1/p$ we can always guarantee $1/p < r_1 < 1+1/p$. Complex interpolation, see [45], yields

$$\begin{aligned} &\| \, I - A(m,d,\vec{I}_1)\, |S^{r_1}_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \\ &\lesssim \|I - A(m,d,\vec{I}_1)|S^r_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\|^{1-\theta} \|I - A(m,d,\vec{I}_1)|S^{r_0}_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\|^{\theta}. \end{aligned}$$

Hence

$$\lim_{m\to\infty} 2^{mr_1}\, m^{-(d-1)(1-1/p)}\, \| \, I - A(m,d,\vec{I}_1)\, |S^{r_1}_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| = 0\,.$$

Because of $1/p < r_1 < 1+1/p$ this is in contradiction with Step 1.1.

Step 2. Proof of (ii). We have to combine Corollary 3.2 with Theorem 4.9. □

Remark 2 : A final remark to sampling and interpolation on $\mathbb{R}^d$. In view of the examples treated in (30) and (31) in would be interesting to clarify what is the correct order of the error of the associated Smolyak algorithm's. Of course, Theorem 4.8 applies and Theorem 4.5 does not.

5.2. *Sinus Cardinalis*

Using the Fourier-analytic definition (see Appendix A.1) of the spaces $B^s_{p,q}(\mathbb{R})$ and $S^r_{p,q}B(\mathbb{R}^d)$, combined with Remark *A1*, we obtain $(\bar{D}_r(p,q))$ for $r > 1/p$, $1 \le q \le \infty$ and $1 < p < \infty$.

Corollary 5.4: *Let $1 < p < \infty$, $1 \le q < \infty$ and $1/p < r < \infty$. Then we have*

$$\| \, I - A(m,d,\mathrm{sinc}\,)\, |S^r_{p,q}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \asymp 2^{-mr}\, m^{(d-1)(1-1/q)}\,, \quad m \in \mathbb{N}_0\,. \quad (51)$$

Proof: Theorem 4.5 yields the estimate from above. in case of the sinc-function. In order to avoid the technical difficulties with an application of Lemma 4.2 one may temporarily work with functions from $C_0^\infty(\mathbb{R}^d)$. This gives (51), just by a density argument. For the lower bound we first observe that (42) holds, since

$$\mathrm{sinc}\,(t) - \mathrm{sinc}\,(2t) = \mathrm{sinc}\,(t)(1-\cos(\pi t)) \ge 0\,, \qquad t \in \mathbb{R}\,.$$

However, sinc does not have compact support. Checking the proof of Theorem 4.6 we have to modify the arguments in Step 2 and the estimate (44). Concerning the latter one observes

$$\begin{aligned}
&\| f_m - A(m,d,\vec{Q}) f_m \,|L_p(\mathbb{R}^d)\| \\
&\quad = \Big\| \sum_{(\bar{j},\bar{k})\in I_m} \Phi(2^{j_1}x_1 - 2k_1)\cdot \,\ldots\, \cdot \Phi(2^{j_d}x_d - 2k_d) \,\Big| L_p([-1,1]^d)\Big\| \\
&\quad \ge 2^{-d(1-1/p)} \Big\| \sum_{(\bar{j},\bar{k})\in I_m} \Phi(2^{j_1}x_1 - 2k_1)\cdot \,\ldots\, \cdot \Phi(2^{j_d}x_d - 2k_d) \,|L_1([-1,1]^d)\Big\| \\
&\quad \gtrsim \sum_{(\bar{j},\bar{k})\in I_m} 2^{-|\bar{j}|_1} \int_{-2^{j_1}-2k_1}^{2^{j_1}-2k_1} |\Phi(y)|\,dy \cdot \ldots \cdot \int_{-2^{j_d}-2k_d}^{2^{j_d}-2k_d} |\Phi(y)|\,dy \\
&\quad \gtrsim m^{d-1}\,.
\end{aligned}$$

In the last step we used

$$\inf\Big\{ \int_{-2^j-2k}^{2^j-2k} |\Phi(y)|\,dy : \quad j \in \mathbb{N}_0\,,\; 1 \le k \le 2^{j-2}\Big\} > 0\,.$$

It remains to estimate $\| f_m \,|S^r_{p,q}B(\mathbb{R}^d)\|$. This is an obvious consequence of the Lizorkin representation, see Remark *A1* in the appendix below. We obtain

$$\|f_m|S^r_{p,q}B(\mathbb{R}^d)\| \lesssim 2^{rm} m^{(d-1)/q}\,,$$

which completes the proof. □

6. Interpolation on the cube

This section is devoted to the approximation of functions on the cube $[0,1]^d$ by using only function values. For the definition of the used function spaces we refer to Appendix B.

6.1. *A general result on interpolation on the cube*

To begin with we make use of the obvious fact that the restriction of a spline to the cube is again a spline. By $\overline{\mathcal{V}}_m(\Lambda,(0,1)^d)$ we denote the restriction of $\overline{\mathcal{V}}_m(\Lambda)$ to the cube $(0,1)^d$, see (33).

Proposition 6.1: *Let $n \in \mathbb{N}$. Let $1 \le p \le \infty$ and $1/p < r < 2n$. For all $f \in S^r_{p,p}B((0,1)^d)$ there exists a spline $g \in \overline{\mathcal{V}}_m(\mathcal{N}_{2n},(0,1)^d)$ such that*

$$f(x) = g(x) \qquad \text{for all} \quad x \in \mathcal{G}(m,d)^{\square}$$

and

$$\| f - g \,|L_p(\mathbb{R}^d)\| \lesssim 2^{-mr}\, m^{(d-1)(1-1/p)}\, \| f \,|S^r_{p,p}B((0,1)^d)\|\,, \quad m \in \mathbb{N}_0\,.$$

Proof: The assertion is an immediate consequence of Corollary 5.2 and Definition B.1 . We associate to $f \in S^r_{p,p}B((0,1)^d)$ an appropriate extension $\mathcal{E}f \in S^r_{p,p}B(\mathbb{R}^d)$ and define $g := A(m,d,\vec{I}_n)\mathcal{E}f$. □

Remark 1 : Of course, there are also counterparts of Proposition 6.1 in the limiting situations. We omit details and refer to Corollary 5.2.

In general, $A(m, d, \vec{I_n})\mathcal{E}f$ uses an infinite number of values of $\mathcal{E}f$. This is not appropriate. For this reason we need to slightly change our algorithm.

6.2. *The Smolyak algorithm on the cube*

We concentrate on interpolation on the grid $\mathcal{G}(m,d)^{\square}$ by splines belonging to $\overline{\mathcal{V}}_m(\mathcal{N}_2, (0,1)^d)$. We define

$$\Gamma_j f(t) := \sum_{k=0}^{2^j} f(k2^{-j})\, \Lambda_2(2^j t - k)\,, \qquad t \in \mathbb{R}\,, \quad j \in \mathbb{N}_0\,,$$

for $f \in C([0,1])$. The corresponding Smolyak algorithm will be denoted by $A(m,d,\vec{\Gamma})^{\square}$. Compared with $A(m,d,\vec{I_1})$, defined in Paragraph 5.1, we obtain immediately

$$A(m,d,\vec{\Gamma})^{\square} f(x) = A(m,d,\vec{I_1}) f(x)\,, \qquad x \in (0,1)^d\,,$$

for all $f \in C(\mathbb{R})$ s.t. $\operatorname{supp} f \subset [0,1]^d$. However, more important for us is the following identity

$$A(m,d,\vec{\Gamma})^{\square} f(x) = A(m,d,\vec{I_1})\, \mathcal{E}f(x)\,, \qquad x \in [0,1]^d\,, \tag{52}$$

where $\mathcal{E}f$ is any continuous extension of f. Our main result is the counterpart on the cube of Corollary 5.3.

Theorem 6.2 : (i) *Let $1 \le p \le \infty$ and $1/p < r < 2$. Then we have*

$$\| I - A(m,d,\vec{\Gamma})^{\square} \,|S^r_{p,p}B((0,1)^d) \to L_p((0,1)^d)\| \asymp 2^{-mr}\, m^{(d-1)(1-1/p)}\,, \quad m \in \mathbb{N}_0\,.$$

(ii) *For $1 < p < \infty$ it holds*

$$\| I - A(m,d,\vec{\Gamma})^{\square} \,|S^2_p H((0,1)^d) \to L_p((0,1)^d)\| \lesssim 2^{-2m}\, m^{(d-1)}\,, \quad m \in \mathbb{N}_0\,. \tag{53}$$

Proof : The estimates from above are consequences of the identity (52) and Proposition 6.1. The estimates from below follow as in proof of Corollary 5.3 and Theorem 4.6, respectively. □

Remark 2 :

(i) Bungartz and Griebel [6] have studied approximation from sparse grids as in Theorem 6.2. In particular, they proved

$$\| I - A(m,d,\vec{\Gamma})^{\square} \,|H^2_{mix}((0,1)^d) \to L_2((0,1)^d)\| \lesssim 2^{-2m}\, m^{(d-1)}\,, \quad m \in \mathbb{N}_0\,. \tag{54}$$

In view of $H^2_{mix}((0,1)^d) = S^2_{2,2}B((0,1)^d)$ (in the sense of equivalent norms, see Proposition B.4), this coincides with (53). Let us mention that Bungartz and Griebel also investigated two further interesting topics in this area, namely, the error in the L_∞-norm (for functions taken from $H^2_{mix}((0,1)^d)$) as well as the dependence of the constants, occurring in (54), on the dimension d.

(ii) It is an open question whether the exponent of m in (53) is optimal.

(iii) In Theorem 6.2 the smoothness of the source space is limited to $1/p < r < 2$. In order to treat the case $r \geq 2$ one could use smoother compactly supported interpolating scaling functions $\Lambda(t)$ instead of the second order cardinal B-spline $\mathcal{N}(t)$. One possibility is the Dubuc/Deslauries wavelet system, see for instance [29, Sect. 2.10]. Another promising approach could be based on Kamont [49]. The paper provides general discrete characterizations of Besov spaces in terms of sampled function values.

6.3. *Sampling and approximation numbers*

We recall two classical concepts of optimality, namely approximation numbers and sampling numbers. Whereas the first concept deals with optimal approximation by linear operators of a fixed rank, the second one deals with optimal approximation by linear operators using a fixed number of function values.

6.3.1. Approximation numbers

Let X and Y be Banach spaces s.t. $Y \hookrightarrow X$. Then we define the n-th approximation number of the embedding operator I as

$$a_n(I, Y, X) := \inf \Big\{ \| I - L_n \,|\, Y \to X \| : \quad \operatorname{rank} L_n < n \Big\} .$$

Proposition 6.3: *Let $r > 0$ and $1 < p < \infty$. Then for $m \in \mathbb{N}$ the relation*

$$a_m(I, S^r_{p,p}B((0,1)^d), L_p((0,1)^d)) \asymp m^{-r} (\log m)^{(d-1)r} \begin{cases} (\log m)^{(d-1)(\frac{1}{2}-\frac{1}{p})} & \text{if} \quad 2 \leq p < \infty , \\ 1 & \text{if} \quad 1 < p \leq 2 \end{cases}$$

holds true.

Remark 3:

(i) A proof of Proposition 6.3 will be provided in [50]. The in order optimal linear operators are hyperbolic cross type partial sums of some Fourier wavelet expansion. Hence, these operators do not use function values, but Fourier coefficients.

(ii) The approximation numbers of the embedding $S^r_{1,1}B((0,1)^d) \hookrightarrow L_1((0,1)^d)$ seem to be unknown. So far, we have an upper estimate if $1 < r < 2$ (see Thm. 6.2, Cor. 6.4). In the periodic case the corresponding upper estimate is known for all $r > 0$, i.e.,

$$a_m(I, S^r_{1,1}B(\mathbb{T}^d), L_1(\mathbb{T}^d)) \lesssim m^{-r} (\log m)^{(d-1)r} \quad , \quad m \in \mathbb{N} . \tag{55}$$

See for instance [19, Thm. 4,5] and [51] for the proof. However, the results of Romanyuk [15, 16] indicate that the right-hand side in (55) is probably the correct behavior.

(iii) Let $1 \leq p_0, p_1 \leq \infty$. Approximation numbers for embeddings $I : S^r_{p_0,q}B(\mathbb{T}^n) \to L_{p_1}(\mathbb{T}^n)$ or $I : S^r_{p_0}H(T^n) \to L_{p_1}(T^n)$ have a certain history, in particular in the Russian literature. We refer to [3], [4], [52] and [13–16] and the references given there. Of course, we expect that the asymptotic behavior of these quantities for the periodic context coincide with those for the non-periodic context.

6.3.2. Sampling numbers

We follow [4, 4.5], but see also [53], [54], [22] and [19]. For fixed $M \in \mathbb{N}$ we denote by

$$\Psi_M(f,\xi)(x) := \sum_{j=1}^{M} f(\xi^j)\, \psi_j(x)$$

a general sampling operator for a class F of continuous functions defined on $(0,1)^d$, where

$$\xi := \left\{\xi^1, ..., \xi^M\right\}, \qquad \xi^i \in (0,1)^d\,, \quad i = 1,2,\ldots,M\,,$$

is a fixed set of sampling points and $\psi_j : (0,1)^d \to \mathbb{C}$, $j = 1,\ldots,M$, are fixed continuous functions. Then the quantity

$$\rho_M(F, L_p((0,1)^d)) := \inf_{\xi} \inf_{\psi_1,\ldots,\psi_M} \sup_{\|f|F\|\le 1} \| f - \Psi_M(f,\xi)f \,| L_p((0,1)^d)\|$$

measures the optimal rate of approximate recovery of the functions taken from F. In [53], [54] and [22] these quantities are called sampling numbers of the embeddings $I : F \to L_p((0,1)^d)$.

The sequence of operators $A(m,d,\vec{\Gamma})^{\square}$ yields an upper bound for the asymptotic decay of the sampling numbers $\rho_M(S^r_{p,p}B((0,1)^d), L_p((0,1)^d))$. Indeed, the operator $A(m,d,\vec{\Gamma})^{\square}$ takes only samples from the sparse grid

$$\mathcal{G}(m,d)^{\square} = \left\{(2^{-j_1}k_1, \ldots, 2^{-j_d}k_d) : \ |\bar{j}|_1 = m\ , \ 0 \le k_i \le 2^{j_i}\ , \ i = 1,...,d\right\}.$$

The cardinality of $\mathcal{G}(m,d)^{\square}$ is easily checked. One has

$$M := |\mathcal{G}(m,d)^{\square}| \asymp 2^m\, m^{d-1}\,,$$

see e.g. [19]. Here $'' \asymp ''$ has to be interpreted in the sense that the constants behind it do not depend on m. Furthermore, we shall use the estimate

$$2^{-rm}\, m^{(d-1)(1-1/p)} \lesssim M^{-r}(\log\, M)^{(d-1)(r+1-1/p)}$$

in order to see the upper bound in the following corollary.

Corollary 6.4: *Let $1 \le p \le \infty$ and $1/p < r < 2$. Then the following relation*

$$\rho_M(S^r_{p,p}B((0,1)^d), L_p((0,1)^d)) \lesssim M^{-r}(\log M)^{(d-1)(r+1-1/p)}\,, \qquad M \in \mathbb{N}\,, \quad (56)$$

holds true.

Remark 4:

(i) Comparing approximation and sampling numbers we get

$$1 \quad \le \quad \frac{\rho_M(S^r_{p,p}B((0,1)^d), L_p((0,1)^d))}{a_M(S^r_{p,p}B((0,1)^d), L_p((0,1)^d))}$$

$$\lesssim \quad \begin{cases} (\log M)^{(d-1)/2} & \text{if} \quad 2 \le p < \infty\,, \\ (\log M)^{(d-1)(1-1/p)} & \text{if} \quad 1 < p \le 2\,, \end{cases} \quad , \qquad M \in \mathbb{N}\,.$$

For $1 \le p \le \infty$ the correct asymptotic behavior of the sampling numbers ρ_M seems to be an open problem. According to Remark **3**/(ii) there is some reason to expect $\rho_M \asymp a_M$ if $p = 1$ and $1 < r < 2$.

(ii) The Smolyak algorithm uses samples of a very specific structure, see (1). Corollary 6.4 and Proposition 6.3 tell us that we can not do much better allowing arbitrary sets of sampling points since $a_M \le \rho_M$. The difference is at most $(\log M)^{(d-1)/2}$.

(iii) The result in Corollary 6.4 is not very surprising since it is known in the periodic setting; see [19, 20] as well as [4, 10, 55] .

(iv) In his new book [48, Thm. 4.15] Triebel established similar results for sampling numbers in case $d = 2$ and $1/p < r < 1 + 1/p$. For the spaces $S^r_{p,p}B((0,1)^2)$ he obtained the same upper bounds as given here and he was even able to shorten the gap between lower bound (approximation numbers) and upper bound in a particular situation. Namely, if $2 < p < \infty$ and $1/p < r < 1/2$ then

$$\rho_M(S^r_{p,p}\mathcal{B}((0,1)^2), L_p((0,1)^2)) \gtrsim M^{-r}(\log M)^{1-1/p}$$

gives a slightly better lower bound than Corollary 6.4 . His treatment is based on a characterization of $S^r_{p,p}B((0,1)^2)$ by a tensor product Faber system. Let us mention that his method allows to treat also sampling numbers for more general pairs like $\rho_M(S^r_{p_0,q_0}B((0,1)^2), L_{p_1}((0,1)^2))$.

Appendix A. Function spaces on $\mathbb{R}^d$

A.1. *Besov type spaces on* $\mathbb{R}$ *and* $\mathbb{R}^d$

Here we recall the definition and a few properties of Besov and Sobolev spaces defined on $\mathbb{R}$. We shall use the Fourier analytic approach. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\varphi(t) = 1$ in an open set containing the origin. Then by means of

$$\varphi_0(t) = \varphi(t)\,, \qquad \varphi_j(t) = \varphi(2^{-j}t) - \varphi(2^{-j+1}t)\,, \qquad t \in \mathbb{R}\,, \quad j \in \mathbb{N}\,,$$

we get a smooth dyadic decomposition of unity. First we deal with Besov spaces.

Definition A.1: Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. The Besov space $B^s_{p,q}(\mathbb{R})$ is then the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|\, f \,|B^s_{p,q}(\mathbb{R})\| := \Big(\sum_{j=0}^\infty 2^{jsq}\, \|\, \mathcal{F}^{-1}[\varphi_j \mathcal{F} f](\,\cdot\,)\, |L_p(\mathbb{R})\|^q \Big)^{1/q}$$

is finite (modification if $q = \infty$).

If φ_j, $j \in \mathbb{N}_0$, is a smooth dyadic decomposition on $\mathbb{R}$, then by means of

$$\varphi_{\bar{j}} := \varphi_{j_1} \otimes \ldots \otimes \varphi_{j_d}\,, \qquad \bar{j} = (j_1, \ldots , j_d) \in \mathbb{N}_0^d\,,$$

we obtain a smooth decomposition of unity on $\mathbb{R}^d$.

Definition A.2: Let $0 < p, q \le \infty$ and $r \in \mathbb{R}$. Then the Besov space $S^r_{p,q}B(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\| f \,|S^r_{p,q}B(\mathbb{R}^d)\| := \Big(\sum_{\bar{j}\in\mathbb{N}_0^d} 2^{r|\bar{j}|_1 q} \, \| \mathcal{F}^{-1}[\varphi_{\bar{j}}\, \mathcal{F}f](\,\cdot\,)|L_p(\mathbb{R}^d)\|^q \Big)^{1/q}$$

is finite (modification if $q = \infty$).

Remark *A1*: Lizorkin representations. As long as we restrict us to $1 < p, q < \infty$ we may replace φ in the above construction by the characteristic function of the interval $(-1, 1)$ ending up with equivalent norms in $B^s_{p,q}(\mathbb{R})$ and $S^r_{p,q}B(\mathbb{R}^d)$, respectively. We refer to [32, 2.5.4] for the isotropic case, to [18] for the dominating mixed situation with $d = 2$, and to the remarks, given for the periodic case, in [19, Lem. 10].

In a similar way one could introduce Sobolev spaces of fractional order. However, here we prefer the interpretation as potential spaces.

Definition A.3: Let $1 < p < \infty$ and $s \in \mathbb{R}$. The fractional Sobolev space $H^s_p(\mathbb{R})$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\| f \,|H^s_p(\mathbb{R})\| := \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2}\, \mathcal{F}f(\xi)](\,\cdot\,)|L_p(\mathbb{R})\|$$

is finite.

Detailed treatments of Besov as well as Sobolev spaces of dominating mixed smoothness are given at various places, we refer to the monographs [56, 57], the survey [58] as well as to the booklet [45].

Remark *A2*: If $s = m \in \mathbb{N}$ then the spaces $H^m_p(\mathbb{R})$ can be equivalently characterized by the quantity

$$\Big(\sum_{\alpha=0}^{m} \|D^\alpha f|L_p(\mathbb{R})\|^p \Big)^{1/p} \tag{A1}$$

(see for instance [32]). In the sequel we shall denote by $H^m(\mathbb{R})$ the space $H^m_2(\mathbb{R})$ normed with (A1).

The corresponding Sobolev spaces with dominating mixed smoothness are defined as follows.

Definition A.4: Let $1 < p < \infty$ and $r \in \mathbb{R}$. The fractional Sobolev space with dominating mixed smoothness $S^r_pH(\mathbb{R}^d)$ is then the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\| f \,|S^r_pH(\mathbb{R}^d)\| := \Big\| \mathcal{F}^{-1}\Big[\prod_{i=1}^{d} (1 + |\xi_i|^2)^{r_i/2}\, \mathcal{F}f(\xi) \Big](\,\cdot\,) \Big| L_p(\mathbb{R}^d) \Big\|$$

is finite.

Remark *A3***:** If $r = m \in \mathbb{N}$ the space $S^m_p H(\mathbb{R}^d)$ can be equivalently characterized by the quantity

$$\Big(\sum_{\alpha \le m} \|D^{\bar{\alpha}} f | L_p(\mathbb{R}^d)\|^p\Big)^{1/p}.$$

This is the original approach of Nikol'skij to function spaces of dominating mixed smoothness, see [5]. For further historical remarks we refer to the monographs [56] and [57].

A few further properties of spaces of dominating mixed smoothness are collected in the following proposition. We always assume $1 \le p, q \le \infty$ in the B-case and $1 < p < \infty$ in the H-case.

Proposition A.5:

(i) For all $r \in \mathbb{R}$ we have the coincidence $S^r_2 H(\mathbb{R}^d) = S^r_{2,2} B(\mathbb{R}^d)$ (equivalent norms).
(ii) If $r > 0$, then $S^r_{p,q} B(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d)$ and $S^r_p H(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d)$.
(iii) If $r > 1/p$, then $S^r_{p,q} B(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$ and $S^r_p H(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$.

A.2. *Tensor products of Besov and Sobolev spaces*

For $1 < p < \infty$ we denote by δ_p the p-nuclear tensor norm. If $p = 1$ the symbol δ_1 denotes the projective tensor norm. For details we refer to [44] and [21]. The following theorem is taken from [21]. We shall use the conventions $S^r_{p,p} B(\mathbb{R}) := B^r_{p,p}(\mathbb{R})$ and $S^r_p H(\mathbb{R}) := H^r_p(\mathbb{R})$, respectively.

Theorem A.6: *Let $d \ge 1$ and $r \in \mathbb{R}$.*

(i) If $1 \le p \le \infty$ we have

$$\begin{aligned} B^r_{p,p}(\mathbb{R}) \otimes_{\delta_p} S^r_{p,p} B(\mathbb{R}^d) &= S^r_{p,p} B(\mathbb{R}^d) \otimes_{\delta_p} B^r_{p,p}(\mathbb{R}) \\ &= S^r_{p,p} B(\mathbb{R}^{d+1}) \end{aligned} \tag{A2}$$

(ii) and if $1 < p < \infty$

$$\begin{aligned} H^r_p(\mathbb{R}) \otimes_{\delta_p} S^r_p H(\mathbb{R}^d) &= S^r_p H(\mathbb{R}^d) \otimes_{\delta_p} H^r_p(\mathbb{R}) \\ &= S^r_p H(\mathbb{R}^{d+1}) \end{aligned}$$

in the sense of equivalent norms.

Remark *A4***:** The above identities supplement the well-know relations

$$L_p(\mathbb{R}^d) \otimes_{\delta_p} L_p(\mathbb{R}) = L_p(\mathbb{R}^{d+1}), \tag{A3}$$

see e.g. [44].

As an immediate consequence of the above theorem, the fact that the involved tensor product norms are uniform, and (A3) one obtains the following.

Corollary A.7: *Let $d > 1$ and let $r \in \mathbb{R}$.*
(i) *Let $1 \le p < \infty$. Suppose $T_j \in \mathcal{L}(B^r_{p,p}(\mathbb{R}), L_p(\mathbb{R}))$, $j = 1, \ldots, d$. Then the tensor product operator $T_1 \otimes \ldots \otimes T_d$ belongs to $\mathcal{L}(S^r_{p,p}(\mathbb{R}^d), L_p(\mathbb{R}^d))$ and the inequality*

$$\| T_1 \otimes \ldots \otimes T_d \,| S^r_{p,p}B(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \le \prod_{j=1}^{d} \| T_j \,| B^r_{p,p}(\mathbb{R}) \to L_p(\mathbb{R})\|$$

holds.
(ii) *Let $1 < p < \infty$. Suppose $T_j \in \mathcal{L}(H^r_p(\mathbb{R}), L_p(\mathbb{R}))$, $j = 1, \ldots, d$. Then the tensor product operator $T_1 \otimes \ldots \otimes T_d$ belongs to $\mathcal{L}(S^r_pH(\mathbb{R}^d), L_p(\mathbb{R}^d))$ and the inequality*

$$\| T_1 \otimes \ldots \otimes T_d \,| S^r_pH(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\| \le \prod_{j=1}^{d} \| T_j \,| H^r_p(\mathbb{R}) \to L_p(\mathbb{R})\|$$

holds.

A.3. *B-splines and associated spline-wavelet bases*

Most of the material on B-splines can be found in [37, 4.2].
Let $n = 1, 2, \ldots$. Let $\mathcal{X}$ denote the characteristic function of the interval $[0, 1]$. Then the n-fold convolution of this function $\mathcal{X}$

$$\mathcal{N}_n(t) \;=\; (\mathcal{X}_{[0,1]} * \,\ldots\, * \mathcal{X}_{[0,1]})(t)\,,$$

is called the cardinal B-spline of order n. We collect a few properties:

(1) $\operatorname{supp}\mathcal{N}_n \;=\; [0, n]$ and $\sum_{\ell=0}^{n} \mathcal{N}_n(\ell) = 1$;
(2) $\mathcal{N}_n$ is a piecewise polynomial function more exactly, the restriction of $\mathcal{N}_n$ on intervals $[k,\, (k+1)]$, $k \in \mathbb{Z}$ is a polynom of order at most $n-1$;
(3) $\mathcal{N}_n \in B^{n-1/2}_{2,\infty}(\mathbb{R})$;
(4) $\mathcal{F}\mathcal{N}_n(\xi) = \frac{1}{\sqrt{2\pi}}\, e^{-in\xi/2}\, (\frac{\sin(\xi/2)}{(\xi/2)})^n$;
(5) $\big(\mathcal{F}\mathcal{N}_n\big)^{(\ell)}(2\pi k) = 0$, falls $k \in \mathbb{Z} \setminus 0$, $\ell = 0, 1, \ldots, n-1$.
(6) The refinement equation. The Fourier transform of the B-splines satisfies the following relation

$$\mathcal{F}\mathcal{N}_n(2\xi) = m_n(\xi)\, \mathcal{F}\mathcal{N}_n(\xi) \qquad \text{with} \qquad m_n(\xi) := 2^{-n} \sum_{j=0}^{n} \binom{n}{j} e^{-ij\xi}\,.$$

By

$$\varphi_n(t) := \frac{1}{\sqrt{2\pi}}\, \mathcal{F}^{-1}\Big[\frac{\mathcal{F}\mathcal{N}_n(\xi)}{\Big(\sum\limits_{k=-\infty}^{\infty} |\mathcal{F}\mathcal{N}_n(\xi + 2\pi k)|^2 \Big)^{1/2}} \Big](t)\,, \qquad t \in \mathbb{R}\,,$$

we obtain an orthonormal scaling function which is again a spline of order n. Finally, by

$$\psi_n(t) := \sum_{k=-\infty}^{\infty} \langle\, \varphi_n(y/2), \varphi_n(y-k) \rangle\, (-1)^k\, \varphi_n(2t + k + 1)$$

we obtain the generator of an orthonormal wavelet system. For $n = 1$ it is easily checked that $-\psi_1(t-1)$ is the Haar wavelet. In general these functions ψ_n have the following properties:

(7) ψ_n restricted to intervals $[\frac{k}{2}, \frac{k+1}{2}]$, $k \in \mathbb{Z}$, is a polynomial of degree at most $n-1$.

(8) $\psi_n \in C^{n-2}(\mathbb{R})$ if $n \geq 2$.

(9) $\psi_n^{(n-2)}$ is uniformly Lipschitz continuous on $\mathbb{R}$ if $n \geq 2$.

(10) There exist positive numbers τ_n and sequences $(c_k)_k$ and $(d_k)_k$ such that

$$\psi_n(t) = \sum_{k=-\infty}^{\infty} c_k \, \mathcal{N}_n(2t-k)\,, \qquad \mathcal{N}_n(t) = \sum_{k=-\infty}^{\infty} d_k \, \varphi_n(t-k)\,, \qquad t \in \mathbb{R}\,,$$

and

$$\sup_{k \in \mathbb{Z}} (|c_k| + |d_k|)\, e^{\tau_n |k|} < \infty \qquad \text{and} \qquad \max_{0 \leq \ell \leq n-2} \sup_{t \in \mathbb{R}} |\psi_n^{(\ell)}(t)|\, e^{\tau_n |t|} < \infty\,.$$

(11) The functions ψ_n satisfy a moment condition of order $n-1$, i.e.

$$\int_{-\infty}^{\infty} t^\ell \, \psi_n(t)\, dt = 0\,, \qquad \ell = 0, 1, \ldots, n-1\,.$$

It will be convenient for us to use the following abbreviations:

$$\psi_{0,k}^n(t) := \varphi_n(t-k) \qquad \text{and} \qquad \psi_{j+1,k}^n(t) := 2^{j/2}\, \psi_n(2^j t - k)\,,$$

where $t \in \mathbb{R}, k \in \mathbb{Z}$ and $j \in \mathbb{N}_0$. Based on this, we define the tensor product system by

$$\psi_{\bar{j},\bar{k}}^n(x_1, ..., x_d) = \psi_{j_1,k_1}^n(x_1) \cdot \ldots \cdot \psi_{j_d,k_d}^n(x_d) \quad , \quad x \in \mathbb{R}^d, \bar{j} \in \mathbb{N}_0^d, \bar{k} \in \mathbb{Z}^d\,.$$

A.4. *Spline wavelets and the discretization of Besov spaces*

Let us consider the mappings

$$R_n \;:\quad f \mapsto (\langle f, \psi_{j,k}^n \rangle)_{j,k}\,.$$

and

$$\bar{R}_n \;:\quad f \mapsto (\langle f, \psi_{\bar{j},\bar{k}}^n \rangle)_{\bar{j},\bar{k}}\,.$$

To formulate the results from [21] we need the following sequence spaces.

Definition A.8: Let $0 < p \leq \infty$.
(i) We put for $r \in \mathbb{R}$

$$b_p^r := \Big\{ (a_{j,k})_{j,k} \subset \mathbb{C} : \quad \| \, a \, | b_p^r \| := \Big(\sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{j(r+\frac{1}{2}-\frac{1}{p})p} \, |a_{j,k}|^p \Big)^{1/p} < \infty \Big\}$$

(modification if $p = \infty$) .
(ii) Furthermore, let $r \in \mathbb{R}$, and

$$s_p^r b := \Big\{ (a_{\bar{j},\bar{k}})_{\bar{j},\bar{k}} \subset \mathbb{C} : \| a \,| s_p^r b \| := \Big(\sum_{\bar{j} \in \mathbb{N}_0^d} \sum_{\bar{k} \in \mathbb{Z}^d} 2^{|\bar{j}|_1 (r + \frac{1}{2} - \frac{1}{p}) p} \, |a_{\bar{j},\bar{k}}|^p \Big)^{1/p} < \infty \Big\} .$$

The following important result is due to Bourdaud [59], but see also Cohen [29] and Lemarie and Kahane [60].

Proposition A.9: *Let $n \in \mathbb{N}$, $1 \le p < \infty$ and $-n + 1/p < r < n - 1 + 1/p$. Then the mapping R_n generates an isomorphism of $B^r_{p,p}(\mathbb{R})$ onto b^r_p.*

Counterparts for spaces with dominating mixed smoothness can be found in Kamont [61], Oswald [12] and [21] . For the following we refer to [21].

Theorem A.10: *Let $d > 1$, $1 \le p < \infty$ and $-n + 1/p < r < n - 1 + 1/p$. Then the mapping $\bar{R}_n$ is an isomorphism from $S^r_{p,p}B(\mathbb{R}^d)$ to $s^r_p b$.*

A.5. *Atomic decompositions*

We essentially follow [45]. For $\bar{\ell} \in \mathbb{N}_0^d$ and $\bar{m} \in \mathbb{Z}^d$ we denote by $Q_{\bar{\ell},\bar{m}}$ the rectangle with center $(2^{-\ell_1} m_1, ..., 2^{-\ell_d} m_d)$ and sides parallel to the coordinate axes of length $2^{-\ell_1}, ..., 2^{-\ell_d}$.

Definition A.11: Let $n \in \mathbb{N}$. A function $a \in C^{n-1}(\mathbb{R}^d)$ is called an n-atom centered at $Q_{\bar{\ell},\bar{m}}$ if

(1) (i) $\operatorname{supp} a \in Q_{\bar{\ell},\bar{m}}$

(2) (ii) All its distributional derivatives $D^\alpha a$, $\alpha_i \le n$, $i = 1, \dots, d$ are integrable functions and satisfy the inequalities

$$\sup_{x \in \mathbb{R}^d} |D^{\bar{\alpha}} a(x)| \le 2^{\bar{\alpha}\bar{\ell}} .$$

The following result is a slightly modified version of [45, Thm. 2.4/(i)].

Theorem A.12: *Let $1 \le p \le \infty$ and $0 < r < n$. If $\Lambda := \{\lambda_{\bar{\ell},\bar{m}}\}_{\bar{\ell} \in \mathbb{N}_0^d, \bar{m} \in \mathbb{Z}^d} \in s^r_p b$ and $a_{\bar{\ell},\bar{m}}$ are n-atoms centered at $Q_{\bar{\ell},\bar{m}}$, then*

$$\sum_{\bar{\ell} \in \mathbb{N}_0^d} \sum_{\bar{m} \in \mathbb{Z}^d} \lambda_{\bar{\ell},\bar{m}} a_{\bar{\ell},\bar{m}}$$

converges in $\mathcal{S}'(\mathbb{R}^d)$, its limits f belongs to the space $S^r_{p,p}B(\mathbb{R}^d)$ and the inequality

$$\|f | S^r_{p,p}B(\mathbb{R}^d)\| \le c \, \| \{2^{-|\bar{\ell}|/2} \, \lambda_{\bar{\ell},\bar{m}}\}_{\bar{\ell},\bar{m}} | s^r_p b\| \tag{A4}$$

holds with some constant c independent of λ and $\{a_{\bar{\ell},\bar{m}}\}_{\bar{\ell},\bar{m}}$.

Proof: In comparison with [45] we have weakened the definition of the atoms. Vybíral has worked with C^n functions. However, the present version is still sufficient without changing the proof, see in particular Step 2 of the proof of Theorem 2.4 in the quoted paper. □

Remark *A5***:** If one replaces condition (i) in Def. A.11 by

$$\text{(i)'} \qquad \operatorname{supp} a \in C\, Q_{\bar{\ell},\bar{m}}\,,$$

where C is a general positive constant (and $C\,Q$ denotes the cube with the same center as Q, the sides are parallel to the axes but have length C times the side length of Q), then inequality (A4) remains true with a new universal constant c'.

Appendix B. Function spaces on $(0,1)^d$

B.1. *Definitions*

Definition B.1: Let $1 \le p, q \le \infty$ and $r > 0$ then $S^r_{p,q}B((0,1)^d)$ is the space of all $f \in L_p((0,1)^d)$ such that there exists a $g \in S^r_{p,q}B(\mathbb{R}^d)$ satisfying $f = g|_{(0,1)^d}$. It is endowed with the quotient norm

$$\|f|S^r_{p,q}B((0,1)^d)\| = \inf\{\|g|S^r_{p,q}B(\mathbb{R}^d)\| \;:\; g|_{(0,1)^d} = f\}\,.$$

For $m \in \mathbb{N}_0$ let us also define the spaces $H^m(0,1)$ and $H^m_{mix}((0,1)^d)$.

Definition B.2: Let $m \in \mathbb{N}_0$.

(i) The space $H^m(0,1)$ is the collection of all m-times weakly differentiable $L_2(0,1)$-functions such that

$$\|f|H^m(0,1)\| = \Big(\sum_{\alpha=0}^{m} \|D^\alpha f|L_2((0,1))\|^2\Big)^{1/2}$$

is finite.

(ii) The space $H^m_{mix}((0,1)^d)$ is the collection of all $L_2((0,1)^d)$-functions such that

$$\|f|H^m_{mix}((0,1)^d)\| = \Big(\sum_{\substack{\alpha_i \le m\\ i=1,\dots,d}} \|D^{\bar{\alpha}} f|L_2((0,1)^d)\|^2\Big)^{1/2}$$

is finite. Here $D^{\bar{\alpha}}f$ has to be understood in the distributional sense.

Next we turn to intrinsic descriptions.

B.2. *Intrinsic characterizations*

Let $f : (0,1)^d \to \mathbb{C}$. For $m \in \mathbb{N}_0$, $h \in \mathbb{R}$ and $i \in \{1,...,d\}$ we define the (directional) difference operator $\Delta^m_{h,i}$ by

$$\Delta^m_{h,i} f(x) = \begin{cases} \sum\limits_{j=0}^{m} (-1)^j \binom{m}{j} f(x_1,...,x_i + (m-j)h,...,x_d) & : (Q) \text{ holds} \\ 0 & : \text{otherwise} \end{cases}. \tag{B1}$$

Here (Q) is satisfied, if, and only if,

$$(x_1,...,x_i + \ell h,...,x_d) \in (0,1)^d\,, \qquad \ell = 0,...,m\,.$$

We are interested in intrinsic characterizations using mixed directional moduli of smoothness of type

$$\omega_A^{\bar{m},(0,1)^d}(f;t_1,...,t_n)_p := \sup_{\substack{|h_i|\le t_i\\ i=1,...,n}} \big\| \big(\Delta^{m_{\alpha_1}}_{h_1,\alpha_1} \circ \cdots \circ \Delta^{m_{\alpha_n}}_{h_n,\alpha_n}\big) f(\,\cdot\,)\, |L_p((0,1)^d)\big\| .$$

Here $A = \{\alpha_1, ..., \alpha_n\} \subset \{1, ..., d\}$ denotes a non-empty set of "directions".

The following theorem will be published in a forthcoming paper of the second named author, [62]. It gives an intrinsic characterization of Besov spaces of dominating mixed smoothness on the cube.

Theorem B.3: *Let $1 \le p, q \le \infty$, $r > 0$, $0 < \beta < 1$ and $r < m \in \mathbb{N}$. Then the space $S^r_{p,q}B((0,1)^d)$ is the collection of all functions $f \in L_p((0,1)^d)$ such that the quantity*

$$\|f|S^r_{p,q}B((0,1)^d)\|^\Delta = \|f|L_p((0,1)^d)\| + \sum_{\substack{A\subset\{1,...,d\}\\ A\neq\emptyset}} \Big[\int_0^\beta \cdots \int_0^\beta t_1^{-rq} \cdots t_n^{-rq} \omega_A^{\bar{m},(0,1)^d}(f;t_1,...,t_n)_p^q \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \Big]^{1/q} \qquad \text{(B2)}$$

is finite. Moreover, $\|\cdot|S^r_{p,q}B((0,1)^d)\|^\Delta$ represents an equivalent norm in $S^r_{p,q}B((0,1)^d)$.

Remark *B1*: (i) Of course, the main term on the right-hand side of (B2) is given by

$$\Big[\int_0^\beta \cdots \int_0^\beta t_1^{-rq} \cdots t_d^{-rq}\, \omega_{1,\dots,d}^{\bar{m},(0,1)^d}(f;t_1,...,t_d)_p^q \frac{dt_1}{t_1} \cdots \frac{dt_d}{t_d} \Big]^{1/q} .$$

However, the lower order terms related to sets A, $|A| < d$, can not be ignored.
(ii) Another proof of (B2), but restricted to $d = 2$, has been given in the monograph [48, Thm. 1.67].

Finally, we compare $H^m_{mix}((0,1)^d)$ with $S^m_{2,2}B((0,1)^d)$.

Proposition B.4: *Let $m \in \mathbb{N}$. Then we have the coincidence*

$$S^m_{2,2}B((0,1)^d) = H^m_{mix}((0,1)^d)$$

in the sense of equivalent norms.

Proof: For $d = 2$ a proof has been given in [48, Thm. 1.67]. This method extends to the case of general d. □

B.3. *Tensor products of function spaces on the cube*

Defining spaces on cubes by restrictions allows a more or less immediate transfer of Theorem A.6. As in Subsection A.2 we shall use the conventions $S^r_{p,p}B(0,1) := B^r_{p,p}(0,1)$ and $S^r_pH(0,1) := H^r_p(0,1)$, respectively.

Theorem B.5: *Let $d \ge 1$ and $r \in \mathbb{R}$.*

(i) If $1 \le p \le \infty$ *we have*

$$B^r_{p,p}(0,1) \otimes_{\delta_p} S^r_{p,p}B((0,1)^d) = S^r_{p,p}B((0,1)^d) \otimes_{\delta_p} B^r_{p,p}(0,1) = S^r_{p,p}B((0,1)^{d+1})$$

(ii) and if $1 < p < \infty$

$$H^r_p(0,1) \otimes_{\delta_p} S^r_pH((0,1)^d) = S^r_pH((0,1)^d) \otimes_{\delta_p} H^r_p(0,1) = S^r_pH((0,1)^{d+1})$$

in the sense of equivalent norms.

Proof: We concentrate on the proof of (i). The proof of (ii) follows by the same type of arguments. First we deal with the most simple case $d = 1$ and $p = 1$. For the case $d > 1$ we iterate the following arguments.

Step 1. Let $\varepsilon > 0$ be given. Let h be an element of the algebraic tensor product $B^r_{1,1}(0,1) \otimes B^r_{1,1}(0,1)$ s.t.

$$h(x,y) = \sum_{i=1}^n f_i(x)\, g_i(y)\,, \quad f_i, g_i \in B^r_{1,1}(0,1)\,, \quad i = 1,\dots n\,,$$

and

$$\delta_1(h, B^r_{1,1}(0,1), B^r_{1,1}(0,1)) \le \sum_{i=1}^n \|\, f_i\,|B^r_{p,p}(0,1)\|\,\|\, g_i\,|B^r_{p,p}(0,1)\| \le \delta_1(h, B^r_{1,1}(0,1), B^r_{1,1}(0,1)) + \varepsilon\,.$$

Then, there exist appropriate extensions $\mathcal{E}f_i\,, \mathcal{E}g_i \in B^r_{p,p}(\mathbb{R})$ such that

$$\|\, f_i\,|B^r_{1,1}(0,1)\| \le \|\,\mathcal{E}f_i\,|B^r_{1,1}(\mathbb{R})\| \le \|\, f_i\,|B^r_{1,1}(0,1)\| + \frac{\varepsilon}{n\, \max_i(1,\, \|\, g_i\,|B^r_{1,1}(0,1)\|)}$$

$$\|\, g_i\,|B^r_{1,1}(0,1)\| \le \|\,\mathcal{E}g_i\,|B^r_{1,1}(\mathbb{R})\| \le \|\, g_i\,|B^r_{1,1}(0,1)\| + \frac{\varepsilon}{n\, \max_i(1,\, \|\, f_i\,|B^r_{1,1}(0,1)\|)}\,.$$

Of course, $\sum_{i=1}^n \mathcal{E}f_i \otimes \mathcal{E}g_i$ is an extension of h and we will denote it by $\mathcal{E}h$. This implies

$$\delta_1(\mathcal{E}h, B^r_{1,1}(\mathbb{R}), B^r_{1,1}(\mathbb{R})) \le \sum_{i=1}^n \|\,\mathcal{E}f_i\,|B^r_{p,p}(\mathbb{R})\|\,\|\,\mathcal{E}g_i\,|B^r_{p,p}(\mathbb{R})\| \le \delta_1(h, B^r_{p,p}(0,1), B^r_{p,p}(0,1)) + 3\,\varepsilon + \varepsilon^2\,. \tag{B3}$$

Let $u_i, v_i \in B^r_{1,1}(\mathbb{R})$ s.t. $\mathcal{E}h = \sum_{i=1}^m u_i \otimes v_i$. The restrictions of u_i and v_i to $(0,1)$ we denote by $\mathrm{re}\, u_i$ and $\mathrm{re}\, v_i$, respectively. By definition of the norm $\|\cdot|B^r_{1,1}(0,1)\|$ we conclude

$$\delta_1(h, B^r_{1,1}(0,1), B^r_{1,1}(0,1)) \le \delta_1(\mathcal{E}h, B^r_{1,1}(\mathbb{R}), B^r_{1,1}(\mathbb{R}))\,. \tag{B4}$$

Hence, combining (B3) and (B4), we find $\delta_1(h, B^r_{1,1}(0,1), B^r_{1,1}(0,1)) =$

$\delta_1(\mathcal{E}h, B^r_{1,1}(\mathbb{R}), B^r_{1,1}(\mathbb{R}))$. Moreover, by using Theorem A.6 we get

$$\begin{aligned}\| h\,|S^r_{1,1}B((0,1)^2)\| &\le \|\mathcal{E}h\,|S^r_{1,1}B(\mathbb{R}^2)\| \\ &\asymp \delta_1(\mathcal{E}h, B^r_{1,1}(\mathbb{R}), B^r_{1,1}(\mathbb{R})) = \delta_1(h, B^r_{1,1}(0,1), B^r_{1,1}(0,1))\end{aligned}$$

for all $h \in B^r_{1,1}(0,1) \otimes B^r_{1,1}(0,1)$.

Step 2. This time we start with $h \in S^r_{1,1}B((0,1)^2)$. For all $\varepsilon > 0$ there exists an extension $\mathcal{E}h$ of h such that

$$\| h\,|S^r_{1,1}B((0,1)^2)\| \le \|\mathcal{E}h\,|S^r_{1,1}B(\mathbb{R}^2)\| \le \| h\,|S^r_{1,1}B((0,1)^2)\| + \varepsilon\,.$$

Then, using again Theorem A.6 and the density of the algebraic tensor product, there exists an element $(\mathcal{E}h)_\varepsilon \in B^r_{1,1}(\mathbb{R}) \otimes B^r_{1,1}(\mathbb{R})$ s.t.

$$\begin{aligned}\|\mathcal{E}h\,|B^r_{1,1}(\mathbb{R}) \otimes_{\delta_1} B^r_{1,1}(\mathbb{R})\| - \varepsilon &\le \delta_1((\mathcal{E}h)_\varepsilon, B^r_{1,1}(\mathbb{R}), B^r_{1,1}(\mathbb{R})) \\ &\le \|\mathcal{E}h\,|B^r_{1,1}(\mathbb{R}) \otimes_{\delta_1} B^r_{1,1}(\mathbb{R})\| + \varepsilon\,.\end{aligned}$$

We put $h_\varepsilon := \mathrm{re}\,(\mathcal{E}h)_\varepsilon$. Then

$$\begin{aligned}\delta_1(h_\varepsilon, B^r_{1,1}(0,1), B^r_{1,1}(0,1)) &\le \delta_1((\mathcal{E}h)_\varepsilon, B^r_{1,1}(\mathbb{R}), B^r_{1,1}(\mathbb{R})) \\ &\asymp \|\,(\mathcal{E}h)_\varepsilon\,|S^r_{1,1}B((0,1)^2)\| \\ &\asymp \|\mathcal{E}h\,|S^r_{1,1}B((0,1)^2)\| \\ &\le c\,\| h\,|S^r_{1,1}B((0,1)^2)\|\end{aligned}$$

for ε being sufficiently small. Here c does not depend on h. We complete the proof by obvious density arguments.

Step 3. Let $1 < p < \infty$. The same strategy applies but with a few technical changes since we have to work with the p-nuclear norm now. We omit details. □

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