Function Spaces with Dominating Mixed Smoothness

Characterization by Differences

Tino Ullrich [1]

Abstract

This paper deals with function spaces with dominating mixed smoothness properties of Besov and Lizorkin-Triebel type on $\mathbb{R}^d$ as well as on the d-torus $\mathbb{T}^d$. The main result is the characterization of these classes in terms of integral means of differences for the largest possible range of parameters. Moreover we obtain further characterizations based on this result using well-known classical moduli of smoothness.

Keywords function space, dominating mixed smoothness, Besov space, Lizorkin-Triebel spaces, characterization by differences, integral means of differences
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[1] Friedrich-Schiller-Universität Jena, Mathematisches Institut, Ernst-Abbe-Platz 3, D-07740, Germany
ullricht@minet.uni-jena.de

Contents

1 Introduction

This paper deals with function spaces of dominating mixed smoothness, especially the scales $S^{\bar{r}}_{p,q}F$ and $S^{\bar{r}}_{p,q}B$ on $\mathbb{R}^d$ as well as on the d-torus $\mathbb{T}^d$. Recently there is an increasing interest in function spaces of this type, see e.g. Bazarkhanov [Ba] or Vybíral [Vy]. Sobolev spaces of this type were first introduced by S. M. Nikol'skij in [Ni 1] and [Ni 2] in the early sixties. Here

$$S^{\bar{r}}_pW(\mathbb{R}^2) = \Big\{ f \in L_p(\mathbb{R}^2) : \|f|S^{\bar{r}}_pW(\mathbb{R}^2)\| = \|f|L_p(\mathbb{R}^2)\| + \Big\| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} |L_p(\mathbb{R}^2) \Big\| + \Big\| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} |L_p(\mathbb{R}^2) \Big\| + \Big\| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}} |L_p(\mathbb{R}^2) \Big\| < \infty \Big\} ,$$

where $1 < p < \infty$ and $r_i = 0, 1, 2, ...$ $(i = 1, 2)$. The mixed derivative $\frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}}$ plays the dominant role and gave the name to this scale of spaces. Later on, these classes as well as corresponding Besov spaces, have been extensively studied in the former Soviet Union, for example by T. I. Amanov, O. V. Besov, P. I. Lizorkin, S. M. Nikol'skij and M. K. Potapov. We refer mainly to the monograph [Am] for a detailed study of this topic in terms of differences and derivatives. As far as the Fourier-analytic treatment is considered we refer to H.-J. Schmeisser, H. Triebel and the book [ST, Ch.2], where the spaces $S^{\bar{r}}_{p,q}B(\mathbb{R}^2)$ and $S^{\bar{r}}_{p,q}F(\mathbb{R}^2)$ of Besov and Lizorkin-Triebel type are introduced. Their approach is based on the decomposition of a tempered distribution into Fourier-analytic building blocks using a certain tensor product of smooth dyadic resolutions of unity. There were also given characterizations by differences for sufficiently large smoothness. The aim of the present paper is to characterize the classes $S^{\bar{r}}_{p,q}B$ and $S^{\bar{r}}_{p,q}F$ by differences for the largest possible range (except some limiting cases) of the parameters $\bar{r} = (r_1, ..., r_d)$, given by

$$0 < p < \infty \ , \quad 0 < q \le \infty \ , \quad r_i > \max(0, 1/p - 1, 1/q - 1) \quad , \quad i = 1, ..., d$$

in the F-case. In case $d = 2$ our main result reads as follows.
$S^{\bar{r}}_{p,q}F(\mathbb{R}^2)$ is the collection of all $f \in L_p(\mathbb{R}^2) \cap S'(\mathbb{R}^2)$ such that the quantity

$$\begin{aligned} \|f|S^{\bar{r}}_{p,q}F(\mathbb{R}^2)\|^R = \|f|L_p(\mathbb{R}^2)\| &+ \Big\| \Big[\int_0^\infty t^{-r_1 q} \Big(\int_{-1}^1 |\triangle^{m_1}_{th,1} f(x)|\, dh \Big)^q \frac{dt}{t} \Big]^{1/q} \Big| L_p(\mathbb{R}^2) \Big\| \\ &+ \Big\| \Big[\int_0^\infty t^{-r_2 q} \Big(\int_{-1}^1 |\triangle^{m_2}_{th,2} f(x)|\, dh \Big)^q \frac{dt}{t} \Big]^{1/q} \Big| L_p(\mathbb{R}^2) \Big\| \\ &+ \Big\| \Big[\int_0^\infty \int_0^\infty t_1^{-r_1 q} t_2^{-r_2 q} \Big(\int_{[-1,1]^2} |\triangle^{m_1}_{t_1h_1,1} \circ \triangle^{m_2}_{t_2h_2,2} f(x)|\, dh \Big)^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \Big]^{1/q} \Big| L_p(\mathbb{R}^2) \Big\| \end{aligned}$$

is finite, where $m_i \in \mathbb{N}$ with $m_i > r_i$ $(i = 1, 2)$. The paper is organized as follows. Section 2 contains some basic preparation and notation. The Sections 3 and 4 represent the main part. We distinguish the nonperiodic case, i.e. spaces on $\mathbb{R}^d$ (Section 3), from the periodic case, i.e. spaces on $\mathbb{T}^d$ (Section 4). After defining the spaces and pointing out the connection between differences and maximal functions, we state and prove our main result. This is the characterization of $S^{\bar{r}}_{p,q}F$ and $S^{\bar{r}}_{p,q}B$ by integral means of differences for arbitrary d. On that basis we additionally obtain further difference characterizations, which use for instance classical moduli of smoothness, cf. [ST, 2.3.3, 2.3.4]. Finally we give a partial answer to the question, whether one can replace $\int_0^\infty$ by $\int_0^1$.

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2 Some preparation

First of all we will introduce some basic notation. The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ denote the real numbers, complex numbers, natural numbers, natural numbers including 0 and the integers. The dimension of the underlying Euclidean space for our function spaces is denoted by d. Vector valued quantities (for instance indices) will be denoted by $\bar{k}, \bar{\ell}, \bar{m}, \bar{\ell}, ...$ or $\bar{r}, \bar{h}, \bar{\mu}, \bar{\beta}...$ and elements of the underlying Euclidean space by $x, y, z, ...$ Let us specify the following notations

$$\begin{aligned}
&\bar{k}+\bar{\ell} = (k_1+\ell_1, ..., k_n+\ell_n) \ , \\
&\lambda \cdot \bar{k} = (\lambda \cdot k_1, ..., \lambda \cdot k_n) \ , \ \lambda \in \mathbb{R} \ , \\
&\lambda + \bar{k} = (\lambda + k_1, ..., \lambda + k_n) \ , \ \lambda \in \mathbb{R} \ , \\
&\bar{t}^{\,\bar{r}} = (t_1^{r_1}, ..., t_n^{r_n}) \ , \\
&\lambda^{\bar{r}} = (\lambda^{r_1}, ..., \lambda^{r_n}) \ , \ \lambda > 0 \ , \\
&\bar{1} = (1, ..., 1) \ , \\
&\frac{\lambda}{\bar{t}} = (\frac{\lambda}{t_1}, ..., \frac{\lambda}{t_n}) \ , \ \lambda \in \mathbb{R} \ , \\
&\bar{k} \cdot \bar{r} = k_1 r_1 + ... + k_n r_n \ , \\
&\bar{r} * \bar{s} = (r_1 s_1, ..., r_n s_n) \ , \\
&d\bar{h} = (dh_1, ..., dh_n) \ , \\
&\frac{d\bar{t}}{\bar{t}} = \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \ , \\
&|\bar{r}| = r_1 + ... + r_n \ ,
\end{aligned}$$

where $\bar{k}, \bar{\ell}, \bar{t}, \bar{s}, \bar{r}, \bar{h} \in \mathbb{R}^n$. Furthermore we abbreviate the relations

$$r_i > s_i \quad (r_i \geq s_i) \quad , \quad i = 1, ..., n$$

by $\bar{r} > \bar{s}$ ($\bar{r} \geq \bar{s}$) and by $\bar{r} > s$ ($\bar{r} \geq s$) if additionally $\bar{s} = (s, ..., s)$ with $s \in \mathbb{R}$.
Let us further define the quantities σ_p and $\sigma_{p,q}$ for $0 < p, q \leq \infty$ by

$$\sigma_p := \left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{p,q} := \left(\frac{1}{\min(p,q)} - 1\right)_+ \ , \tag{1}$$

where $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$.
Also the following convention should be helpful. If we have a tuple $\bar{\beta} \in \{0,1\}^d$ such that $|\bar{\beta}| = n \geq 1$, we assign to $\bar{\beta}$ a tuple $\bar{\delta} = (\delta_1, ..., \delta_n)$ with the property

$$1 \leq \delta_1 < \delta_2 < \cdots < \delta_n \leq d \quad \text{and} \quad \beta_{\delta_i} = 1 \quad , \quad i = 1, ..., n. \tag{2}$$

Let us fix right here some often used sets and index-sets. For $k \in \mathbb{Z}$ we put

$$I_k = [-2^k, 2^k] \quad \text{and} \quad I_k^+ = [0, 2^k].$$

Additionally we put for $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}$

$$I_j^\Delta := \left\{ \begin{array}{lcl} I_j \setminus I_{j-1} & : & j \geq 1 \\ I_0 & : & j = 0 \end{array} \right. \quad \text{and} \quad I_k^{+\Delta} := I_k^+ \setminus I_{k-1}^+ \ .$$

For $\bar{\beta} \in \{0,1\}^d$ with $|\bar{\beta}| = n \geq 1$ and $\bar{\delta}$ according to $\bar{\beta}$ we define for $\bar{\mu} \in \mathbb{Z}^d$

$$\begin{aligned} Q_{\bar{\mu},\bar{\beta}} &:= I_{\mu_{\delta_1}} \times ... \times I_{\mu_{\delta_n}} \ , \\ Q^+_{\bar{\mu},\bar{\beta}} &:= I^+_{\mu_{\delta_1}} \times ... \times I^+_{\mu_{\delta_n}} \ , \\ Q^{+\Delta}_{\bar{\mu},\bar{\beta}} &:= I^{+\Delta}_{\mu_{\delta_1}} \times ... \times I^{+\Delta}_{\mu_{\delta_n}} \end{aligned} \tag{3}$$

and for $\bar{\mu} \in \mathbb{N}_0^d$

$$Q^{\Delta}_{\bar{\mu},\bar{\beta}} := I^{\Delta}_{\mu_{\delta_1}} \times ... \times I^{\Delta}_{\mu_{\delta_n}} \ . \tag{4}$$

If $\bar{\beta} = (1, ..., 1)$ we put $Q_{\bar{\mu}} := Q_{\bar{\mu},\bar{\beta}}$, $Q^+_{\bar{\mu}} := Q^+_{\bar{\mu},\bar{\beta}}$, $Q^{+\,\Delta}_{\bar{\mu}} := Q^{+\,\Delta}_{\bar{\mu},\bar{\beta}}$ and $Q^{\Delta}_{\bar{\mu}} := Q^{\Delta}_{\bar{\mu},\bar{\beta}}$.
The mentioned index-sets are given by

$$\begin{aligned} I_{\bar{\beta}} &= \{\bar{\ell} = (\ell_1, ..., \ell_d) \in \mathbb{N}_0^d : \ell_i = 0 \Longleftrightarrow \beta_i = 0 \, , \, i = 1, ..., d\} \ , \\ Z_{\bar{\beta}} &= \big\{(k_1, ..., k_d) \in \mathbb{Z}^d : \beta_i = 0 \implies k_i = 0, \ i = 1, ..., d\big\} \ , \\ \bar{Z}_{\bar{\beta}} &= Z_{\bar{1}-\bar{\beta}} \ , \\ N_{\bar{\beta}} &= Z_{\bar{\beta}} \cap \mathbb{N}_0^d \ , \\ \bar{N}_{\bar{\beta}} &= N_{\bar{1}-\bar{\beta}} \ , \\ E_{\bar{\beta}} &= Z_{\bar{\beta}} \cap \{0,1\}^d \ , \\ \bar{E}_{\bar{\beta}} &= E_{\bar{1}-\bar{\beta}} \ . \end{aligned} \tag{5}$$

And for $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ we define the sets $M^i_1 = \{0, ..., m_i\}$ and $M^i_0 = \{0\}$, $i = 1, ..., d$.
Finally for $\bar{\beta} \in \{0,1\}^d$ we put

$$M_{\bar{\beta}} = M^1_{\beta_1} \times ... \times M^d_{\beta_d}. \tag{6}$$

3 Spaces on $\mathbb{R}^d$

3.1 Preliminaries

Distributions, Fourier transform and Paley-Wiener-Schwartz

As usual we denote by $S = S(\mathbb{R}^d)$ the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. Its topology is generated by the norms

$$\|\varphi\|_{k,\ell} = \sup_{x \in \mathbb{R}^d} (1 + |x|)^k \sum_{|\bar{\alpha}| \leq \ell} |D^{\bar{\alpha}} \varphi(x)|. \tag{7}$$

A linear mapping $f : S(\mathbb{R}^d) \to \mathbb{C}$ is called a tempered distribution, if there exists a constant $c > 0$ and $k, \ell \in \mathbb{N}_0$ such that

$$|f(\varphi)| \leq c \|\varphi\|_{k,\ell}$$

holds for all $\varphi \in S(\mathbb{R}^d)$. The collection of all such mappings is denoted by $S'(\mathbb{R}^d)$. See [Tr 2] for details. As usual the Fourier transform defined on both $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ is given by

$$(\mathcal{F}f)(\varphi) := f(\mathcal{F}\varphi) \quad , \quad \varphi \in S \quad ,$$

where

$$\mathcal{F}\varphi(\xi) := (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{-ix\cdot\xi} \varphi(x) \, dx.$$

$\mathcal{F}$ is a bijection (in both cases) and its inverse is given by

$$\mathcal{F}^{-1}\varphi(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\varphi(x)\,dx \quad , \quad \varphi \in S(\mathbb{R}^d) \; .$$

We say a (tempered) distribution $T \in S'(\mathbb{R}^d)$ is a regular distribution (or a distribution of function type) if there exists a measurable function $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$T(\varphi) = \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx.$$

Now we are well prepared to recall a proper version of the famous theorem by Paley, Wiener and Schwartz.

Theorem 3.1.1 The following is equivalent:

(i) $f \in S'(\mathbb{R}^d)$ and supp $\mathcal{F}f \subset \{y : |y_i| \le b_i, i = 1, ..., d\}$.

(ii) f is a regular distribution which can be holomorphically extended to $\mathbb{C}^d$. Furthermore we have the following growth condition: For an appropriate real number $\lambda > 0$ and any $\varepsilon > 0$ there exists a constant c_ε such that

$$|f(z)| \le c_\varepsilon (1 + |x|)^\lambda e^{(b_1+\varepsilon)|y_1|} \cdot ... \cdot e^{(b_d+\varepsilon)|y_d|} \quad , \quad z = x + iy \quad , \quad x, y \in \mathbb{R}^d.$$

Proof With some obvious modifications in the proof of the classic theorem, see for example [Yo, VI.4], one obtains the above version. □

Remark 3.1.1 An important consequence for us is the following. We consider $f \in S'(\mathbb{R}^d)$ with supp $\mathcal{F}f \subset \{y : |y_i| \le b_i,\ i = 1, ..., d\}$. One direction of Theorem 3.1.1 tells us that f is representable as an entire analytic function $f(z)$ with some growth condition. If we fix one variable, say z_k, in f (to a real number) the outcome is also an entire analytic function on $\mathbb{C}^{d-1}$ with a corresponding growth condition, only without the term $e^{(b_k+\varepsilon)|y_k|}$. Applying the second direction of the equivalence one can interpret this trace function as a tempered distribution with Fourier support in $[-b_1, b_1] \times ... \times [-b_{k-1}, b_{k-1}] \times [-b_{k+1}, b_{k+1}] \times ... \times [-b_d, b_d]$. □

Vector-valued Lebesgue spaces

As usual $L_p(\mathbb{R}^d)$ for $0 < p \le \infty$ denote the Lebesgue spaces on $\mathbb{R}^d$ (quasi-)normed by

$$\|f|L_p(\mathbb{R}^d)\| = \left(\int_{\mathbb{R}^d} |f(x)|^p\,dx \right)^{1/p}$$

with the typical modification if $p = \infty$. Using the interpretation given above we obtain the following continuous embeddings for $1 \le p \le \infty$

$$S(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset S'(\mathbb{R}^d).$$

If $p < 1$ only the first inclusion holds.
Beside these spaces we will need the vector-valued versions, defined by

$$L_p(\mathbb{R}^d, \ell_q) = \Big\{ \{f_k(x)\}_{k\in I} | \, f_k : \mathbb{R}^d \to \mathbb{C}, \text{ measurable and } \|\{f_k\}_{k\in I}|L_p(\mathbb{R}^d, \ell_q)\| < \infty \Big\} \quad ,$$

where $0 < p < \infty$ and $0 < q \le \infty$. I denotes now and in the sequel a countable index-set and

$$\|\{f_k\}_{k\in I}|L_p(\mathbb{R}^d, \ell_q)\| := \left(\int\limits_{\mathbb{R}^d} \left(\sum_{k\in I} |f_k(x)|^q \right)^{p/q} dx \right)^{1/p}$$

with the usual modification in case $q = \infty$. For abbreviation we may write $\|f_k|L_p(\mathbb{R}^d, \ell_q)\|$ and sometimes $\|f_k|L_p(\ell_q)\|$. Clearly $\{f_k\}_{k\in I} \in L_p(\mathbb{R}^d, \ell_q)$ implies $f_k \in L_p(\mathbb{R}^d)$ for every $k \in I$. Theorem 3.1.1 enables the definition of the spaces $L_p^\Omega(\mathbb{R}^d, \ell_q)$ of entire analytic functions for $0 < p < \infty$ and $0 < q \le \infty$. Hereby $\Omega = \{\Omega_k\}_{k\in I}$ denotes a sequence of compact subsets of $\mathbb{R}^d$. We define

$$L_p^\Omega(\mathbb{R}^d, \ell_q) = \{ f = \{f_k\}_{k\in I} : f_k \in S'(\mathbb{R}^d), \ \operatorname{supp} \mathcal{F} f_k \subset \Omega_k, \ k \in I, \ \|f_k|L_p(\mathbb{R}^d, \ell_q)\| < \infty \}.$$

Maximal functions and maximal inequalities

Maximal functions and corresponding inequalities form an important tool for the Fourier-analytic treatment of the spaces we are interested in. Even for our purpose they play an essential role connecting classic differences with modern Fourier analysis.
For a locally integrable function f we denote by $Mf(x)$ the Hardy-Littlewood maximal function defined by

$$(Mf)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int\limits_Q |f(y)|\, dy \quad , \quad x \in \mathbb{R}^d \quad , \tag{8}$$

where the supremum is taken over all cubes centered at x with sides parallel to the coordinate axes. A vector valued generalization of the famous Hardy-Littlewood maximal inequality is due to C. Fefferman and E. M. Stein [FS].

Theorem 3.1.2 For $1 < p < \infty$ and $1 < q \le \infty$ there exists a constant $c > 0$, such that

$$\|Mf_k|L_p(\mathbb{R}^d, \ell_q)\| \le c \|f_k|L_p(\mathbb{R}^d, \ell_q)\| \tag{9}$$

holds for all sequences $\{f_k\}_{k\in I}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$. □

Now we define a one-dimensional version of (8)

$$(M_i f)(x) = \sup_{s>0} \frac{1}{2s} \int\limits_{x_i - s}^{x_i + s} |f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d)|\, dt \quad , \quad x \in \mathbb{R}^d. \tag{10}$$

Precisely one has to prove, that this operator maps two equivalent representatives of f to the same equivalence class. Having this in mind, one can prove the following version of Theorem 3.1.2:

Theorem 3.1.3 For $1 < p < \infty$ and $1 < q \leq \infty$ there exists a constant $c > 0$ such that

$$\|M_i f_k | L_p(\mathbb{R}^d, \ell_q)\| \leq c \|f_k | L_p(\mathbb{R}^d, \ell_q)\| \quad , \quad i = 1, ..., d \ ,$$

holds for all sequences $\{f_k\}_{k\in I}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

Proof One only has to split the integration over $\mathbb{R}^d$ into d integrations over $\mathbb{R}^1$ and apply Theorem 3.1.2 to the integration according to i. □

The following construction of a maximal function is due to Peetre, Fefferman and Stein. Let $\bar{b} = (b_1, ..., b_d), \bar{s} = (s_1, ..., s_d) \in \mathbb{R}^d$ with $\bar{b}, \bar{s} > 0$ and $f \in S'$ with $\mathcal{F}f$ compactly supported. We define the maximal function $P_{\bar{b},\bar{s}} f$ by

$$P_{\bar{b},\bar{s}} f(x) = \sup_{z\in\mathbb{R}^d} \frac{|f(x-z)|}{(1+|b_1 z_1|^{s_1}) \cdot ... \cdot (1+|b_d z_d|^{s_d})} \quad . \tag{11}$$

Theorem 3.1.1 justifies this definition. Additionally we need one more maximal inequality. Details concerning the following assertions can be found in [ST, 1.6.4].

Lemma 3.1.1 Let $\Omega \subset \mathbb{R}^d$ be compact, $\bar{s} = (s_1, ..., s_d) > 0$ and $\bar{\alpha} = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$. Then there exist two constants $c_1, c_2 > 0$ (independently of f) such that

$$\begin{aligned} P_{\bar{1},\bar{s}}(D^{\bar{\alpha}} f)(x) &\leq c_1 P_{\bar{1},\bar{s}} f(x) \\ &\leq c_2 \big(M_d \big(M_{d-1} \big(... \big(M_1 |f|^{1/s_1}\big)^{s_1/s_2} ...\big)^{s_{d-2}/s_{d-1}}\big)^{s_{d-1}/s_d}\big)^{s_d}(x) \end{aligned} \tag{12}$$

holds for all $f \in S'(\mathbb{R}^d)$ with supp $\mathcal{F}f \subset \Omega$ and all $x \in \mathbb{R}^d$. □

This lemma leads to the following important maximal inequality.

Theorem 3.1.4 Let $0 < p < \infty$ and $0 < q \leq \infty$. Let further $\bar{b}^\ell = (b_1^\ell, ..., b_d^\ell) > 0$ for $\ell \in I$ and $\Omega = \{\Omega_\ell\}_{\ell\in I}$, such that

$$\Omega_\ell \subset \{\xi \in \mathbb{R}^d \ : \ |\xi_i| \leq b_i^\ell, \ i = 1, ..., d\}$$

is compact for $\ell \in I$. Finally we fix a tuple $\bar{s} = (s_1, ..., s_d) > \frac{1}{\min(p,q)}$. Under these assumptions there exists a constant $c > 0$ (independently of f and Ω) such that

$$\left\| P_{\bar{b}^\ell,\bar{s}} f_\ell | L_p(\mathbb{R}^d, \ell_q) \right\| \leq c \left\| f_\ell | L_p(\mathbb{R}^d, \ell_q) \right\|$$

holds for all systems $f = \{f_\ell\}_{\ell\in I} \subset L_p^\Omega(\mathbb{R}^d, \ell_q)$.

Proof This assertion is a direct consequence of the previous lemma, Theorem 3.1.3 and a homogeneity argument: Let $\tilde{f}_\ell\,(x_1, ..., x_d) := f_\ell\,(x_1/b_1^\ell, ..., x_d/b_d^\ell)$.
Then it holds

$$P_{\bar{b}^\ell,\bar{s}} f_\ell\,(x_1, ..., x_d) = P_{\bar{1},\bar{s}} \tilde{f}_\ell\,(b_1^{\bar{\ell}}\, x_1, ..., b_d^{\bar{\ell}}\, x_d).$$

Because of supp $\mathcal{F}\tilde{f}_\ell \subset [-1, 1]^d$, the constant c can be chosen independently of Ω. □

For our purposes we need the following modification of (11). Let $\bar{\alpha} \in \{0,1\}^d$ and $\bar{b}$, $\bar{s}$, f the same as in (11). Then we define the modified maximal function $P_{\bar{b},\bar{s},\bar{\alpha}} f$ by

$$P_{\bar{b},\bar{s},\bar{\alpha}} f(x) := \sup_{z \in \mathbb{R}^d} \frac{|f(x - \bar{\alpha} * z)|}{(1 + |b_1 \alpha_1 z_1|^{s_1}) \cdot \ldots \cdot (1 + |b_d \alpha_d z_d|^{s_d})} \quad . \tag{13}$$

Obviously the maximal operator defined in (11) is applied to that directions where the corresponding component of $\bar{\alpha}$ equals one. With the same arguments used for the maximal operator M and M_i, respectively, we obtain the following modification.

Theorem 3.1.5 Assume $p, q, \bar{b}^\ell, \bar{s}$ and $\Omega = \{\Omega_\ell\}_{\ell \in I}$ as in Theorem 3.1.4. Let further $\bar{\alpha} \in \{0,1\}^d$. Then there exists a constant $c > 0$ (independently of f and Ω) such that

$$\big\| P_{\bar{b}^\ell,\bar{s},\bar{\alpha}} f_\ell | L_p(\mathbb{R}^d, \ell_q) \big\| \leq c \big\| f_\ell | L_p(\mathbb{R}^d, \ell_q) \big\| \tag{14}$$

holds for all systems $f = \{f_\ell\}_{\ell \in I} \subset L_p^\Omega(\mathbb{R}^d, \ell_q)$.

Proof We argue analogously to Theorem 3.1.3. Then the assertion follows immediately from Theorem 3.1.4, Theorem 3.1.1 and Remark 3.1.1. $\square$

3.2 Definition and basic properties

Our function spaces are defined by using a smooth dyadic decomposition of unity. For our purpose it is useful to modify the typical definition given for instance in [ST, 2.2.1]. See also [SS, 2.1]. Nevertheless the result is essentially the same. In the following we denote by $C_0^\infty(\mathbb{R}^d)$ the class of all complex-valued infinitely differentiable compactly supported functions on $\mathbb{R}^d$.

Definition 3.2.1 A system $\varphi = \{\varphi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R})$ belongs to the class $\Phi(\mathbb{R})$ if and only if:

(i) It exists $A > 0$, such that supp $\varphi_0 \subset [-A, A]$.

(ii) There are constants $0 < B < C$, such that supp $\varphi_j \subset \{x \in \mathbb{R} : B2^j \leq |x| \leq C2^j\}$.

(iii) For all $\alpha \in \mathbb{N}_0$ holds

$$\sup_{x \in \mathbb{R}, j \in \mathbb{N}_0} 2^{j\alpha} |D^\alpha \varphi_j(x)| \leq c_\alpha < \infty \qquad \text{and}$$

(iv) $\sum\limits_{j=0}^{\infty} \varphi_j(x) = 1.$

$\square$

Remark 3.2.1 The class $\Phi(\mathbb{R})$ is not empty. Consider the following example. Let $D > 0$ and $\varphi_0(x) \in C_0^\infty(\mathbb{R})$ with $\varphi_0(x) = 1$ on $[-D, D]$. For $j > 0$ we define

$$\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x).$$

Now it is easy to see, that the system $\varphi = \{\varphi_j(x)\}_{j=0}^\infty$ satisfies (i) - (iv). $\square$

We need another useful notation. Assume d systems $\varphi^i = \{\varphi^i_j(x)\}_{j=0}^\infty \in \Phi(\mathbb{R})$, $i = 1,...,d$. We define a certain kind of tensor product of these systems. For $\bar{\ell} = (\ell_1,...,\ell_d) \in \mathbb{N}_0^d$ the function $(\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(x) \in C_0^\infty(\mathbb{R}^d)$ is defined by

$$(\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(x) := \varphi^1_{\ell_1}(x_1) \cdot ... \cdot \varphi^d_{\ell_d}(x_d) \quad , \quad x \in \mathbb{R}^d.$$

Because of (iv) in Definition 3.2.1 this leads to the following decomposition of $f \in S'(\mathbb{R}^d)$. Let $f_{\bar{\ell}}(x)$, $\bar{\ell} \in \mathbb{N}_0^d$, be the entire analytic function defined by

$$f_{\bar{\ell}}(x) := \mathcal{F}^{-1}[(\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(\xi)\mathcal{F}f](x).$$

Then it holds

$$f = \sum_{\bar{\ell} \in \mathbb{N}_0^d} f_{\bar{\ell}}(x) \quad , \quad \text{convergence in } S'(\mathbb{R}^d) \ . \tag{15}$$

We introduce the function spaces $S^{\bar{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$ by using these Fourier-analytic building blocks. For details we refer mainly to [ST, Ch.2] and [Vy, Ch.1].

Definition 3.2.2 Let $\bar{r} = (r_1,...,r_d) \in \mathbb{R}^d$, $0 < q \leq \infty$ and $\varphi^i = \{\varphi^i_j(x)\}_{j=0}^\infty \in \Phi(\mathbb{R})$, $i = 1,...,d$.

(i) Let $0 < p \leq \infty$. Then $S^{\bar{r}}_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\|^{\bar{\varphi}} = \left(\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{\bar{r}\cdot\bar{\ell}q} \big\| \mathcal{F}^{-1}[(\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(\xi)\mathcal{F}f](x)|L_p(\mathbb{R}^d)\big\|^q \right)^{1/q} \tag{16}$$

is finite (modification if $q = \infty$).

(ii) Let $0 < p < \infty$. Then $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\|^{\bar{\varphi}} = \left\| \left(\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{\bar{r}\cdot\bar{\ell}q} |\mathcal{F}^{-1}[(\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(\xi)\mathcal{F}f](x)|^q \right)^{1/q} \middle| L_p(\mathbb{R}^d) \right\| \tag{17}$$

is finite (modification if $q = \infty$).

□

Remark 3.2.2 We first observe, that all these classes are (quasi-)Banach spaces (Banach spaces in case $\min(p,q) \geq 1$) which are (quasi-)normed by (16) and (17), respectively. Furthermore the spaces are independent of the chosen decomposition of unity in the sense of equivalent (quasi-)norms. This is essentially a consequence of certain maximal inequalities (cf. Paragraph 3.1) and Fourier-multiplier assertions, proved for instance in [ST, 1.10.3] for the case $d = 2$. □

Let $S^{\bar{r}}_{p,q}A(\mathbb{R}^d)$ be either $S^{\bar{r}}_{p,q}B(\mathbb{R}^d)$ or $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$ (with $p < \infty$ in the F-case). One can prove the following continuous embeddings.

Lemma 3.2.1 Let $(r_1, ..., r_d) \in \mathbb{R}^d$ and $0 < p \le \infty$.

(i) Let $0 < q \le v \le \infty$. Then it holds

$$S^{\bar{r}}_{p,q}A(\mathbb{R}^d) \hookrightarrow S^{\bar{r}}_{p,v}A(\mathbb{R}^d).$$

(ii) Let $0 < q, v \le \infty$ and $\bar{\varepsilon} = (\varepsilon_1, ..., \varepsilon_d)$ with $\bar{\varepsilon} \ge 0$. Then

$$S^{\bar{r}+\bar{\varepsilon}}_{p,q}A(\mathbb{R}^d) \hookrightarrow S^{\bar{r}}_{p,q}A(\mathbb{R}^d).$$

If $\bar{\varepsilon}$ satisfies $\bar{\varepsilon} > 0$, then even the embedding

$$S^{\bar{r}+\bar{\varepsilon}}_{p,q}A(\mathbb{R}^d) \hookrightarrow S^{\bar{r}}_{p,v}A(\mathbb{R}^d)$$

holds.

Proof In [ST, Ch. 2] the case $d = 2$ is treated in detail. The proof in case $d > 2$ is essentially the same. □

The next theorem tells in particular, under which restrictions to the parameters the classes $S^{\bar{r}}_{p,q}A(\mathbb{R}^d)$ consist of exclusively regular distributions. Of course this is of interest, since characterization by differences only makes sense in this case.

Lemma 3.2.2 Let $0 < p, q \le \infty$ and $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$. The following continuous embeddings hold true.

(i) If $\bar{r} > \sigma_p$ (see (1)) then

$$S^{\bar{r}}_{p,q}A(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \cap L_{\max(p,1)}(\mathbb{R}^d)$$

holds with the usual interpretation.

(ii) If even $\bar{r} > \frac{1}{p}$ then

$$S^{\bar{r}}_{p,q}A(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d).$$

$C(\mathbb{R}^d)$ denotes the space of all uniformly continuous, bounded functions, normed with the supremum-norm.

Proof See [ST, 2.2.3] for details. □

Remark 3.2.3 In case $0 < p < 1$ it can easily be shown that the δ-distribution belongs to the space $S^{\bar{r}}_{p,\infty}B(\mathbb{R}^d)$ with $r_i = \sigma_p$, $i = 1, ..., d$. □

3.3 Differences versus maximal functions

Definitions

We define Mth order as well as corresponding mixed differences. Essentially the same notation as stated in [ST, 2.3.3] is used. Fix $h \in \mathbb{R}$. Under a first order difference with steplength h of a function $f : \mathbb{R} \to \mathbb{C}$ we want to understand the function $\triangle_h f$ defined by

$$\triangle_h f(x) = f(x + h) - f(x) \quad , \quad x \in \mathbb{R}.$$

Iteration leads to Mth order differences, given by

$$\triangle_h^M f(x) = \triangle_h(\triangle_h^{M-1} f)(x) \quad , \quad M \in \mathbb{N} \quad , \quad \triangle_h^0 = Id. \tag{18}$$

Using mathematical induction one can show the explicit formula

$$\triangle_h^M f(x) = \sum_{j=0}^{M} (-1)^j \binom{M}{j} f(x + (M-j)h). \tag{19}$$

For our special purpose we need differences with respect to a certain component of f as well as mixed differences. Let us first define the operator $\triangle_{h,i}^m f$ applied to a function $f : \mathbb{R}^d \to \mathbb{C}$. With an eye on (19) we define

$$\triangle_{h,i}^m f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x_1, ..., x_i + (m-j)h, x_{i+1}, ..., x_d) \quad , \tag{20}$$

where $m \in \mathbb{N}_0,\ h \in \mathbb{R},\ i = 1, ..., d$ and $x = (x_1, ..., x_d) \in \mathbb{R}^d$. Hence one obtains similar to (18) for $m \in \mathbb{N}$ the recursion formula

$$\triangle_{h,i}^m f(x) = \triangle_{h,i}^1(\triangle_{h,i}^{m-1} f)(x) \quad , \quad i = 1, ..., d.$$

The combination of such operators (acting on different components) is called mixed difference. For later use we need a good abbreviating symbol. Let $\bar{h} \in \mathbb{R}^d$, $\bar{\alpha} \in \{0,1\}^d$ and $\bar{\delta}$ be assigned to $|\bar{\alpha}| = n \geq 1$ in the sense of (2). Let us further define the operator

$$\triangle_{\bar{h},\bar{\alpha}}^{\bar{m}} := \Big(\prod_{i=1}^{n} \triangle_{h_{\delta_i},\delta_i}^{m_{\delta_i}}\Big) := \begin{cases} \triangle_{h_{\delta_1},\delta_1}^{m_{\delta_1}} \circ \cdots \circ \triangle_{h_{\delta_n},\delta_n}^{m_{\delta_n}} & : \ |\bar{\alpha}| = n \\ Id & : \ |\bar{\alpha}| = 0 \end{cases} . \tag{21}$$

We try to avoid the product-symbol. Nevertheless in few cases it is useful. If this is the case we will refer to (21) to recall the exact definition.

Differences and maximal functions

We want to develop some tools to estimate differences by maximal functions.
The first inequality is obvious but nevertheless essential. Let $m \in \mathbb{N}_0$ and f locally integrable. Then it holds for almost all $x \in \mathbb{R}^d$

$$\int_{-1}^{1} |f(x + mh)|\, dh \leq 2 \cdot Mf(x). \tag{22}$$

In the following we concentrate on estimating differences by maximal functions, defined in (11) and (13). See also [ST, Lemma 2.3.3].

Lemma 3.3.1 Let $a, b > 0$, $m \in \mathbb{N}$, $h \in \mathbb{R}$ and $f \in S'(\mathbb{R})$ with supp $\mathcal{F}f \subset [-b, b]$. Then there exists a constant $c > 0$ independently of f, b and h such that

$$|\triangle_h^m f(x)| \leq c \max(1, |bh|^a) \min(1, |bh|^m) P_{b,a} f(x)$$

holds for all $x \in \mathbb{R}$.

Proof The main ingredients of the proof are the left-hand inequality in Lemma 3.1.1 and the mean-value theorem of calculus. Because of Theorem 3.1.1 the distribution f is an entire analytic function and hence the restriction to $\mathbb{R}$ is C^∞. So the mean value theorem gives us ξ with $|\xi - x| \leq h$ such that

$$| \triangle_h^m f(x)| = f'(\xi) \cdot h$$

Iteration of this argument leads to

$$\begin{aligned} | \triangle_h^m f(x)| &= | \triangle_h^1 (\triangle_h^{m-1} f)(x)| \\ &\leq \sup_{|y|\leq h} | \triangle_h^{m-1} f'(x-y)| \cdot |h| \\ &\vdots \\ &\leq |h|^m \sup_{|y|\leq mh} \frac{|f^{(m)}(x-y)|}{1+|by|^a}(1+|by|^a). \end{aligned}$$

Hence we obtain

$$\begin{aligned} | \triangle_h^m f(x)| &\leq |h|^m \sup_{y\in\mathbb{R}} \frac{|f^{(m)}(x-y)|}{1+|by|^a} \cdot \sup_{|y|\leq mh} (1+|by|^a) \\ &= |h|^m \sup_{y\in\mathbb{R}} \frac{|f^{(m)}(x-y)|}{1+|by|^a} \cdot (1+m^a|bh|^a) \\ &\leq \underbrace{2m^a}_{=:c_{m,a}} \cdot |h|^m \max(1,|bh|^a) \sup_{y\in\mathbb{R}} \frac{|f^{(m)}(x-y)|}{1+|by|^a}. \end{aligned} \tag{23}$$

The rest is a consequence of a homogeneity argument. To see this let $g \in S'(\mathbb{R})$ s.t. supp $\mathcal{F}g \subset [-1,1]$. Then it follows by (23)

$$| \triangle_h^m g(x)| \leq c_{m,a}|h|^m \max(1,|h|^a) \cdot \sup_y \frac{|g^{(m)}(x-y)|}{1+|y|^a}. \tag{24}$$

We apply (24) to the function $g(x) = f(x/b)$. It is easy to see, that $g \in S'(\mathbb{R})$ and supp $\mathcal{F}g \subset [-1,1]$. Furthermore it holds

$$\triangle_h^m f(x) = \triangle_{bh}^m g(bx).$$

Hence (24) gives us

$$\begin{aligned} | \triangle_h^m f(x)| &= | \triangle_{bh}^m g(bx)| \\ &\leq c_{m,a}|bh|^m \max(1,|bh|^a) \cdot \sup_y \frac{|g^{(m)}(bx-y)|}{1+|y|^a}. \end{aligned}$$

At next we use the left hand side of Lemma 3.1.1 and obtain

$$\begin{aligned} | \triangle_h^m f(x)| &\leq c'_{m,a}|bh|^m \max(1,|bh|^a) \sup_y \frac{|g(bx-y)|}{1+|y|^a} \\ &= c'_{m,a}|bh|^m \max(1,|bh|^a) \sup_y \frac{|g(b(x-y))|}{1+|by|^a} \\ &= c'_{m,a}|bh|^m \max(1,|bh|^a) P_{b,a} f(x). \end{aligned} \tag{25}$$

On the other hand we can estimate directly using (19). This yields

$$
\begin{aligned}
|\triangle_h^m f(x)| &= \left|\sum_{j=0}^{m}(-1)^j\binom{m}{j}f(x+(m-j)h)\right| \\
&\le c_m \sup_{|y|\le mh}\frac{|f(x-y)|}{1+|by|^a}(1+|by|^a) \\
&\le 2c''_m \sup_{y\in\mathbb{R}}\frac{|f(x-y)|}{1+|by|^a}\cdot\max(1,|mhb|^a) \\
&\le c''_{m,a}\max(1,|hb|^a)P_{b,a}f(x).
\end{aligned}
\tag{26}
$$

And finally (25) together with (26) completes the proof. □

The following lemma generalizes the previous one to d dimensions. For sake of brevity we define

$$A(s,t) := \max(1,s)\min(1,t) \quad , \quad \text{for } s,t\ge 0. \tag{27}$$

A simple property consists in the following. For fixed $v,w>0$ there is a constant $c_{v,w}>0$ such that

$$A(v\cdot s, w\cdot t)\le c_{v,w}A(s,t) \tag{28}$$

for all $s,t\ge 0$.

Lemma 3.3.2 Let $\bar{a}=(a_1,...,a_d),\bar{b}=(b_1,...,b_d)\in\mathbb{R}^d$ satisfying $\bar{a},\bar{b}>0$. Let further be $\bar{m}=(m_1,...,m_d)\in\mathbb{N}^d$, $\bar{h}=(h_1,...,h_d)\in\mathbb{R}^d$ and $f\in S'(\mathbb{R}^d)$ with supp $\mathcal{F}f\subset Q_{\bar{b}}$, where

$$Q_{\bar{b}} := [-b_1,b_1]\times ...\times[-b_d,b_d].$$

Then there exists a constant $c>0$ (independent of f, $\bar{b}$ and $\bar{h}$) such that

$$
\begin{aligned}
&|(\triangle_{h_1,1}^{m_1}\circ\triangle_{h_2,2}^{m_2}\circ\cdots\circ\triangle_{h_d,d}^{m_d}f)(x)| \\
&\le c\cdot A(|b_1h_1|^{a_1},|b_1h_1|^{m_1})\cdot ...\cdot A(|b_dh_d|^{a_d},|b_dh_d|^{m_d})\cdot P_{\bar{b},\bar{a}}f(x)
\end{aligned}
$$

holds for all $x\in\mathbb{R}^d$.

Proof The idea is to iterate the previous lemma. We define the function g by

$$g=(\triangle_{h_2,2}^{m_2}\circ\cdots\circ\triangle_{h_d,d}^{m_d}f)(x).$$

Because of

$$\mathcal{F}g=(e^{i\xi_2h_2}-1)^{m_2}\cdot ...\cdot(e^{i\xi_dh_d}-1)^{m_d}\mathcal{F}f$$

the inclusion supp $\mathcal{F}g\subset$ supp $\mathcal{F}f\subset Q_{\bar{b}}$ holds. Let us now fix the components $x_2,...,x_d$ and consider the function

$$\tilde{g}:=g(\cdot,x_2,...,x_d).$$

Remark 3.1.1 after Theorem 3.1.1 gives us $\tilde{g}\in S'(R)$ and supp $\mathcal{F}\tilde{g}\subset[-b_1,b_1]$. Finally we use Lemma 3.3.1 with $\triangle_{h_1}^{m_1}\tilde{g}(x_1)=\triangle_{h_1,1}^{m_1}g(x_1,...,x_d)$ and obtain

$$
\begin{aligned}
|(\triangle_{h_1,1}^{m_1}\circ\triangle_{h_2,2}^{m_2}\circ\cdots\circ\triangle_{h_d,d}^{m_d}f)(x)| &= \triangle_{h_1}^{m_1}\tilde{g}(x_1) \\
&\le c_{a_1,m_1}A(|b_1h_1|^{a_1},|b_1h_1|^{m_1})\cdot P_{b_1,a_1}\tilde{g}(x_1) \\
&= c_{a_1,m_1}A(|b_1h_1|^{a_1},|b_1h_1|^{m_1})\cdot\sup_{y_1}\frac{|g(x_1-y_1,x_2,...,x_d)|}{(1+|b_1y_1|^{a_1})}.
\end{aligned}
$$

One continues by estimating $|g(x_1-y_1,x_2,...,x_d)|$ analogously. Iteration of this procedure finishes the proof. $\square$

Remark 3.3.1 With exactly the same arguments one proves a version of the previous lemma to estimate mixed differences of type $\triangle^{\bar{m}}_{\bar{h},\bar{\alpha}}f(x)$ by the maximal function $P_{\bar{b},\bar{a},\bar{\alpha}}f$. $\square$

3.4 Integral means of differences for $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$

Our main result is a characterization of the spaces $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$, $0<p<\infty$, $0<q\le\infty$ and $\bar{r}>\sigma_{p,q}$, using integral means (rectangle means) of differences. In some sense it is the counterpart of [Tr 2, Th. 2.5.11], where the isotropic scale is treated in terms of ball means. It will be also used as basis for further difference characterization later in this paper.

Theorem 3.4.1 Let $0<p<\infty$, $0<q\le\infty$, $\bar{r}=(r_1,...,r_d)\in\mathbb{R}^d$ and $\bar{m}=(m_1,...,m_d)\in\mathbb{N}^d$ with $\bar{m}>\bar{r}>\sigma_{p,q}$. Under these conditions the space $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all functions $f\in L_p(\mathbb{R}^d)\cap S'(\mathbb{R}^d)$, such that

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\|^R=\|f|L_p(\mathbb{R}^d)\|+\sum_{\bar{\beta}\in\{0,1\}^d,|\beta|\ge1}S^R_{\bar{\beta}}(f)<\infty \quad , \tag{29}$$

where for $|\bar{\beta}|=n\ge1$

$$S^R_{\bar{\beta}}(f)=\Big\|\Big[\int_{(0,\infty)^n}\Big(\prod_{i=1}^n t_i^{-r_{\delta_i}q}\Big)\Big(\int_{[-1,1]^n}\big|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{t_nh_n,\delta_n}f)(x)\big|d\bar{h}\Big)^q\frac{d\bar{t}}{\bar{t}}\Big]^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| \tag{30}$$

with $\bar{\delta}$ assigned to $\bar{\beta}$ in the sense of (2). In case $q=\infty$ one has to replace (30) by

$$S^R_{\bar{\beta}}(f)=\Big\|\sup_{\bar{t}\in(0,\infty)^n}\Big(\prod_{i=1}^n t_i^{-r_{\delta_i}}\Big)\int_{[-1,1]^n}\Big|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{t_nh_n,\delta_n}f)(x)\Big|d\bar{h}\Big|L_p(\mathbb{R}^d)\Big\|.$$

Moreover (29) is an equivalent (quasi-)norm in $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$.

Proof The proof will be divided into 2 steps.

Step 1: We fix $\bar{\varphi}=(\varphi^1,...,\varphi^d)\in\Phi(\mathbb{R})^d$. The aim is to show that it exists a constant $c>0$ such that

$$\|f|S^{\bar{r}}_{p,q}F\|^R\le c\|f|S^{\bar{r}}_{p,q}F\|^{\bar{\varphi}}$$

holds for every $f\in S'(\mathbb{R}^d)\cap L_p(\mathbb{R}^d)$. The basic idea is to use (15) and proceed by estimating the appearing differences by maximal functions via (22), Lemma 3.3.1 and Lemma 3.3.2. The rest will be done by exploiting the maximal inequalities, Theorem 3.1.3 and Theorem 3.1.5. We follow [Tr 2, 2.5.11] and apply the isotropic strategy in a certain sense to every direction. Let us make some further preparation. We choose a tuple $\bar{a}=(a_1,...,a_d)>\frac{1}{\min(p,q)}$ and a number $0<\lambda<\min(p,q)$ such that $\bar{r}>(1-\lambda)\bar{a}$. It is easy to see that this is possible: As a consequence of $\bar{r}>\sigma_{p,q}$ we have in case $\min(p,q)\le1$

$$r_i>\frac{1}{\min(p,q)}(1-\min(p,q)) \quad , \quad i=1,...,d.$$

In case $\min(p,q) > 1$ we simply choose $\lambda = 1$.
Assume first $q < \infty$. We start by considering the expression $A_{\bar{\beta}}(x)$ defined by

$$A_{\bar{\beta}}(x) = \int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^{n} t_i^{-r_{\delta_i} q}\Big)\bigg(\int\limits_{[-1,1]^n} \big|\big(\triangle^{m_{\delta_1}}_{t_1 h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n,\delta_n} f\big)(x)\big| d\bar{h}\bigg)^q \frac{d\bar{t}}{\bar{t}} \quad , \tag{31}$$

where $\bar{\beta}$ with $|\bar{\beta}| = n > 0$ (and corresponding $\bar{\delta}$) is fixed. In order to discretisise (31) we obtain with elementary calculations

$$\begin{aligned} A_{\bar{\beta}}(x) &\le c \sum_{\bar{k}\in Z_{\bar{\beta}}} \int\limits_{Q^{+\Delta}_{-\bar{k},\bar{\beta}}} 2^{\bar{r}\bar{k}q} \bigg(2^{\bar{k}} \int\limits_{Q_{-\bar{k},\bar{\beta}}} \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big|\, d\bar{h}\bigg)^q \frac{d\bar{t}}{2^{-\bar{k}}} \\ &\le c \sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}\bar{k}q} \bigg(2^{\bar{k}} \int\limits_{Q_{-\bar{k},\bar{\beta}}} \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big|\, d\bar{h}\bigg)^q \\ &= c \sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}\bar{k}q} \bigg(\int\limits_{[-1,1]^n} \big|(\triangle^{m_{\delta_1}}_{2^{-k_{\delta_1}} h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{2^{-k_{\delta_n}} h_n,\delta_n} f)(x)\big|\, d\bar{h}\bigg)^q \\ &\le c \sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}\bar{k}q} \bigg(\int\limits_{[-1,1]^d} \big|\, \triangle^{\bar{m}}_{2^{-\bar{k}} * \bar{h},\bar{\beta}} f(x)\big|\, d\bar{h}\bigg)^q . \end{aligned} \tag{32}$$

For the used notation we refer to (3). Recall the Fourier analytic decomposition of f in

$$f_{\bar{\ell}}(x) := \mathcal{F}^{-1}[(\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(\xi)\mathcal{F}f](x) \quad , \quad \bar{\ell} \in \mathbb{N}_0^d.$$

See (15) for details. Obviously it holds for every $\bar{k} \in Z_{\bar{\beta}}$

$$f = \sum_{\bar{\ell}\in Z_{\bar{\beta}}} \sum_{\bar{u}\in \bar{N}_{\bar{\beta}}} f_{\bar{k}+\bar{\ell}+\bar{u}} \tag{33}$$

in $S'(\mathbb{R}^d)$, where we put $\varphi^i_\ell \equiv 0$, $i = 1, ..., d$ if $\ell < 0$. Hence we obtain for $q \le 1$

$$A_{\bar{\beta}}(x) \le c \sum_{\bar{\ell}\in Z_{\bar{\beta}}} \sum_{\bar{u}\in \bar{N}_{\bar{\beta}}} \sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}\bar{k}q} \bigg(\int\limits_{[-1,1]^d} \big|\, \triangle^{\bar{m}}_{2^{-\bar{k}} * \bar{h},\bar{\beta}} f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\big|\, d\bar{h}\bigg)^q . \tag{34}$$

Precisely, we used the unconditional $L_1([-1,1]^n)$-convergence (with respect to $\bar{h}$) of the sum

$$\sum_{\bar{\ell}\in Z_{\bar{\beta}}} \sum_{\bar{u}\in \bar{N}_{\bar{\beta}}} \triangle^{\bar{m}}_{2^{-\bar{k}} * \bar{h},\bar{\beta}} f_{\bar{k}+\bar{\ell}+\bar{u}}(x) \quad ,$$

where x is fixed. This follows from their absolute L_1-convergence and (33). In some sense the arguments are justified if one reads the estimates backwards.
If $q > 1$ the triangle inequality in ℓ_q gives us the estimate

$$A_{\bar{\beta}}(x)^{1/q} \le c \sum_{\bar{\ell}\in Z_{\bar{\beta}}} \sum_{\bar{u}\in \bar{N}_{\bar{\beta}}} \bigg[\sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}\bar{k}q} \bigg(\int\limits_{[-1,1]^d} \big|\, \triangle^{\bar{m}}_{2^{-\bar{k}} * \bar{h},\bar{\beta}} f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\big|\, d\bar{h}\bigg)^q \bigg]^{1/q} \tag{35}$$

instead of (34). We continue estimating $A_{\bar\beta}(x)$ in case $q \le 1$. For the moment we postpone the case $q > 1$. At next we decompose RHS(34) into the following blocks with respect to the $\bar\ell$-sum:

$$A^{\bar\alpha}_{\bar\beta}(x) = \sum_{\bar\ell \in Z^{\bar\alpha}_{\bar\beta}} \sum_{\bar u \in \bar N_{\bar\beta}} \sum_{\bar k \in Z_{\bar\beta}} 2^{\bar r \bar k q} \Bigg(\int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\beta} \, f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h \Bigg)^q \quad , \tag{36}$$

where $\bar\alpha \le \bar\beta$,

$$\begin{aligned} Z^{\bar\alpha}_{\bar\beta} = \big\{ (k_1, ..., k_d) \in \mathbb{Z}^d : (\beta_i = 0 \implies k_i = 0) \\ \wedge\ ((\alpha_i, \beta_i) = (0,1) \implies k_i \ge 0) \\ \wedge\ ((\alpha_i, \beta_i) = (1,1) \implies k_i \le 0)\ ,\ i = 1, ..., d \big\} \end{aligned}$$

and hence $Z_{\bar\beta} \subset \bigcup\limits_{\bar\alpha \le \bar\beta} Z^{\bar\alpha}_{\bar\beta}$. Consequently

$$A_{\bar\beta}(x) \le c \sum_{\bar\alpha \le \bar\beta} A^{\bar\alpha}_{\bar\beta}(x). \tag{37}$$

We investigate the behavior of the integral

$$\int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\beta} \, f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h = \int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\alpha} \circ \triangle^{\bar m}_{2^{-\bar k} * \bar h, (\bar\beta - \bar\alpha)} \, f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h \tag{38}$$

inside RHS(36) for a fixed $\bar\alpha \le \bar\beta$. To avoid technical difficulties we only consider the special situation $\bar\beta = (1, ..., 1, 0, ..., 0)$ and $\bar\alpha = (1, ..., 1, 0, ..., 0)$, where $|\bar\alpha| \le |\bar\beta|$. All other cases can be treated analogously. One only has to change the order of difference operators appropriately. With the help of Lemma 3.3.2 and Remark 3.3.1, respectively, we get rid of the first part $\triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\alpha}$ of the mixed difference in (38). Estimation by a maximal function of type (13) using Lemma 3.3.2 and (28) yields

$$\begin{aligned} &\int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\alpha} \circ \triangle^{\bar m}_{2^{-\bar k} * \bar h, (\bar\beta - \bar\alpha)} \, f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h \\ &\le c_1 A(|2^{\ell_1 + k_1} 2^{-k_1}|^{a_1}, |2^{\ell_1 + k_1} 2^{-k_1}|^{m_1}) \cdot ... \cdot A(|2^{\ell_{|\bar\alpha|} + k_{|\bar\alpha|}} 2^{-k_{|\bar\alpha|}}|^{a_{|\bar\alpha|}}, |2^{\ell_{|\bar\alpha|} + k_{|\bar\alpha|}} 2^{-k_{|\bar\alpha|}}|^{m_{|\bar\alpha|}}) \\ &\quad \cdot \int\limits_{[-1,1]^d} P_{\bar b, \bar a, \bar\alpha} (\triangle^{\bar m}_{2^{-\bar k} * \bar h, (\bar\beta - \bar\alpha)} f_{\bar k + \bar\ell + \bar u})(x) \, d\bar h \quad , \end{aligned} \tag{39}$$

where $\bar a$ was chosen together with λ right at the beginning. Additionally we put $\bar b = (2^{k_1 + \ell_1}, ..., 2^{k_d + \ell_d})$. Recall the fact $\ell_1, ..., \ell_{|\bar\alpha|} \le 0$. In case $|\bar\alpha| < |\bar\beta| = n$ this simplifies estimate (39) to

$$\begin{aligned} &\int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\alpha} \circ \triangle^{\bar m}_{2^{-\bar k} * \bar h, (\bar\beta - \bar\alpha)} \, f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h \\ &\le c_1 2^{\ell_1 m_1} \cdot ... \cdot 2^{\ell_{|\bar\alpha|} m_{|\bar\alpha|}} \int\limits_{[-1,1]^d} P_{\bar b, \bar a, \bar\alpha} (\triangle^{\bar m}_{2^{-\bar k} * \bar h, (\bar\beta - \bar\alpha)} f_{\bar k + \bar\ell + \bar u})(x) \, d\bar h \ , \end{aligned} \tag{40}$$

whereas we otherwise (in case $|\bar{\alpha}| = |\bar{\beta}|$) end up with

$$\int\limits_{[-1,1]^d} \big| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,\bar{\alpha}} \circ \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\big|\, d\bar{h} \le c_1 2^{\ell_1 m_1} \cdot \ldots \cdot 2^{\ell_n m_n} P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})(x)$$

instead of (40). To proceed with (40) we need a new strategy. The remaining difference operator inside (40) acts on the components of $f_{\bar{k}+\bar{\ell}+\bar{u}}(x)$, that correspond to the $\bar{\ell}$-components in $Z^{\bar{\alpha}}_{\bar{\beta}}$, which run over $\mathbb{N}_0$. Of course it can be estimated in the same manner as done above. We obtain by Lemma 3.3.2

$$P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})(x) \le c 2^{\ell_{|\bar{\alpha}|+1} a_{|\bar{\alpha}|+1}} \cdot \ldots \cdot 2^{\ell_n a_n} P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})(x). \tag{41}$$

Let us first split the integrand in RHS(40) into the product

$$\begin{aligned} &P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})(x) \\ &= |P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})|^{1-\lambda}(x) \cdot |P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})|^{\lambda}(x). \end{aligned} \tag{42}$$

In case $\min(p,q) > 1$ there is no such splitting necessary, as $\lambda = 1$ indicates. Moreover the difference $\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}}$ can be expanded to a sum with the help of (20). Using an obvious subadditivity property of the operator $P_{\bar{b},\bar{a},\bar{\alpha}}$, we can write the sum in front of it. This yields

$$P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})(x) \le c_2 \sum_{\bar{w}\in M_{\bar{\beta}-\bar{\alpha}}} c_{\bar{w}} P_{\bar{b},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{\ell}+\bar{u}})(x + \bar{w} * 2^{-\bar{k}} * \bar{h}) \quad , \tag{43}$$

where we refer to (6) for the index notation. Combining (41), (43) and (42) we obtain

$$\begin{aligned} &\int\limits_{[-1,1]^d} \big| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,\bar{\alpha}} \circ \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\big|\, d\bar{h} \\ &\le c_3 2^{\ell_1 m_1} \cdot \ldots \cdot 2^{\ell_{|\bar{\alpha}|} m_{|\bar{\alpha}|}} 2^{\ell_{|\bar{\alpha}|+1} a_{|\bar{\alpha}|+1}(1-\lambda)} \cdot \ldots \cdot 2^{\ell_n a_n (1-\lambda)} |P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{1-\lambda}(x) \times \\ &\times \sum_{\bar{w}\in M_{\bar{\beta}-\bar{\alpha}}} c_{\bar{w}} \int\limits_{[-1,1]^d} |P_{\bar{b},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{\lambda}(x + \bar{w} * 2^{-\bar{k}} * \bar{h})\, d\bar{h}. \end{aligned}$$

And finally Lemma 22 leads to

$$\begin{aligned} &\int\limits_{[-1,1]^d} \big| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,\bar{\alpha}} \circ \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\big|\, d\bar{h} \\ &\le c_4 2^{\ell_1 m_1} \cdot \ldots \cdot 2^{\ell_{|\bar{\alpha}|} m_{|\bar{\alpha}|}} 2^{\ell_{|\bar{\alpha}|+1} a_{|\bar{\alpha}|+1}(1-\lambda)} \cdot \ldots \cdot 2^{\ell_n a_n (1-\lambda)} |P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{1-\lambda}(x) \\ &\quad \cdot M_{|\bar{\alpha}|+1} \circ \ldots \circ M_n(|P_{\bar{b},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{\lambda})(x). \end{aligned} \tag{44}$$

We return to $A^{\bar{\alpha}}_{\bar{\beta}}(x)$. Together with (44) we obtain in our special situation

$$\begin{aligned} A^{\bar{\alpha}}_{\bar{\beta}}(x) \le & c_4 \sum_{\bar{\ell}\in Z^{\alpha}_{\bar{\beta}}} 2^{\ell_1(m_1-r_1)q} \cdot \ldots \cdot 2^{\ell_{|\bar{\alpha}|}(m_{|\bar{\alpha}|}-r_{|\bar{\alpha}|})q} \\ & \cdot 2^{\ell_{|\bar{\alpha}|+1}\left[a_{|\bar{\alpha}|+1}(1-\lambda)-r_{|\bar{\alpha}|+1}\right]q} \cdot \ldots \cdot 2^{\ell_n[a_n(1-\lambda)-r_n]q} \\ & \cdot \sum_{\bar{u}\in \bar{N}_{\bar{\beta}}} \sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}(\bar{k}+\bar{\ell})q} |P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{(1-\lambda)q}(x) \\ & \cdot \big|M_{|\bar{\alpha}|+1} \circ \ldots \circ M_n(|P_{\bar{b},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{\lambda})\big|^q(x). \end{aligned} \tag{45}$$

After modifying the sum over $\bar{k}$ such that it gets independent of $\bar{\ell}$, the sum over $\bar{\ell}$ is nothing more than a geometric series and breaks down to a constant. This is a consequence of $\bar{m} > \bar{r}$ and the second condition to λ (and $\bar{a}$). Hence we arrive at

$$\begin{aligned} A^{\bar{\alpha}}_{\bar{\beta}}(x) \leq & c_5 \sum_{\bar{u}\in\bar{N}_{\bar{\beta}}} \sum_{\bar{k}\in N_{\bar{\beta}}} 2^{\bar{r}\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{u}})|^{(1-\lambda)q}(x) \cdot \big|M_{|\bar{\alpha}|+1} \circ ... \circ M_n(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{u}})|^{\lambda})\big|^q(x) \\ = & c_5 \sum_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}} f_{\bar{k}}|^{(1-\lambda)q}(x) \cdot \big|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}|^{\lambda})\big|^q(x) \quad , \end{aligned}$$

where $\mathbf{M}_{\bar{\gamma}} = M_{\delta_1} \circ ... \circ M_{\delta_{|\bar{\gamma}|}}$ and $\bar{\delta}$ belongs to $\bar{\gamma} \in \{0,1\}^d$ as usual. Altogether we obtained the following estimates for (C_1) $|\bar{\alpha}| < |\bar{\beta}|$ and for (C_2) $|\bar{\alpha}| = |\bar{\beta}|$.

$$A^{\bar{\alpha}}_{\bar{\beta}}(x) \leq c_5 \begin{cases} \sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}} f_{\bar{k}}|^{(1-\lambda)q}(x) \cdot \big|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}|^{\lambda})\big|^q(x) & : \ C_1 \\ \sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}}(f_{\bar{k}})|^q(x) & : \ C_2 \end{cases} . \tag{46}$$

The case $\bar{\alpha} = (0,...,0)$, i.e. $|\bar{\alpha}| = 0$, fits into (C_1). One only has to replace $P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}$ simply by $f_{\bar{k}}$. With obvious modifications we obtain (46) for arbitrary $\bar{\beta}$ and $\bar{\alpha} \leq \bar{\beta}$. Inequality (46) invites us to exploit the Theorems 3.1.3 and 3.1.5 for estimating $\|A^{\bar{\alpha}}_{\bar{\beta}}(x)^{1/q}|L_p\|$. Hence we obtain in case (C_2)

$$\begin{aligned} \|A^{\bar{\alpha}}_{\bar{\beta}}(x)^{1/q}|L_p\| \leq & c_5 \left\| \Big(\sum_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}}(f_{\bar{k}})|^q(x) \Big)^{1/q} \right| L_p \Big\| \\ = & c_5 \, \big\| P_{2^{\bar{k}},\bar{a},\bar{\beta}}(2^{(\bar{r}*\bar{\beta})\cdot\bar{k}} f_{\bar{k}}) \big| L_p(\ell_q) \big\| \\ \leq & c_6 \, \big\| 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}} f_{\bar{k}} \big| L_p(\ell_q) \big\| \\ \leq & c_7 \, \|f|S^{\bar{r}}_{p,q}F\|^{\bar{\varphi}}. \end{aligned} \tag{47}$$

Let us now consider the case (C_1). We begin using Hölder's inequality for sums with the exponents $\frac{1}{\lambda}$ and $\frac{1}{1-\lambda}$ and obtain

$$A^{\bar{\alpha}}_{\bar{\beta}}(x) \leq c_5 \Big(\sum_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} \big|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}|^{\lambda})\big|^{q/\lambda}(x) \Big)^{\lambda} \Big(\sum_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}} f_{\bar{k}}|^q(x) \Big)^{1-\lambda}.$$

Applying the L_p-(quasi-)norm to the previous inequality it follows

$$\begin{aligned} & \|A^{\bar{\alpha}}_{\bar{\beta}}(x)^{1/q}|L_p\| \\ & \leq c_5 \left\| \Big(\sum_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} \big|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}|^{\lambda})\big|^{q/\lambda}(x) \Big)^{\frac{\lambda p}{q}} \times \Big(\sum_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}} f_{\bar{k}}|^q(x) \Big)^{\frac{(1-\lambda)p}{q}} \right| L_1 \Big\|^{1/p} \\ & \leq c_5 \left\| \Big(\sum_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} \big|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}|^{\lambda})\big|^{q/\lambda}(x) \Big)^{\frac{p}{q}} \right| L_1 \Big\|^{\lambda/p} \times \\ & \qquad \times \left\| \Big(\sum_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}} f_{\bar{k}}|^q(x) \Big)^{\frac{p}{q}} \right| L_1 \Big\|^{(1-\lambda)/p}. \end{aligned}$$

Again we used Hölder's inequality with the same exponents, but this time for integrals. Rewriting of the last inequality shows

$$\begin{aligned}\|A^{\bar\alpha}_{\bar\beta}(x)^{1/q}|L_p\| \le & c_5\big\|\mathbf{M}_{\bar\beta-\bar\alpha}(|P_{2^{\bar k},\bar a,\bar\alpha}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|^{\lambda})(x)\big|L_{p/\lambda}(\ell_{q/\lambda})\big\|\times \\ & \times\big\|P_{2^{\bar k},\bar a,\bar\beta}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})(x)\big|L_p(\ell_q)\big\|^{1-\lambda}.\end{aligned} \tag{48}$$

The second factor can be estimated analogously to (47), i.e

$$\begin{aligned}\big\|P_{2^{\bar k},\bar a,\bar\beta}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})(x)\big|L_p(\ell_q)\big\|^{1-\lambda} &\le c_5\|2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k}|L_p(\ell_q)\|^{1-\lambda} \\ &\le c_7\|f|S^{\bar r}_{p,q}F\|^{1-\lambda}.\end{aligned} \tag{49}$$

Because of $p/\lambda, q/\lambda > 1$ and Theorem 3.1.3 the first part of the product (48) reduces to

$$\begin{aligned}\Big\|\mathbf{M}_{\bar\beta-\bar\alpha}(|P_{2^{\bar k},\bar a,\bar\alpha}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|^{\lambda})(x)\Big|\,L_{p/\lambda}(\ell_{q/\lambda})\Big\| &\le c_8\,\|\,|P_{2^{\bar k},\bar a,\bar\alpha}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|^{\lambda}|L_{p/\lambda}(\ell_{q/\lambda})\| \\ &= c_8\,\|P_{2^{\bar k},\bar a,\bar\alpha}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|L_p(\ell_q)\|^{\lambda} \\ &\le c_9\,\|2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|L_p(\ell_q)\|^{\lambda} \\ &\le c_{10}\,\|f|S^{\bar r}_{p,q}F\|^{\lambda}.\end{aligned}$$

This, combined with (48) and (49), yields

$$\|A^{\bar\alpha}_{\bar\beta}(x)^{1/q}|L_p\| \le c_{11}\|f|S^{\bar r}_{p,q}F\|^{\bar\varphi}.$$

Hence (37) yields

$$S^R_{\bar\beta}(f) = \|A^{1/q}_{\bar\beta}(x)|L_p\| \le c_{11}\|f|S^{\bar r}_{p,q}F\|^{\bar\varphi}.$$

The rest follows by Theorem 3.2.2/(ii). We finish the case $q\le 1$. To complete step 1 it remains to describe the necessary modifications in case $q>1$. Let us start with (35).
The condition $\bar r>0$ allows the following estimation.

$$\begin{aligned}A^{1/q}_{\bar\beta}(x) &\le c\sum_{\bar\ell\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}}2^{-\bar r\cdot\bar u}2^{\bar r\cdot\bar u}[\cdots]^{1/q} \le c\sum_{\bar\ell\in Z_{\bar\beta}}\sup_{\bar u}2^{\bar r\cdot\bar u}[\cdots]^{1/q}\sum_{\bar u\in\bar N_{\bar\beta}}2^{-\bar r\cdot\bar u} \\ &\le c'\sum_{\bar\ell\in Z_{\bar\beta}}\Bigg(\sum_{\bar u\in\bar N_{\bar\beta}}2^{\bar r\cdot\bar u q}[\cdots]\Bigg)^{1/q} \\ &= c'\sum_{\bar\ell\in Z_{\bar\beta}}\Bigg(\sum_{\bar k\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}}2^{\bar r\cdot\bar k q}2^{\bar r\cdot\bar u q}\Bigg(\int\limits_{[-1,1]^d}\big|\,\triangle^{\bar m}_{2^{-\bar k}*\bar h,\,\bar\beta}\,f_{\bar k+\bar\ell+\bar u}(x)\big|\,d\bar h\Bigg)^{q}\Bigg)^{1/q}.\end{aligned} \tag{50}$$

Next we decompose the sum over $\bar\ell$ in the same way we did in (37) and put

$$A^{\bar\alpha}_{\bar\beta}(x)^{1/q} = \sum_{\bar\ell\in Z^{\bar\alpha}_{\bar\beta}}\Bigg(\sum_{\bar u\in\bar N_{\bar\beta}}\sum_{\bar k\in Z_{\bar\beta}}\Bigg(\int\limits_{[-1,1]^d}\big|\,\triangle^{\bar m}_{2^{-\bar k}*\bar h,\,\bar\beta}\,f_{\bar k+\bar\ell+\bar u}(x)\big|\,d\bar h\Bigg)^{q}\Bigg)^{1/q}.$$

From now on we can carry over the estimates given in case $q\le 1$ almost word by word. The consequence are three cases for an upper bound for the expression $A^{\bar\alpha}_{\bar\beta}(x)^{1/q}$, depending on $|\bar\alpha|$ and $\min(p,q)$ (i.e. λ). We distinguish the case (C_1) $|\bar\alpha|<|\bar\beta|, \lambda<1$ as well as (C_2)

$|\bar{\alpha}| < |\bar{\beta}|, \lambda = 1$ and (C_3) $|\bar{\alpha}| = |\bar{\beta}|$. Because of $q > 1$ additionally the case $\min(p,q) > 1$, i.e $\lambda = 1$, is possible. See also (46). Altogether we obtain

$$A^{\bar{\alpha}}_{\bar{\beta}}(x)^{1/q} \le c'' \left\{ \begin{array}{lcl} \left[\sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}} f_{\bar{k}}|^{(1-\lambda)q}(x) \cdot \left|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}|^{\lambda})\right|^q(x)\right]^{1/q} & : & C_1 \\ \left[\sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{k}q} \left|\mathbf{M}_{\bar{\beta}-\bar{\alpha}} \circ P_{2^{\bar{k}},\bar{a},\bar{\alpha}}(f_{\bar{k}})\right|^q(x)\right]^{1/q} & : & C_2 \\ \left[\sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}}(f_{\bar{k}})|^q(x)\right]^{1/q} & : & C_3 \end{array} \right. . \quad (51)$$

The rest follows similar as above. The case $q = \infty$ can be treated with obvious modifications and much more simpler arguments.

<u>*Step 2:*</u> We show the converse inequality using a classical construction by S. M. Nikol'skij, cf. [Ni 3, 5.2.1]. The basic idea is to show

$$\|f|S^{\bar{r}}_{p,q}F\|^{\bar{\varphi}} \le c\|f|S^{\bar{r}}_{p,q}F\|^R \quad (52)$$

for every $f \in S'(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$, where $\bar{\varphi} = (\varphi^1, ..., \varphi^d)$ denotes an appropriate tuple from $\Phi(\mathbb{R})^d$. These decompositions of unity are adapted to the order of differences used to compute $\|f|S^{\bar{r}}_{p,q}F\|^R$. For every $i \in \{1, ..., d\}$ we put

$$\varphi^i_0(x) = (-1)^{m_i+1} \sum_{\mu=0}^{m_i-1} \binom{m_i}{\mu} (-1)^\mu \psi((m_i - \mu)x) \quad , \quad (53)$$

where $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\psi(x) = \left\{ \begin{array}{lcl} 1 & : & |x| \le 1 \\ 0 & : & |x| > 3/2 \end{array} \right. .$$

Consequently $\varphi^i_0 \in C_0^\infty(\mathbb{R})$ and moreover

$$\varphi^i_0(x) = \left\{ \begin{array}{lcl} 1 & : & |x| \le 1/m_i \\ 0 & : & |x| > 3/2 \end{array} \right. . \quad (54)$$

(54) is clear in case $|x| > 3/2$. In the case $|x| \le 1/m_i$ we have

$$\begin{aligned} \varphi^i_0(x) &= (-1)^{m_i+1} \Big(\sum_{\mu=0}^{m_i} \binom{m_i}{\mu} (-1)^\mu - (-1)^{m_i} \Big) \\ &= (-1)^{m_i+1}((1-1)^{m_i} - (-1)^{m_i}) \\ &= 1. \end{aligned}$$

Therefore the function $\varphi^i_0(x)$ is admissible in the sense of Definition 3.2.1 and the given example to define via

$$\varphi^i_j(x) := \varphi^i_0(2^{-j}x) - \varphi^i_0(2^{-j+1}x) \quad , \quad j \ge 1$$

a decomposition of unity $\varphi^i := \{\varphi^i_j(x)\}_{j=0}^{\infty} \in \Phi(\mathbb{R})$. Now formula (19) points out the connection to differences. Obviously it holds

$$\varphi_0^i(x) = (-1)^{m_i+1}(\triangle_x^{m_i}\psi(0) - (-1)^{m_i})$$

and

$$\varphi_j^i(x) = (-1)^{m_i+1}(\triangle_{2^{-j}x}^{m_i} - \triangle_{2^{-j+1}x}^{m_i})\psi(0) \quad \text{for} \quad j > 0 \ ,$$

respectively. Therefore the effect is the occurrence of differences on the Fourier side of f. We consider the sequence $\{f_{\bar{\ell}}(x)\}_{\bar{\ell}\in\mathbb{N}_0^d}$, where

$$f_{\bar{\ell}}(x) := \mathcal{F}^{-1}[(\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(\xi)\mathcal{F}f](x) \quad , \quad \bar{\ell} \in \mathbb{N}_0^d.$$

It is necessary to divide the index-set $\mathbb{N}_0^d$ into 2^d disjoint subsets in the following way

$$\mathbb{N}_0^d = \bigcup_{\bar{\beta}\in\{0,1\}^d} I_{\bar{\beta}} \quad ,$$

where we refer to (5) for the notation. To avoid technical difficulties we again discuss only the case $\bar{\beta} = (1, ..., 1, 0, ..., 0)$ with $|\bar{\beta}| = n$. Hence for $\bar{\ell} \in I_{\bar{\beta}}$ we arrive at

$$\begin{aligned} |f_{\bar{\ell}}(x)| = \big|\mathcal{F}^{-1}\big[\, & (\triangle_{2^{-\ell_1}\xi_1}^{m_1} - \triangle_{2^{-\ell_1+1}\xi_1}^{m_1})\psi(0) \\ & \cdot (\triangle_{2^{-\ell_2}\xi_2}^{m_2} - \triangle_{2^{-\ell_2+1}\xi_2}^{m_2})\psi(0) \\ & \ \vdots \\ & \cdot (\triangle_{2^{-\ell_n}\xi_n}^{m_n} - \triangle_{2^{-\ell_n+1}\xi_1}^{m_n})\psi(0) \\ & \cdot \varphi_0^{n+1}(\xi_{n+1}) \cdot ... \cdot \varphi_0^d(\xi_d)\mathcal{F}f\,\big](x)\big|. \end{aligned}$$

Together with (53) we obtain

$$\begin{aligned} |f_{\bar{\ell}}(x)| = & \Bigg| \sum_{\mu_{n+1}=0}^{m_{n+1}-1} \cdots \sum_{\mu_d=0}^{m_d-1} C_{\mu_{n+1}}^{m_{n+1}} \cdot ... \cdot C_{\mu_d}^{m_d} \cdot \\ & \mathcal{F}^{-1}\big[\,(\triangle_{2^{-\ell_1}\xi_1}^{m_1} - \triangle_{2^{-\ell_1+1}\xi_1}^{m_1})\psi(0) \cdot ... \cdot (\triangle_{2^{-\ell_n}\xi_n}^{m_n} - \triangle_{2^{-\ell_n+1}\xi_n}^{m_n})\psi(0) \cdot \\ & \psi((m_{n+1} - \mu_{n+1})\xi_{n+1}) \cdot ... \cdot \psi((m_d - \mu_d)\xi_d)\mathcal{F}f\,\big](x) \Bigg| \quad , \end{aligned} \tag{55}$$

where $C_\mu^m := (-1)^\mu \binom{m}{\mu}$ for $0 \leq \mu \leq m$. Since f is a regular distribution, elementary calculations give us the following

$$\mathcal{F}^{-1}[\psi(\eta_1\xi_1)\cdot...\cdot\psi(\eta_d\xi_d)\mathcal{F}f](x) = -(2\pi)^{-d/2}\int\limits_{\mathbb{R}^d} \mathcal{F}\big[(\psi\otimes...\otimes\psi)(\cdot)\big](\bar{h})f(x_1+\eta_1 h_1, ..., x_d+\eta_d h_d)d\bar{h}.$$

The function $g(\bar{h}) := \mathcal{F}\big[(\psi \otimes ... \otimes \psi)(\cdot)\big](\bar{h})$ belongs to $S(\mathbb{R}^d)$. Consequently for every $r > 0$ there exists constant $c_r > 0$ such that for every $\bar{h} \in \mathbb{R}^d$

$$(1 + |h_1|^2)^{r/2} \cdot ... \cdot (1 + |h_d|^2)^{r/2}|g(\bar{h})| \leq c_r \tag{56}$$

holds. See also (7). Applied to (55), the differences on the right-hand side carry over to f, precisely

$$|f_{\bar{\ell}}(x)| \leq \tag{57}$$
$$c \int\limits_{\mathbb{R}^d} |g(\bar{h})| \cdot \left|(\triangle^{m_1}_{2^{-\ell_1}h_1,1} - \triangle^{m_1}_{2^{-\ell_1+1}h_1,1}) \circ ... \circ (\triangle^{m_n}_{2^{-\ell_n}h_n,n} - \triangle^{m_n}_{2^{-\ell_n+1}h_n,n})(L_{\bar{h}}f)(x)\right| d\bar{h} \quad ,$$

where

$$\begin{aligned} L_{\bar{h}}f(x) \quad = \quad & \sum_{\mu_{n+1}=0}^{m_{n+1}-1} \cdots \sum_{\mu_d=0}^{m_d-1} C^{m_{n+1}}_{\mu_{n+1}} \cdot ... \cdot C^{m_d}_{\mu_d} \times \\ & \times f(x_1, ..., x_n, x_{n+1} + (m_{n+1} - \mu_{n+1})h_{n+1}, ..., x_d + (m_d - \mu_d)h_d). \end{aligned} \tag{58}$$

With an eye on (19) we notice, that $L_{\bar{h}}f(x)$ is almost a mixed difference. Appropriate decomposition of RHS(58) gives us precisely a sum of mixed differences. For technical reasons we rewrite (58) in

$$\begin{aligned} L_{\bar{h}}f(x) = \sum_{\mu_{n+1}=1}^{m_{n+1}} \cdots \sum_{\mu_d=1}^{m_d} C^{m_{n+1}}_{m_{n+1}-\mu_{n+1}} \cdot ... \cdot C^{m_d}_{m_d-\mu_d} \times \\ \times f(x_1, ..., x_n, x_{n+1} + \mu_{n+1}h_{n+1}, ..., x_d + \mu_d h_d). \end{aligned} \tag{59}$$

See again (5) for our predefined index-sets. For abbreviation we define the quantity $C^{\bar{m}}_{\bar{\mu}} := \prod_{i=1}^{d} C^{m_i}_{\mu_i}$. Now it follows

$$\begin{aligned} L_{\bar{h}}f(x) = & \sum_{\substack{\bar{\alpha}\in\bar{E}_{\bar{\beta}} \\ |\bar{\alpha}|=d-n}} \sum_{\bar{\mu}\in M_{\bar{\alpha}}} C^{\bar{m}}_{\bar{m}-\bar{\mu}} \cdot f(x + \bar{\alpha} * \bar{\mu} * \bar{h}) \\ & - \sum_{\substack{\bar{\alpha}\in\bar{E}_{\bar{\beta}} \\ |\bar{\alpha}|=d-n-1}} \sum_{\bar{\mu}\in M_{\bar{\alpha}}} C^{\bar{m}}_{\bar{m}-\bar{\mu}} \cdot f(x + \bar{\alpha} * \bar{\mu} * \bar{h}) \\ & + \sum_{\substack{\bar{\alpha}\in\bar{E}_{\bar{\beta}} \\ |\bar{\alpha}|=d-n-2}} \sum_{\bar{\mu}\in M_{\bar{\alpha}}} C^{\bar{m}}_{\bar{m}-\bar{\mu}} \cdot f(x + \bar{\alpha} * \bar{\mu} * \bar{h}) \\ & \quad \vdots \\ & \pm f(x). \end{aligned} \tag{60}$$

Obviously

$$|\triangle^{\bar{m}}_{\bar{h},\bar{\alpha}} f(x)| = \left| \sum_{\bar{\mu}\in M_{\bar{\alpha}}} C^{\bar{m}}_{\bar{m}-\bar{\mu}} \cdot f(x + \bar{\alpha} * \bar{\mu} * \bar{h}) \right|.$$

Together with (60) and (20) this implies

$$L_{\bar{h}}f(x) = \sum_{\bar{\alpha}\in\bar{E}_{\bar{\beta}}} \varepsilon_{\bar{\alpha}} \triangle^{\bar{m}}_{\bar{h},\bar{\alpha}} f(x) \quad , \tag{61}$$

where the $\varepsilon_{\bar{\alpha}} \in \{-1,1\}$ were chosen suitable. Putting (61) into (57) and using Δ-inequality we obtain the following

$$|f_{\bar{\ell}}(x)| \leq c \sum_{\bar{u}\in E_{\bar{\beta}}} \sum_{\bar{\alpha}\in \bar{E}_{\bar{\beta}}} \int_{\mathbb{R}^d} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-(\bar{\ell}-\bar{u})}*\bar{h},\bar{\beta}} \circ \triangle^{\bar{m}}_{\bar{h},\bar{\alpha}} f(x)\big| \, d\bar{h}. \tag{62}$$

We will write $\ell_{q|I}$ instead of ℓ_q for some $I \subset \mathbb{N}_0^d$. This makes clear, which index-set I belongs to ℓ_q in the present formula. Altogether this gives an estimate for $\|2^{\bar{r}\cdot\bar{\ell}} f_{\bar{\ell}}(x)|L_p(\ell_{q|\mathbb{N}_0^d})\|$, namely

$$\begin{aligned} &\|2^{\bar{r}\cdot\bar{\ell}} f_{\bar{\ell}}(x)|L_p(\ell_{q|\mathbb{N}_0^d})\| \\ &\leq c_1 \sum_{\bar{\beta}\in\{0,1\}^d} \|2^{\bar{r}\cdot\bar{\ell}} f_{\bar{\ell}}(x)|L_p(\ell_{q|I_{\bar{\beta}}})\| \\ &\leq c_2 \sum_{\bar{\beta}\in\{0,1\}^d} \sum_{\bar{u}\in E_{\bar{\beta}}} \sum_{\bar{\alpha}\in \bar{E}_{\bar{\beta}}} \left\| 2^{\bar{r}\cdot\bar{\ell}} \int_{\mathbb{R}^d} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-(\bar{\ell}-\bar{u})}*\bar{h},\bar{\beta}} \circ \triangle^{\bar{m}}_{\bar{h},\bar{\alpha}} f(x)\big| \, d\bar{h} \right| L_p(\ell_{q|I_{\bar{\beta}}}) \right\|. \end{aligned} \tag{63}$$

RHS(63) can be increased if we use the index-set $N_{\bar{\beta}}$ instead of $I_{\bar{\beta}}$ in the ℓ_q-norm and replace $\bar{\ell} - \bar{u}$ by $\bar{\ell}$. We obtain

$$\begin{aligned} &\|2^{\bar{r}\cdot\bar{\ell}} f_{\bar{\ell}}(x)|L_p(\ell_{q|\mathbb{N}_0^d})\| \\ &\leq c_2 \sum_{\bar{\beta}\in\{0,1\}^d} \sum_{\bar{u}\in E_{\bar{\beta}}} \sum_{\bar{\alpha}\in \bar{E}_{\bar{\beta}}} \left\| 2^{\bar{r}\cdot\bar{\ell}} \int_{\mathbb{R}^d} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} \circ \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\alpha}} f(x)\big| \, d\bar{h} \right| L_p(\ell_{q|N_{\bar{\beta}}}) \right\| \\ &= c_2 \sum_{\bar{\beta}\in\{0,1\}^d} \sum_{\bar{u}\in E_{\bar{\beta}}} \sum_{\bar{\alpha}\in \bar{E}_{\bar{\beta}}} \left\| 2^{\bar{r}\cdot\bar{\ell}} \int_{\mathbb{R}^d} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} \circ \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\alpha}} f(x)\big| \, d\bar{h} \right| L_p(\ell_{q|N_{\bar{\alpha}+\bar{\beta}}}) \right\|. \end{aligned}$$

Because of

$$\triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} \circ \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\alpha}} f(x) = \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\alpha}+\bar{\beta}} f(x)$$

for $\bar{\alpha} \in \bar{E}_{\bar{\beta}}$ it holds

$$\begin{aligned} \|f|S^{\bar{r}}_{p,q}F)\|^{\bar{\varphi}} &\leq c_3 \sum_{\bar{\beta}\in\{0,1\}^d} \sum_{\bar{\alpha}\in \bar{E}_{\bar{\beta}}} \left\| 2^{\bar{r}\cdot\bar{\ell}} \int_{\mathbb{R}^d} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\alpha}+\bar{\beta}} f(x)\big| \, d\bar{h} \right| L_p(\ell_{q|N_{\bar{\alpha}+\bar{\beta}}}) \right\| \\ &= c_3 \sum_{\bar{\beta}\in\{0,1\}^d} \left\| 2^{\bar{r}\cdot\bar{\ell}} \int_{\mathbb{R}^d} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} \right| L_p(\ell_{q|N_{\bar{\beta}}}) \right\|. \end{aligned} \tag{64}$$

It remains to estimate the summand

$$\left\| 2^{\bar{r}\cdot\bar{\ell}} \int_{\mathbb{R}^d} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} \right| L_p(\ell_{q|N_{\bar{\beta}}}) \right\| \tag{65}$$

for every $\bar{\beta} \in \{0,1\}^d$. In case $|\bar{\beta}| = 0$ it breaks down to

$$\left\| |f(x)| \cdot \int_{\mathbb{R}^d} |g(\bar{h})| \, d\bar{h} \right| L_p \right\| \quad ,$$

which equals $\|g|L_1(\mathbb{R}^d)\| \cdot \|f|L_p(\mathbb{R}^d)\|$. We will finish the proof by estimating (65) from above by $c \cdot S^R_{\bar{\beta}}(f)$ in case $|\bar{\beta}| > 0$. For this purpose we want to discretisise the integral appearing in (65) similar to step 1. For the used notation we refer to (3) and (4). Obviously

$$\int_{\mathbb{R}^d} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} = \sum_{\bar{\mu}\in\mathbb{N}_0^d} \int_{Q^{\Delta}_{\bar{\mu}}} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h}.$$

Together with (56) we obtain for any $s > 0$

$$\begin{aligned}
\int_{\mathbb{R}^d} |g(\bar{h})| \cdot \big| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} &\le c_s \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{-|\bar{\mu}|\cdot s} \int_{Q^{\Delta}_{\bar{\mu}}} \big| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} \\
&\le c_s \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{-|\bar{\mu}|\cdot s} 2^{\bar{\ell}\cdot\bar{\beta}} \int_{Q_{\bar{\mu}-\bar{\beta}*\bar{\ell}}} \big| \triangle^{\bar{m}}_{\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} \\
&= c_s \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{|\bar{\mu}|(1-s)} 2^{-\bar{\beta}\cdot(\bar{\mu}-\bar{\ell})} 2^{-\bar{\mu}(\bar{1}-\bar{\beta})} \int_{Q_{\bar{\mu}-\bar{\beta}*\bar{\ell}}} \big| \triangle^{\bar{m}}_{\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h}.
\end{aligned}$$

Consequently we obtain for (65) in case $q \le 1$

$$(65) \le c_s \Bigg\| \Bigg[\sum_{\bar{\ell}\in N_{\bar{\beta}}} \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q} 2^{|\bar{\mu}|(1-s)q} 2^{-\bar{\beta}\cdot(\bar{\mu}-\bar{\ell})q} 2^{-\bar{\mu}(\bar{1}-\bar{\beta})q} \Bigg(\int_{Q_{\bar{\mu}-\bar{\beta}*\bar{\ell}}} \big| \triangle^{\bar{m}}_{\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} \Bigg)^q \Bigg]^{1/q} \Bigg| L_p \Bigg\|.$$

The next step is to rediscretisise the sum over $\bar{\ell}$ to an integral over $\bar{t}$. Having an arbitrary tuple $\bar{t} \in Q^{+\,\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}$ we can rewrite the previous estimate to

$$(65) \le c'_s \Bigg\| \Bigg[\sum_{\bar{\ell}\in N_{\bar{\beta}}} \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q} 2^{|\bar{\mu}|(1-s)q} \Bigg(\frac{1}{\bar{t}} \int_{[-t_1,t_1]\times\ldots\times[-t_d,t_d]} \big| \triangle^{\bar{m}}_{\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} \Bigg)^q \Bigg]^{1/q} \Bigg| L_p \Bigg\|.$$

We use $2^{\bar{r}\bar{\ell}} = 2^{\bar{\beta}*(\bar{\ell}-\bar{\mu})\bar{r}} \cdot 2^{(\bar{\beta}*\bar{\mu})\cdot\bar{r}} \sim \bar{t}^{-\bar{r}*\bar{\beta}} \cdot 2^{(\bar{\beta}*\bar{\mu})\cdot\bar{r}}$ and obtain

$$\begin{aligned}
(65) &\le c''_s \Bigg\| \Bigg[\sum_{\bar{\ell}\in N_{\bar{\beta}}} \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{(\bar{\beta}*\bar{\mu})\cdot\bar{r}q} 2^{|\bar{\mu}|(1-s)q} \bar{t}^{-(\bar{r}*\bar{\beta})q} \Bigg(\int_{[-1,1]^d} \big| \triangle^{\bar{m}}_{\bar{t}*\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} \Bigg)^q \Bigg]^{1/q} \Bigg| L_p \Bigg\| \\
&= c''_s \Bigg\| \Bigg[\sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{\bar{\mu}\cdot(1-s+\bar{\beta}*\bar{r})q} \sum_{\bar{\ell}\in N_{\bar{\beta}}} \bar{t}^{-(\bar{r}*\bar{\beta})q} \Bigg(\int_{[-1,1]^d} \big| \triangle^{\bar{m}}_{\bar{t}*\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} \Bigg)^q \Bigg]^{1/q} \Bigg| L_p \Bigg\|.
\end{aligned} \tag{66}$$

It should be mentioned that the chosen $\bar{t}$ depends on the summation index $\bar{\ell}$. Because of measure theoretical reasons there must be a $\bar{t}(x,\bar{\ell}) \in Q^{+\,\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}$, such that the integral average of the function

$$h_x(\bar{t}) := \bar{t}^{-(\bar{r}*\bar{\beta})q} \Bigg(\int_{[-1,1]^d} \big| \triangle^{\bar{m}}_{\bar{t}*\bar{h},\bar{\beta}} f(x)\big| \, d\bar{h} \Bigg)^q$$

with respect to the rectangle $Q^{+\,\Delta}_{\bar\mu-\bar\beta*\bar\ell+1}$ is greater than or equal $h_x(\bar t)$. It is also remarkable that h_x is invariant in that components t_i, which correspond to $\beta_i = 0$. We will need this fact later on. Hence we can replace $h_x(\bar t)$ in (66) by

$$\frac{1}{|Q^{+\,\Delta}_{\bar\mu-\bar\beta*\bar\ell+1}|} \int\limits_{Q^{+\,\Delta}_{\bar\mu-\bar\beta*\bar\ell+1}} h_x(\bar t)\, d\bar t \quad ,$$

which can be estimated from above by

$$c \int\limits_{Q^{+\,\Delta}_{\bar\mu-\bar\beta*\bar\ell+1}} h_x(\bar t)\, \frac{d\bar t}{\bar t} \quad ,$$

where c is independent of x and $\bar\ell$. This yields

$$(65) \le c'''_s \Big\| \Big[\sum_{\bar\mu\in\mathbb{N}_0^d} 2^{\bar\mu\cdot(1-s+\bar\beta*\bar r)q} \sum_{\bar\ell\in N_{\bar\beta}} \int\limits_{Q^{+\,\Delta}_{\bar\mu-\bar\beta*\bar\ell+1}} h_x(\bar t)\, \frac{d\bar t}{\bar t} \Big]^{1/q} \Big| L_p \Big\|. \tag{67}$$

Obviously the integration over the "invariance"-components of h_x breaks down to a constant. Finally it holds

$$\sum_{\bar\ell\in N_{\bar\beta}} \int\limits_{Q^{+\,\Delta}_{\bar\mu-\bar\beta*\bar\ell+1}} h_x(\bar t)\, \frac{d\bar t}{\bar t} \le c \int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\Big) \Big(\int\limits_{[-1,1]^n} \big|\big(\triangle^{m_{\delta_1}}_{t_1 h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n,\delta_n} f\big)(x)\big| d\bar h \Big)^q \frac{d\bar t}{\bar t} \,.$$

Thus we lost the $\bar\mu$-dependence. Putting this into (67), the $\bar\mu$-sum is nothing more than a convergent geometric series for sufficiently large s. This finishes the proof in case $q \le 1$. In case $1 < q < \infty$ we use the triangle inequality in ℓ_q to interchange $\bar\mu$- and $\bar\ell$-sum. Finally the modifications in case $q = \infty$ are clear and make the estimates much simpler. □

Remark 3.4.1 Let us give a remark concerning step 1 of the proof. If we would not exploit the integral structure of the rectangle means after (42) and continue similar as before, the method works only in case $\bar r > 1/\min(p,q)$. At exactly this point one gets an idea, why the integral means are more powerful than classical difference constructions (which we will discuss later in this paper). Namely, the integral in RHS(40) allows the use of Hardy-Littlewood maximal functions. Here we need the chosen quantities λ and $\bar a$. On the one hand λ and $\bar a$ force the convergence of the $\bar\ell$-series in (45) and on the other hand $\lambda < \min(p,q)$ is used to apply the corresponding maximal inequalities in (48). □

Remark 3.4.2 It has been proved recently by Christ and Seeger, cf. [CS], that $\bar r > \sigma_{p,q}$ is a necessary condition. See also [Tr 3, 1.11.9] for the isotropic case. □

3.5 Localization

As we have already seen, the philosophy of difference characterization is to test the smoothness of a function, checking the behavior of differences with small step lengths. Now the question

arises, whether one can replace in (30) the $(0,\infty)$-integrals by $(0,1)$- or (what is essentially the same) by $(0,\varepsilon)$-integrals, where $\varepsilon > 0$. This would be a certain type of localization property, which works unrestricted in the isotropic case, cf. [Tr 2, 2.5.11]. We present a partial result in the case of Banach spaces, except the constellation $1 = p < q \le \infty$, using complex interpolation.
Let $0 < p < \infty$, $0 < q \le \infty$ and $\bar{r} > 0$. For $f \in L_p(\mathbb{R}^d)$. Similar to (29) we define the quantity

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\|^{R,L} := \|f|L_p(\mathbb{R}^d)\| + \sum_{|\bar{\beta}|\ge 1} S^{R,L}_{\bar{\beta}}(f) \quad , \tag{68}$$

where (modification in case $q = \infty$)

$$S^{R,L}_{\bar{\beta}}(f) = \Bigg\| \Big[\int_{(0,1)^n} \big(\prod_{i=1}^{n} t_i^{-r_{\delta_i} q}\big) \Big(\int_{[-1,1]^n} \big|(\triangle^{m_{\delta_1}}_{t_1 h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n,\delta_n} f)(x)\big| d\bar{h} \Big)^q \frac{d\bar{t}}{\bar{t}} \Big]^{1/q} \Big| L_p(\mathbb{R}^d) \Bigg\|$$

for $|\bar{\beta}| \ge 1$. We want to compare this to the quantity $\|f|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\|^R$, cf. Theorem 3.4.1, (29).

Proposition 3.5.1 Let $1 \le q \le p < \infty$ and $\bar{r} > 0$. Then there exists a constant $c > 0$ such that

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\|^R \le c\, \|f|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\|^{R,L}$$

for all $f \in L_p(\mathbb{R}^d)$.

Proof We fix without loss of generality $\bar{\beta} = (1,...,1,0,...,0)$ and estimate the quantity $S^R_{\bar{\beta}}(f)$. Obviously it holds

$$S^R_{\bar{\beta}}(f) \le c_1 \sum_{\bar{\alpha}\in E_{\bar{\beta}}} \Bigg\| \Big[\int_{t_1 \in A_{\alpha_1}} \cdots \int_{t_n \in A_{\alpha_n}} \big(\prod_{i=1}^{n} t_i^{-r_i q}\big) \mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x)^q \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \Big]^{1/q} \Big| L_p \Bigg\| \quad ,$$

where $A_0 = (0,1)$, $A_1 = [1,\infty)$ and

$$\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x) = \int_{-1}^{1} \cdots \int_{-1}^{1} \big|(\triangle^{m_{\delta_1}}_{t_1 h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n,\delta_n} f)(x)\big|\, dh_n \cdots dh_1.$$

Let us only discuss the case $\alpha_{\delta_1} \le \alpha_{\delta_2} \le ... \le \alpha_{\delta_n}$. Changing the order of the $\bar{t}$-integration appropriately we obtain for such an $\bar{\alpha}$ with $|\bar{\alpha}| = n - m \ge 1$

$$\begin{aligned} &\Bigg\| \Big[\int_{A_{\alpha_1}\times\cdots\times A_{\alpha_n}} \big(\prod_{i=1}^{n} t_i^{-r_i q}\big) \mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x)^q \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \Big]^{1/q} \Big| L_p \Bigg\| \\ &= \Bigg\| \Big[\int_{[1,\infty)^{n-m}} \big(\prod_{i=m+1}^{n} t_i^{-r_i q}\big) \int_{(0,1)^m} \big(\prod_{i=1}^{m} t_i^{-r_i q}\big) \mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x)^q \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \Big]^{1/q} \Big| L_p \Bigg\|. \end{aligned} \tag{69}$$

Together with (20) it is possible to estimate $\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x)$ in the following way

$$\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x) \le c_3\, T_{\bar{\alpha}}\big(\mathcal{R}^{\bar{m}}_{(\cdot,...,\cdot),\bar{\beta}-\bar{\alpha}} f(\cdot)\big)(x, t_1, ..., t_n) \ ,$$

where $T_{\bar{\alpha}}g(x,t_1,...,t_n)$ acting on a function $g:\mathbb{R}^d\times\mathbb{R}^m\to\mathbb{C}$, $m<n$ is defined in the following way:

$$T_{\bar{\alpha}}g(x,t_1,...,t_n)=\sum_{\bar{\mu}\in M_{\bar{\alpha}}}\int_{-1}^{1}\cdots\int_{-1}^{1} g(x+d(\bar{\alpha},\bar{\mu},\hat{h},\bar{t}),t_1,...,t_m)dh_{m+1}\cdots dh_n\ , \tag{70}$$

with $\hat{h}=(0,...,0,h_{m+1},...,h_n)\in\mathbb{R}^n$ and moreover

$$d(\bar{\alpha},\bar{\mu},\hat{h},\bar{t})_{\delta_i}=\mu_{\delta_i}\cdot h_i\cdot t_i\quad ,\quad i=m+1,...,n$$

with 0 in all remaining components. Later we will use the translation invariance of the Lebesegue measure and drop the quantity $d(\bar{\alpha},\bar{\mu},\hat{h},\bar{t})$. Recalling (69) and using (70) together with $p\ge q\ge 1$ we obtain

$$\begin{aligned}
&\Big\|\Big[\int\limits_{A_{\alpha_1}\times\cdots\times A_{\alpha_n}}(\prod_{i=1}^{n}t_i^{-r_iq})\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x)^q\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n}\Big]^{1/q}\Big|L_p\Big\|\\
&\le c_4\Big\|\Big[\int\limits_{A_{\alpha_1}\times\cdots\times A_{\alpha_n}}(\prod_{i=1}^{n}t_i^{-r_iq})T_{\bar{\alpha}}\big(\mathcal{R}^{\bar{m}}_{(\cdot,...,\cdot),\bar{\beta}-\bar{\alpha}}f(\cdot)\big)^q(x,t_1,...,t_n)\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n}\Big|L_p\Big]^{1/q}\Big\|\\
&\le c_5\sum_{\bar{\mu}\in M_{\bar{\alpha}}}\Big(\int\limits_{[1,\infty)^{n-m}}\frac{dt_{m+1}}{t_{m+1}}\cdots\frac{dt_n}{t_n}(\prod_{i=m+1}^{n}t_i^{-r_iq})\int\limits_{[-1,1]^{n-m}}dh_{m+1}\cdots dh_n\times\\
&\quad\times\Big\|\int\limits_{(0,1)^m}(\prod_{i=1}^{m}t_i^{-r_iq})\mathcal{R}^{\bar{m}}_{(t_1,...,t_m),\bar{\beta}-\bar{\alpha}}f(x+d(\bar{\alpha},\bar{\mu},\hat{h},\bar{t}))^q\frac{dt_1}{t_1}\cdots\frac{dt_m}{t_m}\Big|L_{p/q}\Big\|\Big)^{1/q}\\
&\le c_6\Big\|\Big[\int\limits_{(0,1)^m}(\prod_{i=1}^{m}t_i^{-r_iq})\mathcal{R}^{\bar{m}}_{(t_1,...,t_m),\bar{\beta}-\bar{\alpha}}f(x)^q\frac{dt_1}{t_1}\cdots\frac{dt_m}{t_m}\Big]^{1/q}\Big|L_p\Big\|\\
&=c_6\,S^{R,L}_{\bar{\beta}-\bar{\alpha}}(f).
\end{aligned}\tag{71}$$

Because of the translation invariance of the Lebesgue measure we can leave out the quantity $d(\bar{\alpha},\bar{\mu},\hat{h},\bar{t})$ in (71). Hence the $L_{p/q}$-norm gets independent of the outer integration over $\bar{h}$ and t_{m+1},...,t_n. Since $\bar{r}>0$, the integral over $[1,\infty)^{n-m}$ exists. Consequently we have

$$S^R_{\bar{\beta}}(f)\le c_6\sum_{\bar{\alpha}\in E_{\bar{\beta}}}S^{R,L}_{\bar{\beta}-\bar{\alpha}}(f)\le c_6\|f|S^{\bar{r}}_{p,q}F\|^{R,L}\ ,$$

which finishes the proof. □

At next we consider the case $q=\infty$ and $1<p<\infty$.

Proposition 3.5.2 For $1<p<\infty$ and $\bar{r}>0$ there exists a constant $c>0$ such that

$$\|f|S^{\bar{r}}_{p,\infty}F(\mathbb{R}^d)\|^R\le c\,\|f|S^{\bar{r}}_{p,\infty}F(\mathbb{R}^d)\|^{R,L}$$

for all $f\in L_p(\mathbb{R}^d)$.

Proof We fix again $\bar{\beta} = (1, ..., 1, 0, ..., 0)$ and estimate the quantity $S^R_{\bar{\beta}}(f)$. With the same notations used above we obtain

$$S^R_{\bar{\beta}}(f) \le c_1 \sum_{\bar{\alpha} \in E_{\bar{\beta}}} \Big\| \sup_{A_{\alpha_1} \times \cdots \times A_{\alpha_n}} \big(\prod_{i=1}^{n} t_i^{-r_i}\big) \mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x) \Big| L_p \Big\|. \tag{72}$$

Again we only discuss the case $\alpha_{\delta_1} \le \alpha_{\delta_2} \le ... \le \alpha_{\delta_n}$, $|\bar{\alpha}| = n - m \ge 1$. Similar to (71) it turns out that

$$\begin{aligned}
&\Big\| \sup_{A_{\alpha_1} \times \cdots \times A_{\alpha_n}} \big(\prod_{i=1}^{n} t_i^{-r_i}\big) \mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x) \Big| L_p \Big\| \\
&\le c_2 \Big\| \sup_{A_{\alpha_1} \times \cdots \times A_{\alpha_n}} \big(\prod_{i=1}^{n} t_i^{-r_i}\big) T_{\bar{\alpha}} \big(\mathcal{R}^{\bar{m}}_{(\cdot,...,\cdot),\bar{\beta}-\bar{\alpha}} f(\cdot)\big)(x, t_1, ..., t_n) \Big| L_p \Big\| \\
&\le c_2 \sum_{\bar{\mu} \in M_{\bar{\alpha}}} \Big\| \sup_{\substack{t_i \in [1,\infty] \\ i=m+1...n}} \big(\prod_{i=m+1}^{n} t_i^{-r_i}\big) \sup_{\substack{t_i \in (0,1) \\ i=1,...,m}} \int_{[-1,1]^{n-m}} dh_{m+1} \cdots dh_n \\
&\quad \Big(\big(\prod_{i=1}^{m} t_i^{-r_i}\big) \mathcal{R}^{\bar{m}}_{(t_1,...,t_m),\bar{\beta}-\bar{\alpha}} f(x + d(\bar{\alpha}, \bar{\mu}, \hat{h}, \bar{t})) \Big) \Big| L_p \Big\|.
\end{aligned} \tag{73}$$

Recall the maximal operator $\mathbf{M}_{\bar{\alpha}}$, consisting of Hardy-Littlewood maximal operators corresponding to $\bar{\alpha}$, defined in Theorem 3.4.1, step 1, (46). Using (22) and the scalar version of Theorem 3.1.3 we have

$$\begin{aligned}
&\Big\| \sup_{A_{\alpha_1} \times \cdots \times A_{\alpha_n}} \big(\prod_{i=1}^{n} t_i^{-r_i}\big) \mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x) \Big| L_p \Big\| \\
&\le c_2 \Big\| \sup_{A_{\alpha_1} \times \cdots \times A_{\alpha_n}} \big(\prod_{i=1}^{n} t_i^{-r_i}\big) T_{\bar{\alpha}} \big(\mathcal{R}^{\bar{m}}_{(\cdot,...,\cdot),\bar{\beta}-\bar{\alpha}} f(\cdot)\big)(x, t_1, ..., t_n) \Big| L_p \Big\| \\
&\le c_3 \Big\| \sup_{\substack{t_i \in (0,1) \\ i=1,...,m}} \mathbf{M}_{\bar{\alpha}} \Big(\big(\prod_{i=1}^{m} t_i^{-r_i}\big) \mathcal{R}^{\bar{m}}_{(t_1,...,t_m),\bar{\beta}-\bar{\alpha}} f \Big)(x) \Big| L_p \Big\| \\
&= c_3 \Big\| \mathbf{M}_{\bar{\alpha}} \Big(\sup_{\substack{t_i \in (0,1) \\ i=1,...,m}} \big(\prod_{i=1}^{m} t_i^{-r_i}\big) \mathcal{R}^{\bar{m}}_{(t_1,...,t_m),\bar{\beta}-\bar{\alpha}} f \Big)(x) \Big| L_p \Big\| \\
&\le c_4 \Big\| \sup_{\substack{t_i \in (0,1) \\ i=1,...,m}} \big(\prod_{i=1}^{m} t_i^{-r_i}\big) \mathcal{R}^{\bar{m}}_{(t_1,...,t_m),\bar{\beta}-\bar{\alpha}} f(x) \Big| L_p \Big\| \\
&= c_5 S^{R,L}_{\bar{\beta}-\bar{\alpha}}(f).
\end{aligned} \tag{74}$$

Together with (72) this completes the proof. □

Let us finally consider the case $1 < p < q < \infty$. Recall (71) and (70). Also for $p < q$ we show the estimate

$$\Big\|\Big[\int_{A_{\alpha_1}\times\cdots\times A_{\alpha_n}} (\prod_{i=1}^{n} t_i^{-r_i q}) T_{\bar\alpha}\big(\mathcal{R}^{\bar m}_{(\cdot,\dots,\cdot),\bar\beta-\bar\alpha} f(\cdot)\big)^q(x,t_1,\dots,t_n)\frac{dt_1}{t_1}\cdots\frac{dt_n}{t_n}\Big|L_p\Big]^{1/q}\Big\|$$
$$\le c\Big\|\Big[\int_{(0,1)^m} (\prod_{i=1}^{n} t_i^{-r_i q}) \mathcal{R}^{\bar m}_{(t_1,\dots,t_m),\bar\beta-\bar\alpha} f(x)^q \frac{dt_1}{t_1}\cdots\frac{dt_m}{t_m}\Big]^{1/q}\Big|L_p\Big\| \ . \tag{75}$$

It suffices to show the boundedness of the operator $T_{\bar\alpha}$ as a mapping between two weighted $L_p(L_q)$-spaces, i.e.

$$T_{\bar\alpha} : L_p(L_{q,w_{q,m}}) \longrightarrow L_p(L_{q,w_{q,n}}) \ ,$$

where

$$w_{s,k}(t_1,\dots,t_k) = \prod_{i=1}^{k} t_i^{-r_i-1/s} \ , \ \infty \ge s > 0 \ , \ t_i > 0 \ , \ i = 1,\dots,k \le d.$$

The function $g(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{C}$ belongs to the space $L_p(L_{q,w})$, $w(\bar t) > 0$, if and only if

$$\|g(x,\bar t)\cdot w(\bar t)|L_q(\bar t)|L_p(x)\| < \infty.$$

The definition of the weighted Lebesgue spaces is borrowed from [Tr 1, 1.18.5]. We do not go into detail. The usage in our situation is more or less obvious.
With the techniques from (71) and (74) we are able to prove the boundedness of $T_{\bar\alpha}$ in the situations

$$T_{\bar\alpha} : L_p(L_{p,w_{p,m}}) \longrightarrow L_p(L_{p,w_{p,n}}) \quad \text{and} \quad T_{\bar\alpha} : L_p(L_{\infty,w_{\infty,m}}) \longrightarrow L_p(L_{\infty,w_{\infty,n}}) \ . \tag{76}$$

Recall that $p > 1$ is important for the second mapping. At next we use complex interpolation and refer mainly to H. Triebel and [Tr 1]. The corners are defined by (76). Let $\vartheta \in (0,1)$ such that

$$\frac{1}{q} = \frac{1-\vartheta}{p} + \frac{\vartheta}{\infty} \quad \Longleftrightarrow \quad \frac{1}{q'} = \frac{1-\vartheta}{p'} + \frac{\vartheta}{1} \ ,$$

where $1/p+1/p' = 1$ and $1/q+1/q' = 1$. Using Theorem [Tr 1, 1.18.4] we obtain for $k = m, n$:

$$[L_p(L_{p,w_{p,k}}), L_p(L_{\infty,w_{\infty,k}})]_\vartheta = L_p([L_{p,w_{p,k}}, L_{\infty,w_{\infty,k}}]_\vartheta) = L_p\Big(\big[(L_{p',\frac{1}{w_{p,k}}})', (L_{1,\frac{1}{w_{\infty,k}}})'\big]_\vartheta\Big).$$

Now Theorem [Tr 1, 1.11.3] gives us

$$L_p\Big(\big[(L_{p',\frac{1}{w_{p,k}}})', (L_{1,\frac{1}{w_{\infty,k}}})'\big]_\vartheta\Big) = L_p\Big(\big[L_{p',\frac{1}{w_{p,k}}}, L_{1,\frac{1}{w_{\infty,k}}}\big]'_\vartheta\Big)$$

and finally Theorem [Tr 1, 1.18.5] leads to

$$L_p\Big(\big[L_{p',\frac{1}{w_{p,k}}}, L_{1,\frac{1}{w_{\infty,k}}}\big]'_\vartheta\Big) = L_p\big(\big[L_{q',\frac{1}{w_{q,k}}}\big]'\big) = L_p(L_{q,w_{q,k}}).$$

Hence it holds

$$[L_p(L_{p,w_{p,k}}), L_p(L_{\infty,w_{\infty,k}})]_\vartheta = L_p(L_{q,w_{q,k}})$$

Thus (75) holds also in the case $1 < p < q < \infty$.
Altogether we can state the following theorem.

Theorem 3.5.1 Let $p = q = 1$ or $1 < p < \infty$, $1 \le q \le \infty$. Let further $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ with $\bar{m} > \bar{r} > \sigma_{p,q}$. Then $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$ is the collection of all functions $f \in L_p(\mathbb{R}^d)$ such that

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\|^{R,L} < \infty.$$

Moreover $\|f|S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\|^{R,L}$ is an equivalent (quasi-)norm in $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$.

Proof As we have seen, this is a consequence of Theorem 3.4.1, Proposition 3.5.1, 3.5.2 and complex interpolation. □

3.6 Further characterizations

This paragraph deals with equivalent (quasi-)norms for $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$ using moduli of smoothness and differences itself. The price one has to pay are stricter conditions to $\bar{r}$, namely $\bar{r} > 1/\min(p,q)$, which cannot be essentially relaxed. The aim is to derive the results from [ST, 2.3.3] for arbitrary d. Our proof is based on the characterization by integral means, cf Theorem 3.4.1.
The first step is a small modification of the integral means.

Proposition 3.6.1 Let $p, q, \bar{r}$ be given as in Theorem 3.4.1. In the sense of equivalent (quasi-)norms the integral means

$$\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x) = \int\limits_{[-1,1]^n} \left|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_nh_n,\delta_n} f)(x)\right| d\bar{h} \tag{77}$$

in (30) can be replaced by

$$\bar{\mathcal{R}}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x) = \int\limits_{\substack{1\ge|h_i|>1/2\\ i=1,...,n}} \left|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_nh_n,\delta_n} f)(x)\right| d\bar{h}\ . \tag{78}$$

Proof Let us recall the abbreviating symbol defined in (21). Assume $q < \infty$. We consider for fixed $|\bar{\beta}| > 0$, $x \in \mathbb{R}^d$ and $t_1, ..., t_{n-1} > 0$ the integral

$$\begin{aligned}
I_n &= \int\limits_{t_n=0}^{\infty} t_n^{-r_{\delta_n}q} \mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x)^q \frac{dt_n}{t_n} \\
&= \int\limits_{t_n=0}^{\infty} t_n^{-r_{\delta_n}q} \Big(\int\limits_{-1}^{1} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_ih_i,\delta_i} \Big) f(x) \Big| d(h_1, ..., h_{n-1})\, dh_n \Big)^q \frac{dt_n}{t_n} \\
&\le 2^q \int\limits_{t_n=0}^{\infty} t_n^{-r_{\delta_n}q} \Big(\frac{1}{t_n} \int\limits_{-t_n/2}^{t_n/2} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n-1} \triangle^{m_{\delta_i}}_{t_ih_i,\delta_i} \Big) \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f(x) \Big| d(h_1, ..., h_{n-1})\, dh_n \Big)^q \frac{dt_n}{t_n} \\
&+ 2^q \int\limits_{t_n=0}^{\infty} t_n^{-r_{\delta_n}q} \Big(\frac{1}{t_n} \int\limits_{t_n \ge |h_n| > \frac{t_n}{2}} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n-1} \triangle^{m_{\delta_i}}_{t_ih_i,\delta_i} \Big) \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f(x) \Big| d(h_1, ..., h_{n-1})\, dh_n \Big)^q \frac{dt_n}{t_n}\ .
\end{aligned}$$

An elementary substitution leads to

$$I_n \leq 2^{-r\delta_n q} \cdot I_n + c \int\limits_{t_n=0}^{\infty} t_n^{-r\delta_n q} \Big(\int\limits_{1 \geq |h_n| > \frac{1}{2}} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| \, d(h_1, ..., h_{n-1}) \, dh_n \Big)^q \frac{dt_n}{t_n}. \qquad (79)$$

Hence we have

$$I_n \leq c' \int\limits_{t_n=0}^{\infty} t_n^{-r\delta_n q} \Big(\int\limits_{1 \geq |h_n| > \frac{1}{2}} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| \, d(h_1, ..., h_{n-1}) \, dh_n \Big)^q \frac{dt_n}{t_n}$$

and iteration of this procedure gives us

$$S^R_{\bar\beta}(f) \leq c'' \Big\| \Big[\int\limits_{(0,\infty)^n} \big(\prod_{i=1}^{n} t_i^{-r_{\delta_i} q} \big) \Big(\int\limits_{\substack{1 \geq |h_i| > 1/2 \\ i=1,...,n}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| d\bar h \Big)^q \frac{d\bar t}{\bar t} \Big]^{1/q} \Big| L_p \Big\| =: c'' S^{R'}_{\bar\beta}(f).$$

The usual modifications lead to the same result in case $q = \infty$. So let us define the quantity

$$\|f|S^{\bar r}_{p,q}F(\mathbb{R}^d)\|^{R'} := \|f|L_p(\mathbb{R}^d)\| + \sum_{|\bar\beta| \geq 1} S^{R'}_{\bar\beta}(f).$$

□

Remark 3.6.1 Let us mention one important problem. It is clear, that the argument used in (79) can only be applied if $I_n < \infty$. Assume f belongs to the space $S^{\bar r}_{p,q}F(\mathbb{R}^d)$. Then one only considers such points x where $A_{\bar\beta}(x) < \infty$ (see (31)). Fubini's theorem then tells us, that $I_n(t_1, ..., t_{n-1}) < \infty$ for almost all $(t_1, ..., t_{n-1}) \in (0, \infty)^{n-1}$. So the exception set has measure zero and does not play any role for the integration with respect to $t_1, ..., t_{n-1}$. On the other hand, this argument causes problems, if f does not belong to the space $S^{\bar r}_{p,q}F(\mathbb{R}^d)$. Because of $\|f|S^{\bar r}_{p,q}F\|^{R'} \leq \|f|S^{\bar r}_{p,q}F\|^R$ for $f \in L_p(\mathbb{R}^d) \cap S'(\mathbb{R}^d)$, it is not clear, that $\|f|S^{\bar r}_{p,q}F\|^R = \infty$ implies $\|f|S^{\bar r}_{p,q}F\|^{R'} = \infty$. In that sense $\|f|S^{\bar r}_{p,q}F\|^{R'}$ can only be used as an equivalent norm and not as a characterization. □

The following theorem is in some sense the counterpart of [Tr 2, Th. 2.5.10] and [ST, Th. 2.3.3], respectively.

Theorem 3.6.1 Let $0 < p < \infty$, $0 < q \leq \infty$, $\bar r = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar m = (m_1, ..., m_d) \in \mathbb{N}_0^d$ with $\bar m > \bar r > \frac{1}{\min(p,q)}$. Then the quantities

$$\|f|S^{\bar r}_{p,q}F(\mathbb{R}^d)\|^{M/\Delta} = \|f|L_p(\mathbb{R}^d)\| + \sum_{\bar\beta \in \{0,1\}^d, |\beta| \geq 1} S^{M/\Delta}_{\bar\beta}(f) \quad ,$$

are equivalent (quasi-)norms in $S^{\bar r}_{p,q}F(\mathbb{R}^d)$, where for $|\bar\beta| = n \geq 1$

(i)

$$S^M_{\bar\beta}(f) := \Big\| \Big[\int\limits_{(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big) \sup_{\substack{|h_i|\le t_i\\ i=1,\dots,n}} \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big|^q \frac{d\bar t}{\bar t}\Big]^{1/q} \Big| L_p(\mathbb{R}^d)\Big\| \quad \text{and}$$

(ii)

$$S^\Delta_{\bar\beta}(f) := \Big\| \Big[\int\limits_{\mathbb{R}^n} \big(\prod_{i=1}^n |h_i|^{-r_{\delta_i} q}\big) \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big|^q \frac{d\bar h}{\tilde h}\Big]^{1/q} \Big| L_p(\mathbb{R}^d)\Big\| \qquad (80)$$

with $\tilde h = (|h_1|,...,|h_n|)$ and $\bar\delta$ as above. In case $q=\infty$ one modifies in both cases

$$S^{M/\Delta}_{\bar\beta}(f) = \Big\| \sup_{\substack{\bar h\in\mathbb{R}^n\\ h\neq 0}} \big(\prod_{i=1}^n |h_i|^{-r_{\delta_i}}\big) \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big| \Big| L_p(\mathbb{R}^d)\Big\|.$$

Proof Step 1: We show $\|f|S^{\bar r}_{p,q}F\|^M \le c\|f|S^{\bar r}_{p,q}F\|^{\bar\varphi}$ following step 1 of the proof of Theorem 3.4.1. First we descretisise the $\bar t$-integration and obtain (with the same notations used in Theorem 3.4.1)

$$\begin{aligned} A_{\bar\beta}(x) &= \int\limits_{[0,\infty]^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big) \sup_{\substack{|h_i|\le t_i\\ i=1,\dots,n}} \big|\big(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{h_n,\delta_n} f\big)(x)\big|^q \frac{d\bar t}{\bar t} \\ &\le c_1 \sum_{\bar k\in Z_{\bar\beta}} 2^{\bar r\bar k q} \sup_{\bar h\in[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta} f(x)|^q. \end{aligned}$$

Now we use again (33) and obtain in case $q\le 1$

$$A_{\bar\beta}(x) \le c_2 \sum_{\bar\ell\in Z_{\bar\beta}} \sum_{\bar u\in \bar N_{\bar\beta}} \sum_{\bar k\in Z_{\bar\beta}} 2^{\bar r\bar k q} \sup_{\bar h\in[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta} f_{\bar k+\bar\ell+\bar u}(x)|^q.$$

In case $q>1$ it holds an estimate similar to (35). What follows is much simpler on the one hand, but much more restrictive on the other hand. Having no integral means for estimating them by Hardy-Littlewood maximal functions we argue as follows. $Z_{\bar\beta}$ is decomposed as done before, but now we directly combine the modified estimates (39) and (41), i.e.

$$\begin{aligned} \sup_{\bar h\in[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h} f_{\bar k+\bar\ell+\bar u}(x)| &= \sup_{\bar h\in[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta}\circ\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta-\bar\alpha} f_{\bar k+\bar\ell+\bar u}(x)| \\ &\le c_3 2^{\ell_1 m_1}\cdot\ldots\cdot 2^{\ell_{|\bar\alpha|} m_{|\bar\alpha|}} 2^{\ell_{|\bar\alpha|+1} a_{|\bar\alpha|+1}}\cdot\ldots\cdot 2^{\ell_n a_n} P_{\bar b,\bar a,\bar\beta}(f_{\bar k+\bar\ell+\bar u})(x). \end{aligned}$$

Recall that $\bar\beta=(1,...,1,0,...,0)$ with $|\bar\beta|=n$, $\bar\alpha=(1,...,1,0,...,0)$ with $\bar\alpha\le\bar\beta$ and $\bar b=(2^{k_1+\ell_1},...,2^{k_d+\ell_d})$. Now the stronger condition $\bar r>1/\min(p,q)$ is required. In this case it is possible to choose a tuple $\bar a=(a_1,...,a_d)$ such that $\bar r>\bar a>1/\min(p,q)$. Proceeding similar to Theorem 3.4.1 we get the estimate

$$\begin{aligned} A^{\bar\alpha}_{\bar\beta}(x) \le{}& c_4 \sum_{\bar\ell\in Z^\alpha_{\bar\beta}} 2^{\ell_1(m_1-r_1)q}\cdot\ldots\cdot 2^{\ell_{|\bar\alpha|}(m_{|\bar\alpha|}-r_{|\bar\alpha|})q}\cdot 2^{\ell_{|\bar\alpha|+1}(a_{|\bar\alpha|+1}-r_{|\bar\alpha|+1})q}\cdot\ldots\cdot 2^{\ell_n(a_n-r_n)q} \\ &\cdot \sum_{\bar u\in\bar N_{\bar\beta}} \sum_{\bar k\in Z_{\bar\beta}} 2^{\bar r(\bar k+\bar\ell)q} |P_{\bar b,\bar a,\bar\beta}(f_{\bar k+\bar\ell+\bar u})|^q(x) \end{aligned}$$

instead of (45). The rest is strait forward and essentially a consequence of Theorem 3.1.5.

<u>*Step 2:*</u> It is sufficient to show $\|f|S^{\bar{r}}_{p,q}F\|^{R'} \le c\|f|S^{\bar{r}}_{p,q}F\|^{\Delta}$ for all $f \in S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$.
The case $q=\infty$ is trivial. Let us first consider the case $1 \le q < \infty$. We again refer to (21). For fixed $x \in \mathbb{R}^d$, Fubini's theorem and $L_q \hookrightarrow L_1$ on compact domains give

$$\begin{aligned} &\int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\Big)\Bigg(\int\limits_{\substack{1\ge|h_i|>1/2\\ i=1,\ldots,n}} \Big|\Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{t_i h_i,\delta_i}\Big) f(x)\Big| d\bar{h}\Bigg)^q \frac{d\bar{t}}{\bar{t}} \\ &\le c_1 \int\limits_{\mathbb{R}^n} \Big|\Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i,\delta_i}\Big) f(x)\Big|^q \cdot \int\limits_{\substack{|h_i|\le t_i\le 2|h_i|\\ i=1,\ldots,n}} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\Big)\frac{1}{\bar{t}}\frac{d\bar{t}}{\bar{t}}\, d\bar{h} \quad , \end{aligned} \tag{81}$$

where

$$\int\limits_{\substack{|h_i|\le t_i\le 2|h_i|\\ i=1,\ldots,n}} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\Big)\frac{1}{\bar{t}}\frac{d\bar{t}}{\bar{t}} \;\le\; c_2\,\frac{\prod_{i=1}^n |h_i|^{-r_{\delta_i} q}}{\prod_{i=1}^n |h_i|}\,.$$

It remains to discuss the case $0<q<1$. We have

$$\begin{aligned} \Bigg(\int\limits_{\substack{1\ge|h_i|>1/2\\ i=1,\ldots,n}} \Big|\Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i,\delta_i}\Big) f(x)\Big| d\bar{h}\Bigg)^q &\le \sup_{\substack{|h_i|\le t_i\\ i=1,\ldots,n}} \Big|\Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i,\delta_i}\Big) f(x)\Big|^{q(1-q)} \times \\ &\times \Bigg(\frac{1}{\bar{t}}\int\limits_{\substack{t_i\ge|h_i|>t_i/2\\ i=1,\ldots,n}} \Big|\Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i,\delta_i}\Big) f(x)\Big|^q d\bar{h}\Bigg)^q. \end{aligned}$$

Now we apply Hölder's inequality with the exponents $\frac{1}{q}$ and $\frac{1}{1-q}$ and obtain

$$\begin{aligned} &\Bigg[\int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\Big)\Bigg(\int\limits_{\substack{1\ge|h_i|>1/2\\ i=1,\ldots,n}} \Big|\Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{t_i h_i,\delta_i}\Big) f(x)\Big| d\bar{h}\Bigg)^q \frac{d\bar{t}}{\bar{t}}\Bigg]^{1/q} \\ &\le \Bigg[\int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\Big)\frac{1}{\bar{t}} \int\limits_{\substack{t_i\ge|h_i|>t_i/2\\ i=1,\ldots,n}} \Big|\Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i,\delta_i}\Big) f(x)\Big|^q d\bar{h}\,\frac{d\bar{t}}{\bar{t}}\Bigg] \times \\ &\times \Bigg[\int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\Big) \sup_{\substack{|h_i|\le t_i\\ i=1,\ldots,n}} \Big|\Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i,\delta_i}\Big) f(x)\Big|^q\Bigg]^{\frac{1-q}{q}}. \end{aligned}$$

Using Fubini and Hölder's inequality again, we obtain the estimate

$$S^{R'}_{\bar{\beta}}(f) \le \big(\|f|S^{\bar{r}}_{p,q}F\|^{\Delta}\big)^q \cdot \big(\|f|S^{\bar{r}}_{p,q}F\|^{M}\big)^{1-q}.$$

Having additionally

$$\begin{aligned} \|f|L_p\| &\le \|f|L_p\|^q \cdot \|f|L_p\|^{1-q} \\ &\le \big(\|f|S^{\bar{r}}_{p,q}F\|^{\Delta}\big)^q \cdot \big(\|f|S^{\bar{r}}_{p,q}F\|^{M}\big)^{1-q} \end{aligned}$$

it turns out, that

$$\|f|S^{\bar r}_{p,q}F\|^{R'} \le c_3\big(\|f|S^{\bar r}_{p,q}F\|^{\Delta}\big)^q \cdot \big(\|f|S^{\bar r}_{p,q}F\|^{M}\big)^{1-q}.$$

With the help of step 1 and Proposition 3.6.1 we can use

$$\|f|S^{\bar r}_{p,q}F\|^{M} \le c'\|f|S^{\bar r}_{p,q}F\|^{R'}.$$

to finish the proof. □

Remark 3.6.2 Step 2 proves $\|f|S^{\bar r}_{p,q}F(\mathbb{R}^d)\|^{R'} \le c\|f|S^{\bar r}_{p,q}F(\mathbb{R}^d)\|^{M}$ on the basis of $\|f|S^{\bar r}_{p,q}F(\mathbb{R}^d)\|^{R'} \le c\|f|S^{\bar r}_{p,q}F(\mathbb{R}^d)\|^{\Delta}$. Of course this can also be proven directly by using the $\triangle$-inequality for integrals starting with $\|f|S^{\bar r}_{p,q}F(\mathbb{R}^d)\|^{R}$. Hence we showed even more than stated in the theorem. Under the given conditions $S^{\bar r}_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in S'(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$ such that $\|f|S^{\bar r}_{p,q}F(\mathbb{R}^d)\|^{M}$ is finite. □

Remark 3.6.3 With a strategy similar to the proof of Proposition 3.5.1 one can replace $S^{\Delta}_{\bar\beta}(f)$, $|\bar\beta| \ge 1$, by

$$S^{\Delta,L}_{\bar\beta}(f) := \Bigg\|\Bigg[\int\limits_{[-1,1]^n} \Big(\prod_{i=1}^n |h_i|^{-r_{\delta_i}q}\Big)\big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{h_n,\delta_n}f)(x)\big|^q\,\frac{d\bar h}{\tilde h}\Bigg]^{1/q}\Bigg|L_p(\mathbb{R}^d)\Bigg\|$$

in the case $0 < q \le p \le \infty$. Hence the corresponding quantity $\|\cdot|S^{\bar r}_{p,q}F(\mathbb{R}^d)\|^{\Delta,L}$ is an equivalent (quasi-)norm in $S^{\bar r}_{p,q}F(\mathbb{R}^d)$ for $\bar r > \big(\frac1q - 1\big)_+$. □

3.7 Integral means of differences for $S^{\bar r}_{p,q}B(\mathbb{R}^d)$

This paragraph deals with Besov spaces of dominating mixed smoothness property. We give a characterization of $S^{\bar r}_{p,q}B(\mathbb{R}^d)$, $0 < p \le \infty$, $0 < q \le \infty$ and $\bar r > \sigma_p$, using integral means of differences. Our main theorem is the counterpart of Theorem 3.4.1. In fact, it is not surprising, that our techniques work in B-case too. The situation is much simpler. We use scalar maximal inequalities instead of corresponding inequalities for the vector-valued case, cf. Paragraph 3.1. Consequently the condition to $\bar r$ gets independent of q and therefore it is possible to give a characterization for $\bar r > \sigma_p$.
Our main result reads as follows.

Theorem 3.7.1 Let $0 < p \le \infty$, $0 < q \le \infty$, $\bar r = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar m = (m_1, ..., m_d) \in \mathbb{N}_0^d$ with $\bar m > \bar r > \sigma_p$. Under these conditions the space $S^{\bar r}_{p,q}B(\mathbb{R}^d)$ is the collection of all functions $f \in L_p(\mathbb{R}^d) \cap S'(\mathbb{R}^d)$ such that

$$\|f|S^{\bar r}_{p,q}B(\mathbb{R}^d)\|^{R} = \|f|L_p(\mathbb{R}^d)\| + \sum_{\bar\beta\in\{0,1\}^d, |\beta|\ge 1} S^{R}_{\bar\beta}(f) < \infty \quad ,$$

where for $|\bar\beta| = n \ge 1$

$$S^{R}_{\bar\beta}(f) = \Bigg[\int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i}q}\Big)\Bigg\|\int\limits_{[-1,1]^n} \big|\big(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{t_nh_n,\delta_n}f\big)(x)\big|d\bar h\Bigg|L_p\Bigg\|^q\frac{d\bar t}{\bar t}\Bigg]^{1/q} \qquad (82)$$

with $\bar{\delta}$ in the sense of (2). In case $q=\infty$ one has to replace (82) by

$$S^R_{\bar\beta}(f) = \sup_{\bar t\in(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i}}\Big)\Big\| \int_{[-1,1]^n} \Big|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{t_nh_n,\delta_n} f)(x)\Big|\, d\bar h\Big| L_p(\mathbb{R}^d)\Big\|.$$

Moreover $\|\cdot|S^{\bar r}_{p,q}B(\mathbb{R}^d)\|^R$ is an equivalent (quasi-)norm in $S^{\bar r}_{p,q}B(\mathbb{T}^d)$.

Proof Step 1: We follow the first step of the proof of Theorem 3.4.1. At the point, where λ and $\bar a$ were chosen, we modify in the following way. Choose $\bar a > \frac{1}{p}$ and $0<\lambda<p$ such that $\bar r > \bar a(1-\lambda)$. In case $p>1$ we simply choose $\lambda = 1$. Let us fix a $|\bar\beta|\geq 1$. We continue using the methods from Theorem 3.4.1 with the obvious modifications to discretisise the quantity $S^R_{\bar\beta}(f)$. This leads to

$$S^R_{\bar\beta}(f) \leq \Big(\sum_{\bar k\in Z_{\bar\beta}} 2^{\bar r\bar k q}\Big\| \int_{[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h} f(x)| d\bar h\Big| L_p\Big\|^q\Big)^{1/q}.$$

If $p\geq 1$ we use (33) and the $\triangle$-inequality in $L_p(\mathbb{T}^d)$ to obtain similar estimates as given in (34) and (35). In case $p<1$ we have to modify a bit. It is necessary to use the p-Banach-$\triangle$-inequality to get the sums out of the L_p-norm. That means

$$\begin{aligned}\Big\|\int_{[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta} f(x)|\, d\bar h\Big| L_p\Big\| &\leq \Big\|\sum_{\bar\ell\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}}\int_{[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta} f_{\bar k+\bar\ell+\bar u}(x)|\, d\bar h\Big| L_p\Big\|\\ &\leq \Big(\sum_{\bar\ell\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}}\Big\|\int_{[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta} f_{\bar k+\bar\ell+\bar u}(x)|\, d\bar h\Big| L_p\Big\|^p\Big)^{1/p}.\end{aligned}$$

And therefore it holds

$$S^R_{\bar\beta}(f) \leq c\Big(\sum_{\bar k\in Z_{\bar\beta}} 2^{\bar r\bar k q}\Big[\sum_{\bar\ell\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}}\Big\|\int_{[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta} f_{\bar k+\bar\ell+\bar u}(x)|\, d\bar h\Big| L_p\Big\|^p\Big]^{q/p}\Big)^{1/q}. \qquad (83)$$

In case $q\leq p$ the usual trick leads to

$$S^R_{\bar\beta}(f) \leq c\Big(\sum_{\bar\ell\in Z_{\bar\beta}}\sum_{\bar k\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}} 2^{\bar r\bar k q}\Big\|\int_{[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta} f_{\bar k+\bar\ell+\bar u}(x)|\, d\bar h\Big| L_p\Big\|^q\Big)^{1/q}. \qquad (84)$$

If $q>p$ the triangle-inequality in $\ell_{q/p}$ applied to (83) gives us

$$\begin{aligned}S^R_{\bar\beta}(f) &\leq c\Big\|\sum_{\bar\ell\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}}\Big\| 2^{\bar r\bar k}\int_{[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta} f_{\bar k+\bar\ell+\bar u}(x)|\, d\bar h\Big| L_p\Big\|^p\Big|\ell_{q/p}\Big\|^{1/p}\\ &\leq c_1\Big[\sum_{\bar\ell\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}}\Big(\sum_{\bar k\in Z_{\bar\beta}} 2^{\bar r\bar k q}\Big\|\int_{[-1,1]^d} |\triangle^{\bar m}_{2^{-\bar k}*\bar h,\bar\beta} f_{\bar k+\bar\ell+\bar u}(x)|\, d\bar h\Big| L_p\Big\|^q\Big)^{p/q}\Big]^{1/p}.\end{aligned} \qquad (85)$$

Let us handle (84) and (85) in the same manner as we did it with (34) and (35), respectively. The rest follows analogously to the F-case. We decompose $S^R_{\bar\beta}(f)$ into the summands $S^{R,\bar\alpha}_{\bar\beta}(f)$ by dividing the index-set $Z_{\bar\beta}$ into the subsets $Z^{\bar\alpha}_{\bar\beta}$ (see (37) for details). Hence instead of (46) and (51), respectively, the estimates (depending on $|\bar\alpha|$ and p) look like

$$S^{R,\bar\alpha}_{\bar\beta}(f) \leq c_2 \begin{cases} \Big[\sum\limits_{\bar k\in\mathbb{N}_0^d} 2^{\bar r\bar k q}\big\||P_{2^{\bar k},\bar a,\bar\beta}f_{\bar k}|^{(1-\lambda)}(x)\cdot \mathbf{M}_{\bar\beta-\bar\alpha}(|P_{2^{\bar k},\bar a,\bar\alpha}f_{\bar k}|^{\lambda})(x)\big|L_p\big\|^q\Big]^{1/q} \\ \Big[\sum\limits_{\bar k\in\mathbb{N}_0^d} 2^{\bar r\bar k q}\big\|\mathbf{M}_{\bar\beta-\bar\alpha}\circ P_{2^{\bar k},\bar a,\bar\alpha}(f_{\bar k})(x)\big|L_p\big\|^q\Big]^{1/q} \\ \Big[\sum\limits_{\bar k\in\mathbb{N}_0^d} 2^{\bar r\bar k q}\big\|P_{2^{\bar k},\bar a,\bar\beta}(f_{\bar k})(x)\big|L_p\big\|^q\Big]^{1/q} \end{cases}.$$

In the first case we continue using Hölder's inequality for integrals with respect to $\frac{1}{\lambda}$ and $\frac{1}{1-\lambda}$. The effect is, that we are able to use the scalar Hardy-Littlewood maximal inequality, cf. Theorem 3.1.3, for the space $L_{p/\lambda}$, where $p/\lambda > 1$. In the second case we do this directly because of $\lambda = 1$ i.e. $p > 1$. We finish step 1 by applying the scalar version of Theorem 3.1.5. The case $p = \infty$ is included there.

Step 2: We follow step 2 in the proof of Theorem 3.4.1 until we arrive at (62). From now on we replace $\|\cdots|L_p(\ell_q)\|$ by $\|\cdots|\ell_q(L_p)\|$ and obtain similar to (64)

$$\|f|S^{\bar r}_{p,q}B\|^{\bar\varphi} \leq c \sum_{\bar\beta\in\{0,1\}^d} \Big\|2^{\bar r\cdot\bar\ell}\int\limits_{\mathbb{R}^d} |g(\bar h)|\cdot\big|\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\beta} f(x)\big|\,d\bar h\Big|\ell_{q|N_{\bar\beta}}(L_p)\Big\|.$$

The rest of the proof can again be copied almost word by word from the F-case. We obtain for

$$T_{\bar\beta}(f) := \Big\|2^{\bar r\cdot\bar\ell}\int\limits_{\mathbb{R}^d} |g(\bar h)|\cdot\big|\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\beta} f(x)\big|\,d\bar h\Big|\ell_{q|N_{\bar\beta}}(L_p)\Big\|$$

in the case $|\bar\beta| \geq 1$ for every $s > 0$ the estimate

$$\begin{aligned} T_{\bar\beta}(f) &\leq c_s \sum_{\bar\mu\in\mathbb{N}_0^d} \Big\|2^{\bar r\bar\ell}2^{|\bar\mu|(1-s)}2^{-\bar\beta\cdot(\bar\mu-\bar\ell)}2^{-\bar\mu(\bar 1-\bar\beta)}\int\limits_{Q_{\bar\mu-\bar\beta*\bar\ell}} \big|\triangle^{\bar m}_{\bar h,\bar\beta} f(x)\big|\,d\bar h\Big|\ell_{q|N_{\bar\beta}}(L_p)\Big\| \\ &\leq c'_s \sum_{\bar\mu\in\mathbb{N}_0^d} 2^{\bar\mu\cdot(1-s+\bar\beta*\bar r)}\Big\|\bar t^{-\bar r*\bar\beta}\int\limits_{[-1,1]^d} \big|\triangle^{\bar m}_{\bar t*\bar h,\bar\beta} f(x)\big|\,d\bar h\Big|\ell_{q|N_{\bar\beta}}(L_p)\Big\| \\ &= c'_s \sum_{\bar\mu\in\mathbb{N}_0^d} 2^{\bar\mu\cdot(1-s+\bar\beta*\bar r)}\Big(\sum_{\bar\ell\in N_{\bar\beta}} \bar t^{-(\bar r*\bar\beta)q}\Big\|\int\limits_{[-1,1]^d} \big|\triangle^{\bar m}_{\bar t*\bar h,\bar\beta} f(x)\big|\,d\bar h\Big|L_p\Big\|^q\Big)^{1/q}, \end{aligned} \tag{86}$$

for all $\bar t \in Q^{+\Delta}_{\bar\mu-\bar\beta*\bar\ell+1}$. Choosing s large enough and replacing the $\bar\ell$-sum by an integral over $(0,\infty)^n$ by standard arguments one obtains

$$T_{\bar\beta}(f) \leq c_1\Big(\int\limits_{(0,\infty)^n}\big(\prod_{i=1}^n t_i^{-r\delta_i}\big)\Big\|\int\limits_{[-1,1]^n}\big|\big(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1}\circ\ldots\circ\triangle^{m_{\delta_n}}_{t_nh_n,\delta_n}f\big)(x)\big|\,d\bar h\Big|L_p\Big\|^q\frac{d\bar t}{\bar t}\Big)^{1/q}. \tag{87}$$

This finishes the proof. □

Remark 3.7.1 (Localization) As already done in the F-case we consider the question, whether one can replace the $(0,\infty)$-integrals in (82) by $(0,1)$-integrals. We obtain a positive answer for $1 \le p \le \infty$ and $0 < q \le \infty$. The restriction $p \ge 1$ has only technical reasons (generalized Minkowski's inequality) and does not seem to be a natural condition. In particular for $p < 1$ the problem is open. $\square$

3.8 Further characterizations

The aim of the present paragraph is to prove the results from [ST, 2.3.4] for arbitrary d with help of Theorem 3.7.1. First of all we modify the integral means appropriately. See also Proposition 3.6.1.

Proposition 3.8.1 Let $p, q, \bar{r}$ be given as in Theorem 3.7.1. In the sense of equivalent (quasi-)norms the integral mean $\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x)$ in (82) can be replaced by $\bar{\mathcal{R}}^{\bar{m}}_{\bar{t},\bar{\beta}} f(x)$, cf. (77), (78). We consider for $p < 1$ and $q < \infty$ the integral

$$I_n := \int\limits_{t_n=0}^{\infty} t_n^{-r\delta_n q} \Big\| \int\limits_{-1}^{1} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| \, d(h_1, ..., h_{n-1}) \, dh_n \Big| L_p \Big\|^q \frac{dt_n}{t_n}$$

$$\le \int\limits_{t_n=0}^{\infty} t_n^{-r\delta_n q} \Big(\Big\| \int\limits_{-1/2}^{1/2} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| \, d(h_1, ..., h_{n-1}) \, dh_n \Big| L_p \Big\|^p$$

$$+ \Big\| \int\limits_{|h_n|>1/2} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| \, d(h_1, ..., h_{n-1}) \, dh_n \Big| L_p \Big\|^p \Big)^{q/p} \frac{dt_n}{t_n}.$$

In case $q/p \le 1$ we immediately obtain by the same trick used in the F-case

$$I_n \le c \int\limits_{t_n=0}^{\infty} t_n^{-r\delta_n q} \Big\| \int\limits_{|h_n|>1/2} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| \, d(h_1, ..., h_{n-1}) \, dh_n \Big| L_p \Big\|^q \frac{dt_n}{t_n}. \qquad (88)$$

In case $q/p > 1$ we apply the $\triangle$-inequality for the space $L_{q/p}$ and obtain

$$I_n \le \Big[\Big(\int\limits_{t_n=0}^{\infty} t_n^{-r\delta_n q} \Big\| \int\limits_{-1/2}^{1/2} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| \, d(h_1, ..., h_{n-1}) \, dh_n \Big| L_p \Big\|^q \frac{dt_n}{t_n} \Big)^{p/q}$$

$$+ \Big(\int\limits_{t_n=0}^{\infty} t_n^{-r\delta_n q} \Big\| \int\limits_{|h_n|>1/2} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| \, d(h_1, ..., h_{n-1}) \, dh_n \Big| L_p \Big\|^q \frac{dt_n}{t_n} \Big)^{p/q} \Big]^{q/p}.$$

Consequently it holds

$$I_n^{p/q} \le c \Big(\int\limits_{t_n=0}^{\infty} t_n^{-r\delta_n q} \Big\| \int\limits_{|h_n|>1/2} \int\limits_{[-1,1]^{n-1}} \Big| \Big(\prod_{i=1}^{n} \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Big| \, d(h_1, ..., h_{n-1}) \, dh_n \Big| L_p \Big\|^q \frac{dt_n}{t_n} \Big)^{p/q},$$

which implies (88). The case $p \ge 1$ can be carried over almost word by word from the F-case. We denote the corresponding (quasi-)norm by $\|f|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\|^{R'}$. $\square$

Let us now prove the dominating mixed counterpart of the classical difference characterization for Besov spaces, cf. [Tr 2, 2.5.12]. We need to compute moduli of smoothness with respect to a rectangle, given by

$$(\prod_{i=1}^{n} t_i^{-r_{\delta_i}}) \sup_{\substack{|h_i|\le t_i\\ i=1,...,n}} \big\| \big(\triangle_{h_1,\delta_1}^{m_{\delta_1}} \circ ... \circ \triangle_{h_n,\delta_n}^{m_{\delta_n}} f\big)(x)\big|L_p(\mathbb{R}^d)\big\| \ , \ t_i > 0 \ , \ i = 1, ..., n \ .$$

The following theorem points out especially the situation $1 \le p \le \infty$. Having powerful techniques in this case (generalized Minkowski's inequality), we are able to cover all spaces with $\bar{r} > 0$.

Theorem 3.8.1 Let $1 \le p \le \infty$, $0 < q \le \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}_0^d$ with $\bar{m} > \bar{r} > 0$. The following quantities describe equivalent norms in the space $S^{\bar{r}}_{p,q}B(\mathbb{R}^d)$:

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\|^{M/\Delta} = \|f|L_p(\mathbb{R}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\bar{\beta}|\ge 1} S^{M/\Delta}_{\bar{\beta}}(f) \quad ,$$

where for $|\bar{\beta}| = n \ge 1$ (modification if $q = \infty$)

(i)

$$S^M_{\bar{\beta}}(f) = \Bigg(\int\limits_{(0,\infty)^n} (\prod_{i=1}^{n} t_i^{-r_{\delta_i} q}) \sup_{\substack{|h_i|\le t_i\\ i=1,...,n}} \big\| \big(\triangle_{h_1,\delta_1}^{m_{\delta_1}} \circ ... \circ \triangle_{h_n,\delta_n}^{m_{\delta_n}} f\big)(x)\big|L_p(\mathbb{R}^d)\big\|^q \frac{d\bar{t}}{\bar{t}} \Bigg)^{1/q} \quad \text{and}$$

(ii)

$$S^{\Delta}_{\bar{\beta}}(f) = \Bigg[\int\limits_{\mathbb{R}^n} (\prod_{i=1}^{n} |h_i|^{-r_{\delta_i} q}) \big\|(\triangle_{h_1,\delta_1}^{m_{\delta_1}} \circ \cdots \circ \triangle_{h_n,\delta_n}^{m_{\delta_n}} f)(x)\big|L_p(\mathbb{R}^d)\big\|^q \frac{d\bar{h}}{\tilde{h}} \Bigg]^{1/q} \quad ,$$

with $\tilde{h} = (|h_1|, ..., |h_n|)$ and $\bar{\delta}$ as usual.

Proof <u>*Step 1:*</u> We follow [Tr 2, 2.5.12] to show

$$\|f|S^{\bar{r}}_{p,q}B\|^M \le c\|f|S^{\bar{r}}_{p,q}B\|^{\bar{\varphi}}$$

for $f \in S'(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$. With the same arguments used several times it follows in case $q \le 1$

$$S^M_{\bar{\beta}}(f) \le c_1 \Bigg(\sum_{\bar{\ell}\in Z_{\bar{\beta}}} \sum_{\bar{u}\in \bar{N}_{\bar{\beta}}} \sum_{\bar{k}\in \mathbb{Z}_{\bar{\beta}}} 2^{\bar{r}\bar{k}q} \sup_{\bar{h}\in[-1,1]^d} \big\| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar{\beta}} f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\big|L_p\big\|^q \Bigg)^{1/q} . \tag{89}$$

Because of $p \ge 1$ we exploited the usual Δ-inequality in L_p. The supremum is applied after taking the L_p-norm. Hence we can use the translation invariance of the Lebesgue measure. We obtain on the one hand

$$\big\| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar{\beta}} f_{\bar{k}+\bar{\ell}+\bar{u}}\big|L_p\big\| \le c_{\bar{m}}\|f_{\bar{k}+\bar{\ell}+\bar{u}}|L_p\|. \tag{90}$$

On the other hand Lemma 3.3.2 and Remark 3.3.1 give us for the special case $\bar{\beta} = (1, ..., 1, 0, ..., 0)$ the following (see also (27))

$$\big\| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar{\beta}} f_{\bar{k}+\bar{\ell}+\bar{u}}\big|L_p\big\| \le c'_{\bar{m}} A(2^{\ell_1}, 2^{\ell_1 m_1}) \cdot ... \cdot A(2^{\ell_n}, 2^{\ell_n m_n})\|P_{\bar{b},\bar{1}} f_{\bar{k}+\bar{\ell}+\bar{u}}|L_p\|.$$

Hence the scalar versions of Theorem 3.1.5 leads to the estimate

$$\| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar{\beta}} f_{\bar{k}+\bar{\ell}+\bar{u}}|L_p\| \le c_2 A(2^{\ell_1}, 2^{\ell_1 m_1}) \cdot ... \cdot A(2^{\ell_n}, 2^{\ell_n m_n}) \|f_{\bar{k}+\bar{\ell}+\bar{u}}|L_p\|. \tag{91}$$

We combine (90) and (91) in order to get

$$\| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar{\beta}} f_{\bar{k}+\bar{\ell}+\bar{u}}|L_p\| \le c_3 \min(1, 2^{\ell_1 m_1}) \cdot ... \cdot \min(1, 2^{\ell_n m_n}) \cdot \|f_{\bar{k}+\bar{\ell}+\bar{u}}|L_p\|.$$

Putting this into (89) we derive

$$\begin{aligned} S^M_{\bar{\beta}}(f) \le c_4 \Bigg(\sum_{\bar{\ell}\in Z_{\bar{\beta}}} & 2^{-\ell_1 r_1 q} \min(1, 2^{\ell_1 m_1 q}) \cdot ... \cdot 2^{-\ell_n r_n q} \min(1, 2^{\ell_n m_n q}) \times \\ & \times \sum_{\bar{u}\in \bar{N}_{\bar{\beta}}} \sum_{\bar{k}\in \mathbb{Z}_{\bar{\beta}}} 2^{\bar{r}(\bar{k}+\bar{\ell})q} \|f_{\bar{k}+\bar{\ell}+\bar{u}}|L_p\|^q \Bigg)^{1/q}. \end{aligned} \tag{92}$$

Let us increase RHS(92) by putting $2^{\bar{r}\bar{u}q}$ into the $\bar{u}$-sum. Hence it holds

$$\begin{aligned} S^M_{\bar{\beta}}(f) \le & c_5 \Bigg(\sum_{\bar{\ell}\in Z_{\bar{\beta}}} 2^{-\ell_1 r_1 q} \min(1, 2^{\ell_1 m_1 q}) \cdot ... \cdot 2^{-\ell_n r_n q} \min(1, 2^{\ell_n m_n q}) \times \\ & \times \sum_{\bar{u}\in \bar{N}_{\bar{\beta}}} \sum_{\bar{k}\in \mathbb{Z}_{\bar{\beta}}} 2^{\bar{r}\bar{u}q} 2^{\bar{r}(\bar{k}+\bar{\ell})q} \|f_{\bar{k}+\bar{\ell}+\bar{u}}|L_p\|^q \Bigg)^{1/q} \\ = & c_5 \Bigg(\|f|S^{\bar{r}}_{p,q}B\|^q \cdot \sum_{\bar{\ell}\in Z_{\bar{\beta}}} 2^{-\ell_1 r_1 q} \min(1, 2^{\ell_1 m_1 q}) \cdot ... \cdot 2^{-\ell_n r_n q} \min(1, 2^{\ell_n m_n q}) \Bigg)^{1/q} \quad , \end{aligned}$$

where the $\bar{\ell}$-sum is a convergent geometric series, that breaks down to a constant. With obvious modifications done several times one can also treat the case $q \ge 1$.

<u>Step 2:</u> We see immediately $S^R_{\bar{\beta}}(f) \le c' S^M_{\bar{\beta}}(f)$ for all $f \in L_p(\mathbb{R}^d)$ using generalized Minkowski's inequality. It remains to prove $\|f|S^{\bar{r}}_{p,q}B\|^{R'} \le c\|f|S^{\bar{r}}_{p,q}B\|^{\Delta}$. We postpone it to the proof of the next theorem. □

Remark 3.8.1 Under the assumptions of the last theorem one can characterize the space $S^{\bar{r}}_{p,q}B(\mathbb{R}^d)$ by $\| \cdot |S^{\bar{r}}_{p,q}F(\mathbb{R}^d)\|^M$. See also Remark 3.6.2. □

Our last theorem deals with the case $0 < p < 1$. We were able to give a result for $\bar{r} > 1/p$, but not for $\bar{r} > \sigma_p = 1/p - 1$, as it is possible in the isotropic case, cf. [Tr 2, 2.5.12]. This problem still remains open, cf. also [ST, Remark 2.3.4/2].

Theorem 3.8.2 Let $0 < p, q \le \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}^d_0$ with $\bar{m} > \bar{r} > 1/p$. The following quantities describe equivalent (quasi-)norms in the space $S^{\bar{r}}_{p,q}B(\mathbb{R}^d)$:

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\|^{M/M'/\Delta} = \|f|L_p(\mathbb{R}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\bar{\beta}|\ge 1} S^{M/M'/\Delta}_{\bar{\beta}}(f) \quad ,$$

where additionally to Theorem 3.8.1/(i)/(ii) for $|\bar\beta| = n \geq 1$ (modification if $q=\infty$)

$$S^{M'}_{\bar\beta}(f) = \Big(\int\limits_{(0,\infty)^n} (\prod_{i=1}^n t_i^{-r_{\delta_i} q}) \| \sup_{\substack{|h_i|\leq t_i\\ i=1,\ldots,n}} |(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \ldots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)| \big| L_p(\mathbb{R}^d) \|^q \frac{d\bar t}{\bar t} \Big)^{1/q}$$

with $\bar\delta$ as usual.

Proof The same techniques we used in the proof of Theorem 3.6.1 combined with corresponding arguments out of Theorem 3.7.1 lead us to

$$\|f|S^{\bar r}_{p,q}B\|^{M/M'} \leq c\, \|f|S^{\bar r}_{p,q}B\|^{\bar\varphi}. \tag{93}$$

It remains to show $\|f|S^{\bar r}_{p,q}B\|^{R'} \leq c\|f|S^{\bar r}_{p,q}B\|^{\Delta}$ by estimating

$$\Big\| \int\limits_{\substack{1\geq|h_i|>1/2\\ i=1,\ldots,n}} \big|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_nh_n,\delta_n} f)(x)\big| d\bar h \Big| L_p \Big\|^q. \tag{94}$$

Assume $p<1$. With the usual trick and Hölder's inequality with respect to $\frac{1}{p}$ and $\frac{1}{1-p}$ we have

$$\begin{aligned}\Big\| \int\limits_{\substack{1\geq|h_i|>1/2\\ i=1,\ldots,n}} |\cdots|\, d\bar h \Big| L_p \Big\|^q &\leq \Big\| \int\limits_{\substack{1\geq|h_i|>1/2\\ i=1,\ldots,n}} |\cdots|^p\, d\bar h \cdot \sup_{\substack{|h_i|\leq 1\\ i=1,\ldots,n}} |\cdots|^{1-p} \Big| L_p \Big\|^q \\
&\leq \Big(\int\limits_{\substack{1\geq|h_i|>1/2\\ i=1,\ldots,n}} \big\|\cdots\big|L_p\big\|^p\, d\bar h \Big)^{1/p\,\cdot\, qp} \cdot \big\| \sup_{\substack{|h_i|\leq 1\\ i=1,\ldots,n}} |\cdots| \big| L_p \big\|^{q(1-p)} \\
&\leq c_1 \Big(\int\limits_{\substack{1\geq|h_i|>1/2\\ i=1,\ldots,n}} \big\|\cdots\big|L_p\big\|\, d\bar h \Big)^{qp} \cdot \big\| \sup_{\substack{|h_i|\leq 1\\ i=1,\ldots,n}} |\cdots| \big| L_p \big\|^{q(1-p)}.\end{aligned}$$

Using again Hölder's inequality for the $\bar t$-integral (see Theorem 3.6.1) with respect to $\frac{1}{p}$, $\frac{1}{1-p}$ we obtain

$$S^{R'}_{\bar\beta}(f) \leq c_1 \big(S^{M'}_{\bar\beta}(f)\big)^{1-p} \times \Big[\int\limits_{(0,\infty)^n} (\prod_{i=1}^n t_i^{-r_{\delta_i} q}) \Big(\int\limits_{\substack{1\geq|h_i|>1/2\\ i=1,\ldots,n}} \big\|\cdots\big|L_p\big\|\, d\bar h \Big)^q \frac{d\bar t}{\bar t} \Big]^{1/q\,\cdot\, p}. \tag{95}$$

Now we proceed exactly as in step 2 of Theorem 3.6.1. This gives us in case $q\geq 1$

$$S^{R'}_{\bar\beta}(f) \leq c_2 \big(S^{M'}_{\bar\beta}(f)\big)^{1-p} \cdot \big(S^{\Delta}_{\bar\beta}(f)\big)^p$$

and in case $q<1$

$$S^{R'}_{\bar\beta}(f) \leq c_2 \big(S^{M'}_{\bar\beta}(f)\big)^{1-p} \cdot \big(S^{\Delta}_{\bar\beta}(f)\big)^{qp} \big(S^{M}_{\bar\beta}(f)\big)^{(1-q)p}.$$

Finally we use (93) and Proposition 3.8.1 to obtain in both cases

$$\|f|S^{\bar r}_{p,q}B\|^{R'} \leq c_3 \|f|S^{\bar r}_{p,q}B\|^{\Delta}.$$

It remains the case $p \geq 1$. Here we use generalized Minkowski's inequality to estimate (94) by putting the L_p-(quasi-)norm inside the integral. Afterwards we proceed as above and obtain in case $q < 1$.

$$S_{\bar{\beta}}^{R'}(f) \leq c_4 \big(S_{\bar{\beta}}^{\Delta}(f)\big)^q \big(S_{\bar{\beta}}^{M}(f)\big)^{1-q} \,. \tag{96}$$

Obviously (96) also completes step 2 in the proof of Theorem 3.8.1. □

Remark 3.8.2 We showed even more. Similar to $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$, cf. also Remark 3.6.2, the space $S^{\bar{r}}_{p,q}B(\mathbb{R}^d)$ can be characterized by $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\|^{M'}$. □

Remark 3.8.3 The proof for the isotropic case in [Tr 2, 2.5.12] is based on Jackson type inequalities and characterization by approximation. It is not possible to carry over this idea to the dominating mixed scale, since we even do not have a corresponding characterization by quantities of best approximation. □

Remark 3.8.4 The localized versions $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\|^{M,L}$, $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\|^{\Delta,L}$ of $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\|^{M}$ and $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{R}^d)\|^{\Delta}$ are equivalent (quasi-)norms in any case. One estimates the L_p-(quasi-)norm of the differences of f according to the $(1,\infty)$-integrals simply by the norm of f using the translation invariance. Consequently these integrals vanish. Recall also Proposition 3.5.1. □

4 Spaces on $\mathbb{T}^d$

4.1 Preliminaries

Distributions on the torus, periodic distributions

Let $\mathbb{T}^d$ denote the d-torus, represented in the Euclidean space $\mathbb{R}^d$ by the cube $\mathbb{T}^d = [0, 2\pi]^d$, where opposite points are identified. That means, that $x, y \in \mathbb{T}^d$ are identified if and only if $x - y = 2\pi k$, where $k = (k_1, ..., k_d) \in \mathbb{Z}^d$. Let further $D(\mathbb{T}^d)$ denote the collection of all complex-valued infinitely differentiable functions on $\mathbb{T}^d$. In particular one has $f(x) = f(y)$ if $x - y = 2\pi k$ and $f \in D(\mathbb{T}^d)$. A linear functional $f : D(\mathbb{T}^d) \to \mathbb{C}$ belongs to $D'(\mathbb{T}^d)$, if and only if there is a constant $c_N > 0$ such that

$$|f(\varphi)| \leq c_N \sum_{|\bar{\alpha}| \leq N} \sup_{x \in \mathbb{T}^d} \big|D^{\bar{\alpha}} f(x)\big|$$

holds for all $\varphi \in D(\mathbb{T}^d)$ and for some natural number N. $T \in D'(\mathbb{T}^d)$ is said to be a regular distribution, if there exists a measurable function $f : \mathbb{T}^d \to \mathbb{C}$ such that

$$T(\varphi) = \int_{\mathbb{T}^d} f(x) \cdot \varphi(x)\, dx \quad , \quad \varphi \in D(\mathbb{T}^d).$$

Recall that $x \cdot y = \sum_{j=1}^{d} x_j y_j$ for $x, y \in \mathbb{R}^d$. The Fourier coefficients of a distribution $f \in D'(\mathbb{T}^d)$ are the complex numbers given by

$$\hat{f}(k) = (2\pi)^{-d} f(e^{-ik\cdot x}) \quad , \quad k \in \mathbb{Z}^d.$$

In the sense of convergence in $D'(\mathbb{T}^d)$ it holds

$$f = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)e^{ik\cdot x}.$$

See [ST, Ch. 3] for details. Furthermore $T^\Lambda \subset D(\mathbb{T}^d)$ denotes the collection of all trigonometric polynomials with harmonics in Λ, precisely

$$T^\Lambda = \left\{ t(x) = \sum_{k\in\Lambda} a_k e^{ik\cdot x} \middle| a_k \in \mathbb{C} \right\} ,$$

where $\Lambda \subset \mathbb{Z}^d$ is a finite set. Of course every trigonometric polynomial $t(x) \in T^\Lambda$ can also be interpreted as a distribution from $S'(\mathbb{R}^d)$ in the usual sense

$$\varphi \mapsto \int\limits_{\mathbb{R}^d} t(x)\varphi(x)\,dx \quad , \quad \varphi \in S(\mathbb{R}^d) \ , \tag{97}$$

with supp $\mathcal{F}t \subset \Lambda$. In that context it is called periodic distribution.

Vector-valued Lebesgue spaces

Let us define the periodic counterparts of $L_p^\Omega(\mathbb{R}^d, \ell_q)$. We follow [ST, 3.4.1]. $L_p(\mathbb{T}^d)$ denotes the space of all measurable functions $f : \mathbb{T}^d \to \mathbb{C}$ (that means f is 2π-periodic in each direction) such that

$$\|f|L_p(\mathbb{T}^d)\| = \left(\int\limits_{\mathbb{T}^d} |f(x)|^p\,dx \right)^{1/p} < \infty$$

with the usual modification in case $p = \infty$. Also in this situation we have for $1 \le p \le \infty$ the embeddings

$$D(\mathbb{T}^d) \subset L_p(\mathbb{T}^d) \subset D'(\mathbb{T}^d).$$

Furthermore we define for $0 < p < \infty$ and $0 < q \le \infty$ the quantity

$$\|\{f_k\}_{k\in I}|L_p(\mathbb{T}^d, \ell_q)\| := \left(\int\limits_{\mathbb{T}^d} \left(\sum_{k\in I} |f_k(x)|^q \right)^{p/q} dx \right)^{1/p}$$

(modification if $q = \infty$) where the $f_k : \mathbb{T}^d \to \mathbb{C}$ are Lebesgue-measurable. $L_p(\mathbb{T}^d, \ell_q)$ denotes the corresponding (quasi-)Banach-space. Again we have $f_k \in L_p(\mathbb{T}^d)$ for $k \in I$ if $\{f_k\}_{k\in I} \in L_p(\mathbb{T}^d, \ell_q)$. Finally we define the spaces $L_p^\Lambda(\mathbb{T}^d, \ell_q)$, where $\Lambda = \{\Lambda_j\}_{j\in I}$ denotes a sequence of finite subsets of $\mathbb{Z}^d$. We define for $0 < p < \infty$ and $0 < q \le \infty$

$$L_p^\Lambda(\mathbb{T}^d, \ell_q) = \{t = \{t_j(x)\}_{j\in I} : t_j \in T^{\Lambda_j}, j \in I, \|t_j|L_p(\mathbb{T}^d, \ell_j)\| < \infty\}.$$

Maximal functions and maximal inequalities

The maximal functions Mf, M_if as well as $P_{\bar{b},\bar{a}}f$ and $P_{\bar{b},\bar{a},\bar{\beta}}f$ are defined by (8), (10), (11) and (13). Of course these definitions also make sense in the periodic setting. Therefore the existence of periodic counterparts of the Theorems 3.1.2, 3.1.3, 3.1.4 and 3.1.5 is not surprising. Consider for instance [ST, 3.2.4].

Theorem 4.1.1 Let $1 < p < \infty$ and $1 < q \leq \infty$. Then there exists a positive constant $c > 0$ such that

$$\|Mf_k|L_p(\mathbb{T}^d, \ell_q)\| \leq \|f_k|L_p(\mathbb{T}^d, \ell_q)\|$$

holds for all $\{f_k\}_{k\in I} \in L_p(\mathbb{T}^d, \ell_q)$.

Proof Let $g \in L_p(\mathbb{T}^d)$. Because of the periodicity of g one obtains the inequality

$$Mg(x) \leq cM\tilde{g}(x) \quad \text{for} \quad x \in \mathbb{T}^d \quad ,$$

where $\tilde{g}(x)$ is the restriction of $g(x)$ on $[-3\pi, 3\pi]^d$ and in particular not longer a periodic function. Theorem 3.1.2 then implies

$$\begin{aligned}\|Mf_k|L_p(\mathbb{T}^d, \ell_q)\| &\leq c_1\|M\tilde{f}_k|L_p(\mathbb{R}^d, \ell_q)\| \\ &\leq c_2\|\tilde{f}_k|L_p(\mathbb{R}^d, \ell_q)\| \\ &\leq c_3\|f_k|L_p(\mathbb{T}^d, \ell_q)\|.\end{aligned}$$

□

Using the same arguments like in the proof of Theorem 3.1.3 we derive its periodic version directly from the last theorem. We obtain

Theorem 4.1.2 For $1 < p < \infty$ and $1 < q \leq \infty$ there exists a constant $c > 0$ such that

$$\|M_if_k|L_p(\mathbb{T}^d, \ell_q)\| \leq c\|f_k|L_p(\mathbb{T}^d, \ell_q)\| \quad , \quad i = 1, ..., d.$$

holds for all sequences $\{f_k\}_{k\in I} \in L_p(\mathbb{T}^d, \ell_q)$ on $\mathbb{R}^d$. □

Finally we translate also Theorem 3.1.4 into the periodic setting.

Theorem 4.1.3 Let $0 < p < \infty$ and $0 < q \leq \infty$. Let further $\bar{b}^\ell = (b_1^\ell, ..., b_d^\ell) > 0$ for $\ell \in I$ and $\Lambda = \{\Lambda_\ell\}_{\ell\in I}$ a sequence of finite subsets of $\mathbb{Z}^d$ such that

$$\Lambda_\ell \subset \{\xi \in \mathbb{R}^d \,:\, |\xi_i| \leq b_i^\ell, \, i = 1, ..., d\} \quad , \quad \ell \in I.$$

Finally assume $\bar{s} = (s_1, ..., s_d) > \frac{1}{\min(p,q)}$. Then there exists a constant $c > 0$ (independently of t and $\bar{b}^\ell$) such that

$$\left\|P_{\bar{b}^\ell,\bar{s}}t_\ell|L_p(\mathbb{T}^d, \ell_q)\right\| \leq c\left\|t_\ell|L_p(\mathbb{T}^d, \ell_q)\right\|$$

holds for all systems of trigonometric polynomials $t = \{t_\ell\}_{\ell\in I} \subset L_p^\Lambda(\mathbb{R}^d, \ell_q)$.

Proof The assertion is a direct consequence of Lemma 3.1.1 and Theorem 4.1.2. The arguments in the proof of Theorem 3.1.4 carry over literally. □

Of course also Theorem 3.1.5 has a periodic counterpart. It reads as follows.

Theorem 4.1.4 Assume $p, q, \bar{b}^\ell, \bar{s}$ and $\Lambda = \{\Lambda_\ell\}_{\ell\in I}$ as in Theorem 4.1.3. Let further $\bar{\alpha} \in \{0,1\}^d$. Then there exists a constant $c > 0$ (independently of t and Ω) such that

$$\big\|P_{\bar{b}^\ell,\bar{s},\bar{\alpha}} t_\ell | L_p(\mathbb{T}^d, \ell_q)\big\| \leq c \big\| t_\ell | L_p(\mathbb{T}^d, \ell_q)\big\|$$

holds for all systems of trigonometric polynomials $t = \{t_\ell\}_{\ell\in I} \subset L_p^\Lambda(\mathbb{T}^d, \ell_q)$. □

4.2 Definitions and basic properties

This paragraph deals with the Fourier-analytic definition of the spaces $S^{\bar{r}}_{p,q}B$ and $S^{\bar{r}}_{p,q}F$ on the d-torus. One has to combine the techniques in [ST, Ch. 1] with the periodic versions of the stated maximal inequalities to collect all necessary tools. Analogously to the methods in [ST, Ch. 3] introducing the scales $F^s_{p,q}(\mathbb{T}^d)$ and $B^s_{p,q}(\mathbb{T}^d)$, we define $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ and $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$. Recall Definition 3.2.1 (decomposition of unity) and Definition 3.2.2 (spaces on $\mathbb{R}^d$). For the periodic case the building blocks, cf. (15), are given by

$$f_{\bar{\ell}}(x) = \sum_{k\in\mathbb{Z}^d} (\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(k) \hat{f}(k) e^{ik\cdot x}$$

and obviously it holds

$$f = \sum_{\bar{\ell}\in\mathbb{N}_0^d} f_{\bar{\ell}}(x) \quad , \quad \text{convergence in } D'(\mathbb{T}^d).$$

Definition 4.2.1 Let $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$, $0 < q \leq \infty$ and $\varphi^i = \{\varphi^i_j(x)\}_{j=0}^\infty \in \Phi(\mathbb{R})$, $i = 1, ..., d$.

(i) Let $0 < p \leq \infty$. Then $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ is the collection of all $f \in D'(\mathbb{T}^d)$ such that

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{\bar{\varphi}} = \Bigg(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\cdot\bar{\ell}q} \Bigg\| \sum_{k\in\mathbb{Z}^d} (\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(k) \hat{f}(k) e^{ik\cdot x} \Bigg| L_p(\mathbb{T}^d)\Bigg\|^q\Bigg)^{1/q}$$

is finite (modification in case $q = \infty$).

(ii) Let $0 < p < \infty$. Then $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ is the collection of all $f \in D'(\mathbb{T}^d)$ such that

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{\bar{\varphi}} = \Bigg\|\Bigg(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\cdot\bar{\ell}q} \Bigg| \sum_{k\in\mathbb{Z}^d} (\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(k) \hat{f}(k) e^{ik\cdot x}\Bigg|^q\Bigg)^{1/q} \Bigg| L_p(\mathbb{T}^d)\Bigg\|$$

is finite (modification in case $q = \infty$).

□

Remark 4.2.1 All properties of the corresponding spaces on $\mathbb{R}^d$, i.e. Remark 3.2.2, Lemma 3.2.1 and Lemma 3.2.2, carry over almost literally. Only $\mathbb{R}^d$ has to be replaced by $\mathbb{T}^d$. □

4.3 Integral means of differences for $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$

We are now in a situation similar to Section 3. Because of (97) all techniques developed in Paragraph 3.3 apply in the periodic case as well. In addition, with Theorem 4.1.2 and Theorem 4.1.4, we have analogous maximal inequalities. It is clear, how to modify the proof of Theorem 3.4.1 to obtain the following main characterization.

Theorem 4.3.1 Let $0 < p < \infty$, $0 < q \le \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ with $\bar{m} > \bar{r} > \sigma_{p,q}$. Under these conditions the space $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ is the collection of all functions $f \in L_p(\mathbb{T}^d) \cap D'(\mathbb{T}^d)$ such that

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^R = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\beta|\ge 1} S^R_{\bar{\beta}}(f) < \infty \quad , \tag{98}$$

where for $|\bar{\beta}| = n \ge 1$

$$S^R_{\bar{\beta}}(f) = \Big\| \Big[\int\limits_{(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big) \Big(\int\limits_{[-1,1]^n} \big|(\triangle^{m_{\delta_1}}_{t_1 h_1, \delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n, \delta_n} f)(x)\big| d\bar{h} \Big)^q \frac{d\bar{t}}{\bar{t}} \Big]^{1/q} \Big| L_p(\mathbb{T}^d) \Big\|$$

with $\bar{\delta}$ assigned to $\bar{\beta}$ in the sense of (2). In case $q = \infty$ we put

$$S^R_{\bar{\beta}}(f) = \Big\| \sup_{\bar{t}\in(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i}}\big) \int\limits_{[-1,1]^n} \Big|(\triangle^{m_{\delta_1}}_{t_1 h_1, \delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n, \delta_n} f)(x)\Big| d\bar{h} \Big| L_p(\mathbb{T}^d) \Big\| .$$

Moreover (98) is an equivalent (quasi-)norm in $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$. □

By the same reasons we can carry over Theorem 3.5.1 and obtain its periodic version.

Theorem 4.3.2 Let $p = q = 1$ or $1 < p < \infty$, $1 \le q \le \infty$. Let further $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ with $\bar{m} > \bar{r} > \sigma_{p,q}$. Then $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ is the collection of all functions $f \in L_p(\mathbb{T}^d)$ such that

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{R,L} := \|f|L_p(\mathbb{T}^d)\| + \sum_{|\bar{\beta}|\ge 1} S^{R,L}_{\bar{\beta}}(f) < \infty \quad ,$$

where for $|\bar{\beta}| = n \ge 1$ (modification in case $q = \infty$)

$$S^{R,L}_{\bar{\beta}}(f) = \Big\| \Big[\int\limits_{(0,1)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big) \Big(\int\limits_{[-1,1]^n} \big|(\triangle^{m_{\delta_1}}_{t_1 h_1, \delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n, \delta_n} f)(x)\big| d\bar{h} \Big)^q \frac{d\bar{t}}{\bar{t}} \Big]^{1/q} \Big| L_p(\mathbb{T}^d) \Big\|$$

with $\bar{\delta}$ as usual.
Moreover $\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{R,L}$ is an equivalent (quasi-)norm in $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$. □

4.4 Further characterizations

Let us confine on the main result and state the periodic version of Theorem 3.6.1.

Theorem 4.4.1 Let $0 < p < \infty$, $0 < q \leq \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}_0^d$ with $\bar{m} > \bar{r} > \frac{1}{\min(p,q)}$. Then the quantities

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{M/\Delta} = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\beta|\geq 1} S^{M/\Delta}_{\bar{\beta}}(f) \quad ,$$

are equivalent (quasi-)norms in $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$, where for $|\bar{\beta}| = n \geq 1$

(i)

$$S^M_{\bar{\beta}}(f) := \Bigg\| \Big[\int\limits_{(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big) \sup_{\substack{|h_i|\leq t_i \\ i=1,\ldots,n}} \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big|^q \frac{d\bar{t}}{\bar{t}} \Big]^{1/q} \Bigg| L_p(\mathbb{T}^d) \Bigg\| \quad \text{and}$$

(ii)

$$S^\Delta_{\bar{\beta}}(f) := \Bigg\| \Big[\int\limits_{\mathbb{R}^n} \big(\prod_{i=1}^n |h_i|^{-r_{\delta_i} q}\big) \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big|^q \frac{d\bar{h}}{\tilde{h}} \Big]^{1/q} \Bigg| L_p(\mathbb{T}^d) \Bigg\|$$

with $\tilde{h} = (|h_1|, ..., |h_n|)$ and $\bar{\delta}$ as usual. In case $q = \infty$ one modifies in (i) and (ii) to

$$S^{M/\Delta}_{\bar{\beta}}(f) = \Bigg\| \sup_{\substack{\bar{h}\in\mathbb{R}^n \\ h\neq 0}} \big(\prod_{i=1}^n |h_i|^{-r_{\delta_i}}\big) \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big| \Bigg| L_p(\mathbb{T}^d) \Bigg\|.$$

□

Remark 4.4.1 All results and remarks concerning characterization and localization keep valid. □

4.5 Integral means of differences for $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$

Let us state here and in the next paragraph the corresponding results for the periodic B-scale. The main characterization by integral means reads as follows.

Theorem 4.5.1 Let $0 < p \leq \infty$, $0 < q \leq \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}_0^d$ with $\bar{m} > \bar{r} > \sigma_p$. Under these conditions the space $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ is the collection of all functions $f \in L_p(\mathbb{T}^d) \cap D'(\mathbb{T}^d)$ such that

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^R = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\beta|\geq 1} S^R_{\bar{\beta}}(f) < \infty \quad ,$$

where for $|\bar{\beta}| = n \geq 1$

$$S^R_{\bar{\beta}}(f) = \Big[\int\limits_{(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big) \Bigg\| \int\limits_{[-1,1]^n} \big|\big(\triangle^{m_{\delta_1}}_{t_1 h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n,\delta_n} f\big)(x)\big| d\bar{h} \Bigg| L_p(\mathbb{T}^d) \Bigg\|^q \frac{d\bar{t}}{\bar{t}} \Big]^{1/q}$$

with $\bar{\delta}$ as usual. In case $q = \infty$ one modifies as follows

$$S^R_{\bar{\beta}}(f) = \sup_{\bar{t}\in(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i}}\Big)\Big\| \int\limits_{[-1,1]^n} \Big|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_nh_n,\delta_n} f)(x)\Big|\, d\bar{h}\Big| L_p(\mathbb{T}^d)\Big\|.$$

Moreover $\| \cdot |S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^R$ is an equivalent (quasi-)norm in $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$. □

4.6 Further characterizations

And finally also the classical characterization can be applied for the periodic B-scale. Let us finish with the following two theorems.

Theorem 4.6.1 Let $1 \le p \le \infty$, $0 < q \le \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}^d_0$ with $\bar{m} > \bar{r} > 0$. Then the following quantities describe equivalent (quasi-)norms in the space $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{M/\Delta} = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\bar{\beta}|\ge 1} S^{M/\Delta}_{\bar{\beta}}(f) \quad ,$$

where for $|\bar{\beta}| = n \ge 1$ (modification if $q = \infty$)

(i)

$$S^M_{\bar{\beta}}(f) = \Big(\int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i}q}\Big) \sup_{\substack{|h_i|\le t_i\\ i=1,\dots,n}} \big\|\big(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \ldots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f\big)(x)\big|L_p(\mathbb{T}^d)\big\|^q \,\frac{d\bar{t}}{\bar{t}} \Big)^{1/q} \quad \text{and}$$

(ii)

$$S^\Delta_{\bar{\beta}}(f) := \Big[\int\limits_{\mathbb{R}^n} \Big(\prod_{i=1}^n |h_i|^{-r_{\delta_i}q}\Big) \big\|\big(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f\big)(x)\big|L_p(\mathbb{T}^d)\big\|^q \,\frac{d\bar{h}}{\tilde{h}}\Big]^{1/q} \quad ,$$

with $\tilde{h} = (|h_1|, ..., |h_n|)$ and $\bar{\delta}$ as usual. □

Theorem 4.6.2 Let $0 < p, q \le \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}^d_0$ with $\bar{m} > \bar{r} > 1/p$. Then the following quantities describe equivalent (quasi-)norms in the space $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$:

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{M/M'/\Delta} = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\bar{\beta}|\ge 1} S^{M/M'/\Delta}_{\bar{\beta}}(f) \quad ,$$

where additional to Theorem 4.6.1/(i)/(ii) for $|\bar{\beta}| = n \ge 1$ (modification if $q = \infty$)

$$S^{M'}_{\bar{\beta}}(f) = \Big(\int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i}q}\Big)\Big\| \sup_{\substack{|h_i|\le t_i\\ i=1,\dots,n}} |\big(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \ldots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f\big)(x)|\Big|L_p(\mathbb{T}^d)\Big\|^q \,\frac{d\bar{t}}{\bar{t}} \Big)^{1/q}$$

with $\bar{\delta}$ as usual. □

For sake of completeness we again refer to Remark 4.4.1.

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