Smolyak's Algorithm, Sparse Grid Approximation and Periodic Function Spaces with Dominating Mixed Smoothness

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Zusammenfassung

Die vorliegende Arbeit wurde hauptsächlich durch die Ergebnisse Sickels in den Arbeiten [34, 35] und durch frühere Untersuchungen von Temlyakov in [43, 45] motiviert. Wir studieren periodische Funktionenräume mit dominierend gemischten Glattheitseigenschaften vom Besov-, Triebel-Lizorkin- und Sobolev-Typ. Im Mittelpunkt der Untersuchungen steht das Problem der optimalen Rekonstruktion von Funktionen aus einer endlichen Menge von Funktionswerten. Die Qualität der optimalen Rekonstruktion einer Klasse von Funktionen wird mit Hilfe der Größen ρ_M gemessen. Diese Größen sind mit den bekannten linearen Weiten (oder Approximationszahlen) vergleichbar. Man beschränkt sich allerdings nur auf lineare Abtastoperatoren, deren Rang kleiner oder gleich M ist. In [44] findet man erste Resultate für Räume vom Sobolev- und Nikol'skij-Besov-Typ mit dominierend gemischter Glattheit. Diese wurden in [34] und [35] teilweise verbessert und auf periodische Räume vom Besov- und Triebel-Lizorkin-Typ auf $\mathbb{T}^2$ ausgeweitet.
Seit einiger Zeit besteht ein wachsendes Interesse (insbesondere aus der Finanzmathematik) an der Lösung hochdimensionaler Probleme ($d = 100, 1000, ...$). In vielen Fällen kann die Lösung nicht exakt bestimmt werden, weshalb man sich auf Näherungsverfahren zurückziehen muss. Diese Arbeit dient unter anderem auch der Analyse spezieller derartiger Verfahren. Normalerweise wächst der Aufwand für die Bestimmung einer hinreichend genauen Näherung exponentiell in d. Die Herausforderung besteht nun darin, diesem sogenannten "Fluch der Dimension" durch geeignete Wahl der Funktionenklassen bzw. Algorithmen zu entgehen. Dazu gibt es unter anderen von Wasilkowski und Woźniakowski verschiedene Untersuchungen in [57]. Es stellte sich heraus, dass der Smolyak-Algorithmus und Funktionenräume mit dominierend gemischter Glattheit gut zusammenpassen.
Das Ziel der Arbeit ist die Erweiterung von Sickels Resultaten auf beliebige Dimensionen d. Die Behandlung dieses allgemeinen Falles setzt ein genaues Studium der Theorie periodischer Funktionenräume voraus. Kapitel 1 ist dem Fourier-analytischen Zugang der Räume $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ und $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ auf dem d-Torus gewidmet. Wir berufen uns dabei hauptsächlich auf die Monographie [32] von Schmeißer und Triebel. Dort wurden die genannten Skalen auf dem $\mathbb{R}^2$ eingeführt. Beide Raumskalen enthalten wichtige klassische Funktionenräume dominierend gemischter Glattheit, beispielsweise Hölder-Zygmund-, Nikol'skij-Besov- oder Sobolev-Räume. Letztere wurden erstmalig von Nikol'skij in den frühen sechziger Jahren

wie folgt auf dem $\mathbb{R}^2$ definiert:

$$S_p^{(r_1,r_2)}W(\mathbb{R}^2) = \Big\{ f \in L_p(\mathbb{R}^2) : \|f|S_p^{(r_1,r_2)}W(\mathbb{R}^2)\| = \|f|L_p(\mathbb{R}^2)\| + \Big\|\frac{\partial^{r_1} f}{\partial x_1^{r_1}}\Big|L_p(\mathbb{R}^2)\Big\| + \Big\|\frac{\partial^{r_2} f}{\partial x_2^{r_2}}\Big|L_p(\mathbb{R}^2)\Big\| + \Big\|\frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1}\partial x_2^{r_2}}\Big|L_p(\mathbb{R}^2)\Big\| < \infty \Big\} .$$

Dabei ist $1 < p < \infty$ und $r_i = 0, 1, 2, ...$ $(i = 1, 2)$. Die gemischte Ableitung $\frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1}\partial x_2^{r_2}}$ spielt eine dominierende Rolle und verhalf deshalb der Klasse zu ihrem Namen. Sowohl diese Klasse als auch die verwandten Besov-Räume wurden intensiv in der früheren Sowjetunion untersucht, beispielsweise von Amanov, Besov, Lizorkin, Nikol'skij, Potapov und Temlyakov, um nur einige zu nennen. Eine erste systematische Untersuchung gelang Amanov in [1]. Angelehnt an die Theorie der isotropen Räume entwickelten Schmeißer und Triebel Mitte der achziger Jahre den Fourier-analytischen Rahmen (siehe [32]). Dort werden grundlegende Einbettungen, Spursätze und sogar die Charakterisierung durch Differenzen behandelt. Außerdem führt man die isotropen periodischen Räume $F^s_{p,q}(\mathbb{T}^d)$ und $B^s_{p,q}(\mathbb{T}^d)$ ein. Das Kapitel 1 dieser Arbeit kombiniert beide Zugänge, um eine vollständige Theorie der periodischen Räume mit dominierend gemischter Glattheit zu liefern. Zunächst entwickeln wir die dazu notwendigen Werkzeuge. Die Funktionenräume werden dann in voller Allgemeinheit über Tensorprodukte von klassischen Zerlegungen der Einheit definiert. Später nehmen wir Bezug auf elementare Einbettungen der Räume untereinander und untersuchen im Speziellen die Einbettungen in den Raum der periodischen und stetigen Funktionen. Nach einem Abschnitt zur komplexen Interpolation der F-Räume konzentrieren wir uns auf die Charakterisierung durch Differenzen. Basierend auf der Charakterisierung durch Integralmittel von Differenzen erhalten wir verschiedene äquivalente (Quasi-)Normen. Diese sind unter anderem nötig, um bestehende Resultate in unseren Kontext einzuordnen.
Das eigentliche Vorhaben besteht in der approximationstheoretischen Untersuchung der beschriebenen Klassen durch Abtastoperatoren auf dünnen Gittern. Eine ausführliche Untersuchung dieses Gegenstandes schließt sich in Kapitel 2 an. Die zentrale Methode ist der sogenannte Smolyak-Algorithmus, welcher lineare Abtastoperatoren auf $\mathbb{T}^d$ vom Typ

$$A(m,d)\, f(x) = \sum_{k=1}^{M_m} f(x_\ell^m)\psi_\ell^m(x) \quad , \quad x \in \mathbb{T}^d ,$$

generiert. Die Menge der Abtastknoten $\{x_1^m, ..., x_{M_m}^m\}$ ist fest und stellt ein sogenanntes dünnes Gitter dar, für welches insbesondere $M_m \asymp 2^m m^{d-1}$ gilt. Die grundlegende Idee hinter der Smolyak-Konstruktion ist auch in einem elementaren Kontext verständlich. Wir betrachten dazu monoton wachsende, konvergente Folgen reeller Zahlen $(a_j^1)_{j=0}^\infty, \ldots, (a_j^d)_{j=0}^\infty$. Die entsprechenden Grenzwerte seien $a^1, \ldots, a^d$ genannt. Zusätzlich setzen wir $a_{-1}^\ell = 0$ für $\ell = 1, \ldots, d$. Dann existiert die Reihe $a^\ell = \sum_{j=0}^\infty (a_j^\ell - a_{j-1}^\ell)$ und es gilt demzufolge auch

$$a^1 \cdot \ldots \cdot a^d = \sum_{j_1,\ldots,j_d=0}^{\infty} \prod_{\ell=1}^{d} (a^\ell_{j_\ell} - a^\ell_{j_\ell-1}) \,.$$

Es war Smolyaks [40] Idee, die Folge

$$\sum_{j_1+\ldots+j_d\le m} \prod_{\ell=1}^{d} (a^\ell_{j_\ell} - a^\ell_{j_\ell-1})\,, \qquad m = 0,1,\ldots\,,$$

zur Approximation des Produktes $a^1 \cdot \ldots \cdot a^d$ heranzuziehen. Es spricht nichts dagegen, diese Konstruktion auch für

$$a^1_j = a^2_j = \ldots = a^d_j = L_j f(x)\,, \qquad x \in \mathbb{T}\,,$$

zu verwenden, wobei L_j einen Abtastoperator zu einer gewissen Menge $\mathcal{T}_j$ von Abtastknoten bezeichnet. Der resultierende Abtastprozess verwendet Funktionswerte auf einem dünnen Gitter in $\mathbb{T}^d$. Im Hauptteil der vorliegenden Arbeit untersuchen wir die Approximationsrate solcher Abtastoperatoren für Funktionen aus Besov- und Triebel-Lizorkin-Räumen mit dominierend gemischter Glattheit (aus Kapitel 1). Bezeichnet $A(\mathbb{T}^d)$ einen solchen Raum, dann ist die übliche Norm eine sogenannte Kreuznorm (siehe Abschnitt 1.3 und 1.4), was sich in der Gleichung

$$\| f_1 \otimes \ldots \otimes f_d \,|A(\mathbb{T}^d)\| = \prod_{\ell=1}^{d} \| f_\ell \,|A(\mathbb{T})\|$$

ausdrückt. Die betrachteten Funktionenräume sind daher hinreichend nahe am Tensorprodukt von Räumen auf dem Torus angesiedelt. Diese Eigenschaft erlaubt Abschätzungen für die Approximationsrate des Smolyak-Algorithmus' basierend auf der Rate der L_j. Der Fehler wird dabei in der L_p-Metrik mit $1 \le p \le \infty$ gemessen. Die Hauptresultate in Abschnitt 2.4 sind scharfe Abschätzungen für diesen Fehler. Darauf basierend erhalten wir in Abschnitt 2.6 obere Schranken für die Größen $\rho_M(F, L_p)$ für $F = S^r_{p,q}B(\mathbb{T}^d)$, $1 \le p,q \le \infty$, $r > 1/p$ und $F = S^r_{p,q}F(\mathbb{T}^d)$, $1 < p,q < \infty$, $r > \max(1/p, 1/q)$. Diese sehen in beiden Fällen wie folgt aus

$$\rho_M(F, L_p(\mathbb{T}^d)) \le cM^{-r}(\log M)^{(d-1)(r+1-1/q)} \;.$$

Die Klasse der Sobolev-Räume $S^r_pW(\mathbb{T}^d)$ stellt einen Spezialfall dar, und zwar gilt

$$\rho_M(S^r_p\, W(\mathbb{T}^d), L_p(\mathbb{T}^d)) \le cM^{-r}(\log M)^{(d-1)(r+1/2)}$$

für $1 < p < \infty$ und $r > \max(1/p, 1/2)$. Das Ergebnis verbessert ein entsprechendes Resultat von Temlyakov in [45] um $(d-1)/2$ in der Potenz des Logarihmus' (siehe auch Abschnitt 2.7). Leider ist nicht klar, ob unsere Konstruktion das Optimum realisiert, d.h. ob es auch entsprechende untere Schranken gibt.

Um die Resultate in einem größeren Kontext zu verstehen, wird in diesem Zusammenhang auch auf Faltungsoperatoren eingegangen. Die in Abschnitt 2.3 präsentierte Definition des Smolyak-Algorithmus' ist hinreichend allgemein, sodass sie sogar auf die Fourier-Partialsumme oder auf de la Vallée-Poussin-Mittel anwendbar ist. Zu diesem Thema (Approximation vom hyperbolischen Kreuz) existieren zahlreiche Arbeiten. Wir werden nicht weiter ins Detail gehen und verweisen auf [3, 4], [2], [6], [9], [10, 11, 12], [15], [19], [22], [25], [30], [31], [36], [40], [46], [45] und [39]. Am Ende des zweiten Kapitels werden die erzielten Resultate ausführlich diskutiert und mit denen in [34], [35], [45], [11] und [57] verglichen.
Kapitel 3 enthält schließlich eine Sammlung offener Probleme im Zusammenhang mit den Untersuchungen dieser Arbeit und gibt außerdem verschiedene Anregungen für weiterführende Forschung.

Introduction

This work has essentially been motivated by Sickel's papers [34, 35] and former investigations done by Temlyakov during the late eighties and early nineties, cf. [43] as well as [45]. We concentrate on the study of periodic function spaces with dominating mixed smoothness of Besov, Triebel-Lizorkin and of Sobolev type. In particular, the problem of optimal approximate recovery is studied in detail. Here we measure the quality of recovery of a class of functions from values at a fixed finite set of points. Precisely, for fixed grid size M we consider the quantity ρ_M, which can be seen as a counterpart to the linear widths. We restrict to linear sampling operators with rank less or equal to M. The intention is to derive lower and upper bounds for these quantities.
In [45] results were given for Nikol'skij-Besov and Sobolev type spaces on the d-torus, where d is arbitrary. These results have been partly improved ($d = 2$) and extended by Sickel to the scales of Besov and Triebel-Lizorkin spaces of dominating mixed smoothness, $S^r_{p,q}B(\mathbb{T}^2)$ and $S^r_{p,q}F(\mathbb{T}^2)$.
Recently there has been significant interest in solving problems that involve functions defined on high-dimensional domains ($d = 100, 1000, ...$), for instance in the field of financial mathematics. In most cases solutions can not be computed exactly, but approximated in a certain sense, for example by the methods considered below. Typically, the cost needed to find an approximate solution, increases exponentially with the dimension d. The challenge is to avoid this so called "Curse of Dimension" by defining appropriate function classes and algorithms. For several results in this field we mainly refer to Wasilkowski and Woźniakowski and their paper [57] .
Our intention is to extend the results of Sickel to arbitrary d. The treatment of the general case requires a detailed development of the theory of periodic spaces with dominating mixed smoothness. Chapter 1 is devoted to the Fourier analytic approach to the function spaces $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ and $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ on the d-torus in its full generality following the monograph [32] by Schmeisser and Triebel, where the bivariate nonperiodic case is treated. It is important to mention that these classes cover many well-known classical spaces (e.g. as Hölder-Zygmund, Nikolskij-Besov and Sobolev spaces) with dominating mixed smoothness properties. Spaces of Sobolev type on $\mathbb{R}^2$ were firstly introduced by Nikol'skij in the early

sixties as follows

$$S_p^{(r_1,r_2)}W(\mathbb{R}^2) = \Bigg\{ f \in L_p(\mathbb{R}^2) : \|f|S_p^{(r_1,r_2)}W(\mathbb{R}^2)\| = \|f|L_p(\mathbb{R}^2)\| + \left\| \frac{\partial^{r_1} f}{\partial x_1^{r_1}} |L_p(\mathbb{R}^2) \right\| + \left\| \frac{\partial^{r_2} f}{\partial x_2^{r_2}} |L_p(\mathbb{R}^2) \right\| + \left\| \frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}} |L_p(\mathbb{R}^2) \right\| < \infty \Bigg\} ,$$

where $1 < p < \infty$ and $r_i = 0, 1, 2, ...$ $(i = 1, 2)$. The mixed derivative $\frac{\partial^{r_1+r_2} f}{\partial x_1^{r_1} \partial x_2^{r_2}}$ dominates the norm, which led to the name of this scale of function spaces. Later on, these classes as well as the corresponding Besov spaces have been extensively studied in the former Soviet Union, for example by Amanov, Besov, Lizorkin, Nikol'skij and Potapov. For a first systematical study we refer to the monograph [1]. What concerns the Fourier analytic treatment of Besov and Triebel-Lizorkin type spaces on $\mathbb{R}^2$ we mainly consult [32]. Several types of equivalent quasi-norms, embedding and trace theorems as well as characterizations by differences are proved there. This reference also deals with isotropic periodic spaces, i.e. the scales $F^s_{p,q}(\mathbb{T}^d)$ and $B^s_{p,q}(\mathbb{T}^d)$. In Chapter 1 both fields have been combined in order to present a complete theory of periodic spaces with dominating mixed smoothness on the d-torus. We first discuss several necessary embedding properties including also some limiting cases. After proving a result concerning complex interpolation within the scale of Triebel-Lizorkin type spaces, we concentrate on the characterization by differences. Several characterizations based on integral means involving differences are given. They are important for the treatment of our main problem, which is the approximation of the described classes by linear sampling operators acting on sparse grids presented in Chapter 2. The central construction in this field is Smolyak's algorithm as a method to obtain a sequence of linear sampling operators of type

$$A(m,d)\, f(x) = \sum_{k=1}^{M_m} f(x_\ell^m)\psi_\ell^m(x) \quad , \quad x \in \mathbb{T}^d ,$$

acting on a sparse grid $\{x_1^m, ..., x_{M_m}^m\}$. The size M_m of the grid satisfies $M_m \asymp 2^m m^{d-1}$. The method is based on the tensor product of sampling operators with respect to $\mathbb{T}$. Let us present the main idea in a very elementary context. Consider $(a_j^1)_{j=0}^\infty, \ldots, (a_j^d)_{j=0}^\infty$ as monotone increasing, convergent sequences of real numbers. The respective limits are denoted by $a^1, \ldots, a^d$. In addition we put $a_{-1}^\ell = 0$, $\ell = 1, \ldots, d$. Then $a^\ell = \sum_{j=0}^\infty (a_j^\ell - a_{j-1}^\ell)$ and hence,

$$a^1 \cdot \ldots \cdot a^d = \sum_{j_1,\ldots,j_d=0}^{\infty} \prod_{\ell=1}^{d} (a_{j_\ell}^\ell - a_{j_\ell-1}^\ell) .$$

It has been the idea of Smolyak [40] to use the sequence

$$\sum_{j_1+\ldots+j_d \le m} \prod_{\ell=1}^{d} (a_{j_\ell}^\ell - a_{j_\ell-1}^\ell) , \qquad m = 0, 1, \ldots ,$$

to approximate the product $a^1 \cdot \ldots \cdot a^d$. The idea is to employ this construction with

$$a_j^1 = a_j^2 = \ldots = a_j^d = L_j f(x)\,, \qquad x \in \mathbb{T}\,,$$

where L_j denotes a sampling operator with respect to a certain set $\mathcal{T}_j$ of sample points on the torus. The suggested approximation procedure yields an operator which uses samples just from a sparse grid in $\mathbb{T}^d$. In the main part of the present work we investigate the approximation power of these sampling operators for functions belonging to periodic Besov, Triebel-Lizorkin and Sobolev spaces providing dominating mixed smoothness properties. If $A(\mathbb{T}^d)$ denotes such a space then the norm in these classes is a cross-norm (cf. Section 1.3 and 1.4), i.e.

$$\|\, f_1 \otimes \ldots \otimes f_d \,|A(\mathbb{T}^d)\| = \prod_{\ell=1}^{d} \|\, f_\ell \,|A(\mathbb{T})\|\,.$$

Hence, the function spaces we consider here are sufficiently close to the tensor product of function spaces defined on $\mathbb{T}$. This feature allows to derive sharp estimates for the rate of convergence of Smolyak's algorithm based on the approximation power of L_j, measured in the $L_p(\mathbb{T}^d)$-metric with $1 \le p \le \infty$. Our main result reads essentially as follows. It turns out that

$$\rho_M(F, L_p(\mathbb{T}^d)) \le cM^{-r}(\log M)^{(d-1)(r+1-1/q)}\;, \tag{1}$$

where $F = S^r_{p,q}B(\mathbb{T}^d)$, $1 \le p,q \le \infty$, $r > 1/p$ or $F = S^r_{p,q}F(\mathbb{T}^d)$, $1 < p,q < \infty$, $r > \max(1/p, 1/q)$. In the case of Sobolev spaces $S^r_p W(\mathbb{T}^d)$ we obtain

$$\rho_M(S^r_p W(\mathbb{T}^d), L_p(\mathbb{T}^d)) \le cM^{-r}(\log M)^{(d-1)(r+1/2)}$$

for $1 < p < \infty$ and $r > 0$ as a consequence of the Littlewood-Paley theory and (1). This relation improves the corresponding result by Temlyakov by $(d-1)/2$ in the power of the logarithm (cf. Section 2.7). However, it is not clear whether this construction realizes the optimal rate of approximate recovery. To embed these results in a general context we consider not only sampling operators. The use of a rather general definition of Smolyak's algorithm covers even operators of convolution type (such as Fourier partial sums and de la Vallée Poussin means). We shall not go into detail and will refer mainly to [39]. We finish Chapter 2 with a detailed comparison of our results with those given in [34], [35], [45], [11] and [57] to point out the advances. Finally, Chapter 3 includes a collection of open problems and gives some stimulation for further investigation.

Contents

Chapter 1

Periodic Spaces with Dominating Mixed Smoothness

1.1 Introduction

This chapter deals with the Fourier analytical treatment of the scales of Besov- and Triebel-Lizorkin spaces with dominating mixed smoothness properties on the d-torus, denoted by $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ and $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$. Here we assume $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $0 < p, q \leq \infty$ ($p < \infty$ in the F-case). Our approach is based on the combination of chapter 2 and 3 in [32]. After specifying some notation we recall briefly the concepts of tempered and periodic distributions and point out their connection. In Section 1.3 the main tools are collected, such as the periodic Nikol'skij-inequality and the scalar as well as vector-valued Fourier multiplier assertions. In particular, we will pay some attention to Lizorkin's multiplier theorem (see [21]) which is also one of the main instruments in Chapter 2. Section 1.4 presents the definition of the spaces mentioned above in terms of Fourier analysis. We use tensor products of scalar decompositions of unity on the Fourier side, which is a well-known technique in this field. Afterwards elementary embeddings are considered. Especially, we discuss the problem of necessary and sufficient conditions for the embedding into $C(\mathbb{T}^d)$, the space of continuous functions. This question is important since we want to study sampling operators in Chapter 2. After having remarked on Littlewood-Paley theory and the coincidence of $S^{\bar{r}}_{p,2}F(\mathbb{T}^d)$ and $S^{\bar{r}}_{p}W(\mathbb{T}^d)$ for $1 < p < \infty$ and $\bar{r} > 0$ we consider the subject of complex interpolation. A counterpart of a bivariate interpolation formula, given in [31], will be proved for the spaces $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$. Finally, we treat the technically difficult subject of characterization by differences. The purpose of Section 1.6 is to characterize the scales $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ and $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ by (quasi-)norms involving means of differences for the largest possible range of parameters $\bar{r} = (r_1, ..., r_d)$, given by

$$0 < p < \infty \ , \quad 0 < q \leq \infty \ , \quad r_i > \max(0, 1/p - 1, 1/q - 1) \quad , \quad i = 1, ..., d$$

in the F-case. If $d = 2$ our main result reads as follows.
The space $S^{\bar{r}}_{p,q}F(\mathbb{T}^2)$ is the collection of all $f \in L_p(\mathbb{T}^2) \cap L_1(\mathbb{T}^2)$ satisfying the finiteness of

the quantity

$$\begin{aligned}\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^2)\|^R &= \|f|L_p(\mathbb{T}^2)\|\\ &+\Big\|\Big[\int\limits_0^\infty t^{-r_1q}\Big(\int\limits_{-1}^1 |\triangle^{m_1}_{th,1}f(x)|\,dh\Big)^q\frac{dt}{t}\Big]^{1/q}\Big|L_p(\mathbb{T}^2)\Big\|\\ &+\Big\|\Big[\int\limits_0^\infty t^{-r_2q}\Big(\int\limits_{-1}^1 |\triangle^{m_2}_{th,2}f(x)|\,dh\Big)^q\frac{dt}{t}\Big]^{1/q}\Big|L_p(\mathbb{T}^2)\Big\|\\ &+\Big\|\Big[\int\limits_0^\infty\int\limits_0^\infty t_1^{-r_1q}t_2^{-r_2q}\Big(\int\limits_{[-1,1]^2}|\triangle^{m_1}_{t_1h_1,1}\circ\triangle^{m_2}_{t_2h_2,2}f(x)|\,dh\Big)^q\frac{dt_1}{t_1}\frac{dt_2}{t_2}\Big]^{1/q}\Big|L_p(\mathbb{T}^2)\Big\|\end{aligned}$$

assuming that $m_i \in \mathbb{N}$ with $m_i > r_i$ $(i = 1, 2)$. The d-dimensional case turns out to be more complicated. The corresponding (quasi-)norm is a sum of 2^d summands of the type given above. In addition, we present a selection of the results from [52], where also the non-periodic case is treated in detail. Let us also refer to [32, 2.3.3/2.3.4] and [1]. Characterizations of this type are necessary in order to compare our results with the ones obtained by Temlyakov for periodic Nikols'kij-Besov spaces of dominating mixed smoothness (cf. Section 2.7). We will also employ the results in Section 1.6 for an important bump function argument in Chapter 2 (see Paragraph 2.4.4).

1.2 Preliminaries

1.2.1 Notation

First of all it is necessary to specify some notation. The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ denote the real numbers, complex numbers, natural numbers, natural numbers including 0 and the integers. Furthermore $\mathbb{T}$ denotes the torus, defined in Paragraph 1.2.3. The symbol I will be reserved for identity operators, whereas $\mathcal{I}$ is always used for countable index sets. We shall write $a \asymp b$ if there exists a constant $c > 0$ (independent of the relevant parameters in the context) such that

$$c^{-1}\,a \le b \le c\,a\,.$$

The values of our estimating constants often change from line to line. We will indicate this by adding subscripts. In case a constant represents a fixed value we shall use capital letters like $C_1, C_2, \ldots$. The natural number d is reserved for the dimension of the considered Euclidean spaces $\mathbb{T}^d$ and $\mathbb{R}^d$, where elements of them are denoted by x, y, z and sometimes by ξ. The Euclidean distance is given as usual by $|x|$ (the ℓ_1^d-norm is denoted by $|x|_1$), where $x \cdot y = \sum_{i=1}^d x_i y_i$ denotes the corresponding Euclidean scalar product. Indices of d-dimensional Fourier coefficients are always denoted by $k = (k_1, ..., k_d)$ and the multi-index

$\alpha = (\alpha_1, ..., \alpha_d)$ corresponds to the differential operator D^α, given by

$$D^\alpha = \frac{\partial^{|\alpha|_1}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \,.$$

We often need further vector-type quantities like indices and parameters. They are often denoted by $\bar{\ell}, \bar{s}, \bar{r}$ and $\bar{h}$ with numbered components. Moreover, we fix the notation $\bar{1} = (1, ..., 1)$ and $\bar{0} = (0, ..., 0)$. As usual we put $\bar{r} + \bar{\ell} = (r_1 + \ell_1, ..., r_n + \ell_n)$, $\lambda \cdot \bar{\ell} = (\lambda \cdot \ell_1, ..., \lambda \cdot \ell_n)$, $\lambda \in \mathbb{R}$, $\bar{k} \cdot \bar{r} = k_1 r_1 + ... + k_n r_n$ and $|\bar{r}|_1 = r_1 + ... + r_n$. Furthermore, we shall use the operations $\lambda + \bar{r} = (\lambda + r_1, ..., \lambda + r_n)$, where $\lambda \in \mathbb{R}$, $\bar{t}^{\,\bar{r}} = (t_1^{r_1}, ..., t_n^{r_n})$, $\lambda^{\bar{r}} = (\lambda^{r_1}, ..., \lambda^{r_n})$ with $\lambda > 0$ and $\bar{r} * \bar{s} = (r_1 s_1, ..., r_n s_n)$. We also abbreviate the relations

$$r_i > s_i \quad (r_i \geq s_i) \quad , \quad i = 1, ..., n \,,$$

by $\bar{r} > \bar{s}$ $(\bar{r} \geq \bar{s})$. Often we shortly write $\bar{r} > s$, $s \in \mathbb{R}$, which means $\bar{r} > (s, ..., s)$. During this work the constants σ_p and $\sigma_{p,q}$ are fixed and defined as

$$\sigma_p := \left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{p,q} := \left(\frac{1}{\min(p,q)} - 1\right)_+ \,, \tag{1.1}$$

where $0 < p, q \leq \infty$ and $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$.

1.2.2 Tempered Distributions and Fourier Transform

As usual we denote by $S = S(\mathbb{R}^d)$ the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. Its topology is generated by the (semi-)norms

$$\|\varphi\|_{k,\ell} = \sup_{x \in \mathbb{R}^d} (1 + |x|)^k \sum_{|\alpha|_1 \leq \ell} |D^\alpha \varphi(x)| \quad , \quad k, \ell \in \mathbb{N}_0 \,. \tag{1.2}$$

By $\operatorname{supp} \psi$ we denote the support of the function $\psi \in S(\mathbb{R}^d)$, given by

$$\operatorname{supp} \psi = \overline{\{x \in \mathbb{R}^d : \psi(x) \neq 0\}} \,.$$

A linear mapping $f : S(\mathbb{R}^d) \to \mathbb{C}$ is called a tempered distribution if a constant $c > 0$ and $k, \ell \in \mathbb{N}_0$ exist such that

$$|f(\varphi)| \leq c \|\varphi\|_{k,\ell}$$

holds for all $\varphi \in S(\mathbb{R}^d)$. The collection of all such mappings is denoted by $S'(\mathbb{R}^d)$. The support of a tempered distribution $f \in S'(\mathbb{R}^d)$ is defined by

$$\operatorname{supp} f = \left\{x \in \mathbb{R}^d : \text{ for all } \delta > 0 \text{ exists } \psi \in S(\mathbb{R}^d) \text{ with } \operatorname{supp} \psi \subset K_\delta(x) \text{ and } f(\psi) \neq 0\right\},$$

where $K_\delta(x) = \{y \in \mathbb{R}^d : |x - y| < \delta\}$.
As usual the Fourier transform defined on both $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ is given by

$$(\mathcal{F}f)(\varphi) := f(\mathcal{F}\varphi) \quad , \quad \varphi \in S(\mathbb{R}^d),\ f \in S'(\mathbb{R}^d) \,,$$

where

$$\mathcal{F}\varphi(\xi) := (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{-ix\cdot\xi}\varphi(x)\,dx.$$

$\mathcal{F}$ is a bijection (in both cases) and its inverse is given by

$$\mathcal{F}^{-1}\varphi(\xi) := (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{ix\cdot\xi}\varphi(x)\,dx \quad , \quad \varphi \in S(\mathbb{R}^d)\ .$$

Moreover, $L_p(\mathbb{R}^d)$ denotes the classical Lebesgue space with $0 < p \leq \infty$ and

$$\|f|L_p(\mathbb{R}^d)\| = \Big(\int\limits_{\mathbb{R}^d} |f(x)|^p\,dx \Big)^{1/p} . \tag{1.3}$$

If $p = \infty$ we modify (1.3) by

$$\|f|L_\infty(\mathbb{R}^d)\| = \operatorname*{ess\text{-}sup}_{x\in\mathbb{R}^d} |f(x)|\,.$$

We call a (tempered) distribution $T \in S'(\mathbb{R}^d)$ regular if a locally integrable function $f : \mathbb{R}^d \to \mathbb{C}$ exists such that

$$T(\varphi) = \int\limits_{\mathbb{R}^d} f(x)\varphi(x)\,dx \quad , \quad \varphi \in S(\mathbb{R}^d)\,.$$

In this sense we have the chain of embeddings

$$S(\mathbb{R}^d) \hookrightarrow L_p(\mathbb{R}^d) \hookrightarrow S'(\mathbb{R}^d)\,.$$

The symbol "$\hookrightarrow$" denotes the set-theoretical inclusion as well as the topological embedding. The embedding on the right-hand side does not hold in the case $p < 1$.
Let us mention this version of the famous Nikol'skij inequality, which is due to Stöckert and Uninskij. As usual, derivatives have to be understood in the weak sense.

Proposition 1.1 *Let $\bar{b} > 0$ and $\Omega_{\bar{b}} = \{x = (x_1, ..., x_d) \in \mathbb{R}^d : |x_i| \leq b_i\ ,\ i = 1, ..., d\}$ be a generalized rectangle. Let further $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$ and $0 < p \leq u \leq \infty$. Then a positive constant c (independent of $\bar{b}$) exists such that*

$$\|D^\alpha f|L_u(\mathbb{R}^d)\| \leq c\, b_1^{\alpha_1+1/p-1/u} b_2^{\alpha_2+1/p-1/u} \cdots b_d^{\alpha_d+1/p-1/u} \|f|L_p(\mathbb{R}^d)\|$$

holds for all $f \in L_p(\mathbb{R}^d) \cap S'(\mathbb{R}^d)$ assuming that

$$\operatorname{supp}\mathcal{F}f \subset \Omega_{\bar{b}}\,.$$

Proof A proof can be found in [32, Thm. 1.6.2.] for the bivariate case. The arguments can easily be transferred to the case $d > 2$. See also [42], [54, 55] and[26] . $\square$

1.2.3 Distributions on the Torus, Periodic Distributions

Let $\mathbb{T}^d$ denote the d-torus, represented in the Euclidean space $\mathbb{R}^d$ by the cube $\mathbb{T}^d = [0, 2\pi]^d$, where opposite points are identified. That means $x, y \in \mathbb{T}^d$ are identified if and only if $x - y = 2\pi k$, where $k = (k_1, ..., k_d) \in \mathbb{Z}^d$. In particular one has $f(x) = f(y)$ if $x - y = 2\pi k$ and $f \in D(\mathbb{T}^d)$. Let further $D(\mathbb{T}^d)$ denote the collection of all complex-valued infinitely differentiable functions on $\mathbb{T}^d$. Its topology is generated by the family of norms

$$\|\varphi\|_N = \sum_{|\alpha|_1 \leq N} \sup_{x \in \mathbb{T}^d} \left| D^\alpha \varphi(x) \right| \quad , \quad N \in \mathbb{N}_0 \,.$$

A linear functional $f : D(\mathbb{T}^d) \to \mathbb{C}$ belongs to $D'(\mathbb{T}^d)$, if and only if there is a constant $c_N > 0$ such that

$$|f(\varphi)| \leq c_N \|\varphi\|_N$$

holds for all $\varphi \in D(\mathbb{T}^d)$ and for some natural number N. We endow $D'(\mathbb{T}^d)$ with the weak topology. Precisely, $\{f_n\}_{n=1}^\infty \subset D'(\mathbb{T}^d)$ converges to $f \in D'(\mathbb{T}^d)$ if and only if $\lim_{n\to\infty} f_n(\varphi) = f(\varphi)$ holds for all $\varphi \in D(\mathbb{T}^d)$. The space $D'(\mathbb{T}^d)$ is complete in this topology.

Moreover, the Fourier coefficients of a distribution $f \in D'(\mathbb{T}^d)$ are the complex numbers

$$c_k(f) = (2\pi)^{-d} f(e^{-ik\cdot x}) \quad , \quad k \in \mathbb{Z}^d.$$

In the sense of convergence in $D'(\mathbb{T}^d)$ we have

$$f = \sum_{k \in \mathbb{Z}^d} c_k(f) e^{ik\cdot x}.$$

We call $T \in D'(\mathbb{T}^d)$ a regular distribution if a $\mathbb{T}^d$-integrable function $f : \mathbb{T}^d \to \mathbb{C}$ exists with

$$T(\varphi) = \int_{\mathbb{T}^d} f(x) \cdot \varphi(x)\, dx \quad , \quad \varphi \in D(\mathbb{T}^d).$$

Now the calculation of the Fourier coefficients is performed by the well-known classical integral

$$c_k(f) = c_k(T) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-ikx}\, dx \,.$$

See [32, Chapt. 3] for details.

Let us also mention the δ-distribution, which is not a regular distribution. It is defined by

$$\delta(\psi) := \psi(0) \quad , \quad \psi \in D(\mathbb{T}^d) \,.$$

Of course $\delta \in D'(\mathbb{T}^d)$. Obviously, the Fourier coefficients are given by

$$c_k(\delta) = \frac{1}{(2\pi)^d} \quad , \quad k \in \mathbb{Z}^d \,. \tag{1.4}$$

We will return to that later.
Let Λ be a finite subset of $\mathbb{Z}^d$. Then $T^\Lambda \subset D(\mathbb{T}^d)$ denotes the collection of all trigonometric polynomials with harmonics in Λ, precisely

$$T^\Lambda = \left\{ t(x) = \sum_{k\in\Lambda} a_k e^{ik\cdot x} \middle| a_k \in \mathbb{C}\,,\, k \in \Lambda \right\} .$$

Of course, every trigonometric polynomial $t(x) \in T^\Lambda$ can also be interpreted as a tempered distribution in the sense above

$$t(\varphi) := \int_{\mathbb{R}^d} t(x)\varphi(x)\,dx \quad , \quad \varphi \in S(\mathbb{R}^d)\ . \tag{1.5}$$

In that context t represents a periodic tempered distribution satisfying

$$t\big(\varphi(\cdot + 2\pi k)\big) = t(\varphi) \quad , \quad \varphi \in S(\mathbb{R}^d)\,, k \in \mathbb{Z}^d\,.$$

It turns out that

$$\operatorname{supp} \mathcal{F}t = \{k \in \mathbb{Z}^d : c_k(t) \neq 0\} \subset \Lambda\,.$$

1.2.4 Vector-Valued Lebesgue Spaces

We follow [32, 3.4.1]. Now and subsequently the symbol $\mathcal{I}$ denotes a countable index-set. The class $L_p(\mathbb{T}^d)$ with $0 < p \le \infty$ denotes the space of all measurable functions $f : \mathbb{T}^d \to \mathbb{C}$ (f is 2π-periodic in each direction) satisfying

$$\|f|L_p(\mathbb{T}^d)\| = \Big(\int_{\mathbb{T}^d} |f(x)|^p\,dx \Big)^{1/p} < \infty$$

with the usual modification in case $p = \infty$, see Paragraph 1.2.2. Here we have also the embeddings

$$D(\mathbb{T}^d) \hookrightarrow L_p(\mathbb{T}^d) \hookrightarrow D'(\mathbb{T}^d)$$

if $1 \le p \le \infty$. The second embedding does not hold in case $p < 1$.
Furthermore, we define for $0 < p, q \le \infty$ the quantity

$$\|\{f_k\}_{k\in\mathcal{I}}|L_p(\mathbb{T}^d, \ell_q)\| := \Big\| \Big(\sum_{k\in\mathcal{I}} |f_k(x)|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^d) \Big\| \quad , \tag{1.6}$$

where we replace (1.6) in case $q = \infty$ by

$$\|\{f_k\}_{k\in\mathcal{I}}|L_p(\mathbb{T}^d, \ell_\infty)\| := \Big\| \sup_{k\in\mathcal{I}} |f_k(x)| \Big| L_p(\mathbb{T}^d) \Big\| .$$

The sequence $\{f_k\}$ is supposed to consist of Lebesgue-measurable functions on $\mathbb{T}^d$. In the sequel we may shortly write $\|f_k|L_p(\mathbb{T}^d, \ell_q)\|$ instead of $\|\{f_k\}_{k\in\mathcal{I}}|L_p(\mathbb{T}^d, \ell_q)\|$. The class

$L_p(\mathbb{T}^d, \ell_q)$ denotes the corresponding (quasi-)Banach space. If $\{f_k\}_{k\in\mathcal{I}}$ belongs to $L_p(\mathbb{T}^d, \ell_q)$ then $f_k \in L_p(\mathbb{T}^d)$ follows immediately. Finally, we define the spaces $L_p^\Lambda(\mathbb{T}^d, \ell_q)$, where $\Lambda = \{\Lambda_j\}_{j\in\mathcal{I}}$ denotes a sequence of finite subsets of $\mathbb{Z}^d$. We put

$$L_p^\Lambda(\mathbb{T}^d, \ell_q) = \Big\{ \{t_j(x)\}_{j\in\mathcal{I}} : t_j \in T^{\Lambda_j} \ , \ j \in \mathcal{I} \ , \ \|t_j | L_p(\mathbb{T}^d, \ell_q)\| < \infty \Big\}$$

for $0 < p, q \leq \infty$.

The following two lemmas represent important tools for the sequel. See also [32, 3.3.4] .

Lemma 1.1 *Let the locally integrable function $M : \mathbb{R}^d \to \mathbb{C}$ be a compactly supported tempered distribution satisfying $\mathcal{F}^{-1}M \in L_1(\mathbb{R}^d)$. For $f \in L_1(\mathbb{T}^d)$ the following identity*

$$\sum_{k\in\mathbb{Z}^d} M(k) c_k(f) e^{ikx} = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} (\mathcal{F}^{-1}M)(y) f(x-y)\, dy \tag{1.7}$$

holds true in $L_1(\mathbb{T}^d)$.

Proof By well-known properties of the Fourier transform, M has a continuous representative because of $\mathcal{F}^{-1}M \in L_1(\mathbb{R}^d)$. With that representative the expression $M(k)$ makes sense. Furthermore, M is compactly supported. Therefore, the sum on the left-hand side of (1.7) exists. The function

$$g(x) := (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} (\mathcal{F}^{-1}M)(y) f(x-y)\, dy$$

is 2π-periodic (in each direction) and integrable with respect to the d-torus $\mathbb{T}^d$. This follows by Minkowski's inequality and $\mathcal{F}^{-1} M \in L_1(\mathbb{R}^d)$. We proceed by computing the Fourier coefficients of $g \in L_1(\mathbb{T}^d)$. Let us fix $k \in \mathbb{Z}^d$ and calculate

$$\begin{aligned}
c_k(g) &= (2\pi)^{-d} \int\limits_{\mathbb{T}^d} g(x) e^{-ikx}\, dx \\
&= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} (\mathcal{F}^{-1}M)(y) (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(x-y) e^{-ikx}\, dx\, dy \\
&= (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} (\mathcal{F}^{-1}M)(y) (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(z) e^{-ik(z+y)}\, dz\, dy \\
&= c_k(f) (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} (\mathcal{F}^{-1}M)(y) e^{-iky}\, dy \\
&= c_k(f) M(k) \,,
\end{aligned}$$

which proves the claim. □

We also would like to mention the following version of the famous Poisson summation formula. In some sense this is a discrete counterpart of Lemma 1.1. We refer to [41, Cor. 7.2.6] .

Lemma 1.2 *Let the locally integrable function* $M : \mathbb{R}^d \to \mathbb{C}$ *be a compactly supported tempered distribution satisfying* $\mathcal{F}^{-1}M \in L_1(\mathbb{R}^d)$ *. Then the identity*

$$\sum_{k\in\mathbb{Z}^d} M(k)e^{ikx} = (2\pi)^{d/2} \sum_{\bar{\ell}\in\mathbb{Z}^d} \mathcal{F}^{-1}M(x+2\pi\bar{\ell}) \tag{1.8}$$

holds true in the space $L_1(\mathbb{T}^d)$ *.*

Proof Concerning the existence of the expressions compare with the proof of the previous lemma. Of course, we have

$$\sum_{\bar{\ell}\in\mathbb{Z}^d} \int\limits_{[-\pi,\pi]^d} |\mathcal{F}^{-1}M(x+2\pi\bar{\ell})|\,dx = \int\limits_{\mathbb{R}^d} |\mathcal{F}^{-1}M(x)|\,dx < \infty\,.$$

Hence, the sum on the right-hand side exists in L_1 . We proceed by deriving the Fourier coefficients of the periodic $L_1(\mathbb{T}^d)$-function

$$g(x) = (2\pi)^{d/2} \sum_{\bar{\ell}\in\mathbb{Z}^d} \mathcal{F}^{-1}M(x+2\pi\bar{\ell})$$

with a similar calculation done in the proof of Lemma 1.1 . It turns out that

$$c_k(g) = M(k) \quad , \quad k \in \mathbb{Z}^d \quad ,$$

which finishes the proof. □

1.3 Basic Tools

1.3.1 The Periodic Nikol'skij Inequality

Let us present the periodic Nikol'skij inequality and a counterpart of Proposition 1.1 . Recall the connection between tempered distributions and distributions on the d-torus mentioned in Paragraph 1.2.3 . In the sequel we need the number of elements of the set $\operatorname{supp} \mathcal{F}t^{p_0}$, denoted by $|\operatorname{supp} \mathcal{F}t^{p_0}|$. Here t is a trigonometric polynomial, p_0 a natural number and therefore t^{p_0} again a trigonometric polynomial.

Proposition 1.2 *Let $0 < p \le q \le \infty$ and p_0 be the smallest natural number greater or equal $p/2$. For a trigonometric polynomial $t(x)$ we put*

$$C_{p_0,t} = \frac{1}{2\pi}|supp\, \mathcal{F}t^{p_0}|.$$

Then the following inequality holds for every trigonometric polynomial $t(x)$

$$\|t(x)|L_q(\mathbb{T}^d)\| \le C_{p_0,t}^{1/p-1/q}\|t(x)|L_p(\mathbb{T}^d)\|\,.$$

Proof See again [26] and also [32, Prop. 3.3.2]. □

A useful special case is the following theorem, which is similar to Proposition 1.1.

Theorem 1.1 *Let $0 < p \le q \le \infty$ and*

$$\Lambda \subset \{k \in \mathbb{Z}^d : |k_j| \le N_j, j = 1, ..., d\} \quad ,$$

where the N_j are natural numbers. Then a constant $c > 0$ exists such that

$$\|t(x)|L_q(\mathbb{T}^d)\| \le c\big(\prod_{j=1}^d N_j\big)^{1/p-1/q}\|t(x)|L_p(\mathbb{T}^d)\|$$

holds for every trigonometric polynomial $t \in T^\Lambda$.

1.3.2 Maximal Inequalities

For a locally integrable function $f : \mathbb{R}^d \to \mathbb{C}$ we denote by $Mf(x)$ the classical Hardy-Littlewood maximal function defined by

$$(Mf)(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|\,dy \quad , \quad x \in \mathbb{R}^d \quad , \tag{1.9}$$

where the supremum is taken over all cubes centered at x with sides parallel to the coordinate axes. We firstly refer to the famous Hardy-Littlewood maximal inequality. There is a constant $c > 0$ such that

$$\|Mf|L_p(\mathbb{R}^d)\| \le c\|f|L_p(\mathbb{R}^d)\|$$

holds if $1 < p \le \infty$. A vector valued generalization is due to Fefferman and Stein [14]. For $1 < p < \infty$ and $1 < q \le \infty$

$$\|Mf_k|L_p(\mathbb{R}^d, \ell_q)\| \le c\,\|f_k|L_p(\mathbb{R}^d, \ell_q)\| \tag{1.10}$$

holds for every $\{f_k\}_{k\in\mathbb{N}} \in L_p(\mathbb{R}^d, \ell_q)$. The appearing vector-valued spaces with respect to $\mathbb{R}^d$ are the nonperiodic counterparts of (1.6).
Now we recall a standard procedure to obtain a corresponding assertion also for periodic functions. See also [32, 3.2.4].

Proposition 1.3 *Let $1 < p < \infty$ and $1 < q \leq \infty$. Then a constant $c > 0$ exists with*

$$\|Mf_k|L_p(\mathbb{T}^d,\ell_q)\| \leq c\,\|f_k|L_p(\mathbb{T}^d,\ell_q)\|$$

for all $\{f_k\}_{k\in\mathcal{I}} \in L_p(\mathbb{T}^d,\ell_q)$.

Proof Let $g \in L_p(\mathbb{T}^d)$. Because of the periodicity of g one obtains the inequality

$$Mg(x) \leq cM\tilde{g}(x) \quad \text{for} \quad x \in \mathbb{T}^d \quad ,$$

where $\tilde{g}(x)$ is the restriction of $g(x)$ on $[-3\pi, 3\pi]^d$ and in particular not longer a periodic function. Then (1.10) implies

$$\begin{aligned}\|Mf_k|L_p(\mathbb{T}^d,\ell_q)\| &\leq c_1\|M\tilde{f}_k|L_p(\mathbb{R}^d,\ell_q)\|\\ &\leq c_2\|\tilde{f}_k|L_p(\mathbb{R}^d,\ell_q)\|\\ &\leq c_3\|f_k|L_p(\mathbb{T}^d,\ell_q)\|.\end{aligned}$$

□

Now we define a one-dimensional version of (1.9)

$$(M_i f)(x) = \sup_{s>0} \frac{1}{2s} \int\limits_{x_i-s}^{x_i+s} |f(x_1,...,x_{i-1},t,x_{i+1},...,x_d)|\,dt \quad , \quad x \in \mathbb{R}^d. \tag{1.11}$$

It is necessary to prove that this operator maps two equivalent representatives of f to the same equivalence class. Having this in mind, one can show the following version of Proposition 1.3.

Proposition 1.4 *For $1 < p < \infty$ and $1 < q \leq \infty$ a constant $c > 0$ exists such that*

$$\|M_i f_k|L_p(\mathbb{T}^d,\ell_q)\| \leq c\|f_k|L_p(\mathbb{T}^d,\ell_q)\| \quad , \quad i = 1,...,d\ ,$$

holds for all sequences $\{f_k\}_{k\in\mathcal{I}} \in L_p(\mathbb{T}^d,\ell_q)$.

Proof One only has to split the integration over $\mathbb{R}^d$ into d integrations over $\mathbb{R}^1$ and apply Proposition 1.3 to the integration according to x_i. □

The following construction of a maximal function is due to Peetre. Let $\bar{b} = (b_1,...,b_d)$ and $\bar{s} = (s_1,...,s_d)$ belong to $\mathbb{R}^d$ and satisfy $\bar{b},\bar{s} > 0$. Let further $f \in S'(\mathbb{R}^d)$ be a tempered distribution with compactly supported Fourier transform $\mathcal{F}f$. In this situation the famous theorem of Paley-Wiener-Schwartz implies that f is an entire analytic function (see also Theorem 1.8 in Paragraph 1.6.2). We define the maximal function $P_{\bar{b},\bar{s}}f$ by

$$P_{\bar{b},\bar{s}}f(x) = \sup_{z\in\mathbb{R}^d} \frac{|f(x-z)|}{(1+|b_1z_1|^{s_1})\cdot\ldots\cdot(1+|b_dz_d|^{s_d})}\,. \tag{1.12}$$

In addition, we need another maximal inequality. Details concerning the following assertions can be found in [32, 1.6.4].

Lemma 1.3 *Let $\Omega \subset \mathbb{R}^d$ be compact, $\bar{s} = (s_1, ..., s_d) > 0$ and $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$. Then two constants $c_1, c_2 > 0$ exist (independent of f, but dependent of Ω) such that*

$$\begin{aligned} P_{\bar{1},\bar{s}}(D^\alpha f)(x) &\leq c_1 P_{\bar{1},\bar{s}} f(x) \\ &\leq c_2 \Big(M_d\Big(M_{d-1}\Big(...\Big(M_1|f|^{1/s_1}\Big)^{s_1/s_2}...\Big)^{s_{d-2}/s_{d-1}}\Big)^{s_{d-1}/s_d}\Big)^{s_d}(x) \end{aligned}$$

holds for all $f \in S'(\mathbb{R}^d)$ with $\operatorname{supp} \mathcal{F}f \subset \Omega$ and all $x \in \mathbb{R}^d$.

This lemma leads to the following important maximal inequality.

Proposition 1.5 *Let $0 < p < \infty$ and $0 < q \leq \infty$. Let further $\bar{b}^\ell = (b_1^\ell, ..., b_d^\ell) > 0$ for $\ell \in \mathcal{I}$ and $\Lambda = \{\Lambda_\ell\}_{\ell \in \mathcal{I}}$ be finite subsets of $\mathbb{Z}^d$ satisfying*

$$\Lambda_\ell \subset \{\xi \in \mathbb{R}^d \,:\, |\xi_i| \leq b_i^\ell,\, i = 1, ..., d\} \quad , \quad \ell \in \mathcal{I}\,.$$

Finally, the tuple $\bar{s} = (s_1, ..., s_d) > \frac{1}{\min(p,q)}$ is fixed. Under these assumptions we have a constant $c > 0$ (independent of f and Λ) such that

$$\big\| P_{\bar{b}^\ell,\bar{s}} f_\ell | L_p(\mathbb{T}^d, \ell_q)\big\| \leq c \,\big\| f_\ell \,| L_p(\mathbb{T}^d, \ell_q)\big\|$$

holds for all systems $f = \{f_\ell\}_{\ell \in \mathcal{I}} \subset L_p^\Lambda(\mathbb{T}^d, \ell_q)$.

Proof This assertion is a direct consequence of the previous lemma, Proposition 1.4 and a homogeneity argument: Let $\tilde{f}_\ell\,(x)$ be defined as

$$\tilde{f}_\ell(x_1, ..., x_d) := f_\ell\,(x_1/b_1^\ell, ..., x_d/b_d^\ell)\,.$$

Then

$$P_{\bar{b}^\ell,\bar{s}} f_\ell\,(x_1, ..., x_d) = P_{\bar{1},\bar{s}} \tilde{f}_\ell\,(b_1^\ell\, x_1, ..., b_d^\ell\, x_d)$$

holds true. Because of $\operatorname{supp} \mathcal{F}\tilde{f}_\ell \subset [-1, 1]^d$ the constant c can be chosen independently of Λ. □

1.3.3 Fourier Multipliers

In the following two sections we develop parts of the theory of Fourier multipliers for scalar and vector-valued L_p-spaces. This section deals with Fourier multipliers for $L_p(\mathbb{T}^d)$. Our approach basically is the combination of [32, 1.8.3] and [32, 3.3.4]. First of all we need some spaces of functions on $\mathbb{R}^d$. Let $\bar{\kappa} \geq 0$. Then a function $f \in L_2(\mathbb{R}^d)$ belongs to $S_2^{\bar{\kappa}} H(\mathbb{R}^d)$ if

$$\|\, f \,| S_2^{\bar{\kappa}} H(\mathbb{R}^d)\| := \big\|(1 + |\xi_1|^2)^{\kappa_1/2} \,\ldots\, (1 + |\xi_d|^2)^{\kappa_d/2}\, |\mathcal{F}f(\xi)|\, | L_2(\mathbb{R}^d)\big\| < \infty\,. \tag{1.13}$$

In case $d = 1$ we shall write $H_2^\kappa(\mathbb{R})$ instead of $S_2^{\bar{\kappa}} H(\mathbb{R})$. See also (1.40) in Paragraph 1.4.4 for the periodic counterpart of these spaces.

Remark 1.1 *Let us point out an important property of spaces defined in this way. Consider d functions* $g_i \in H_2^{\kappa_i}(\mathbb{R})$ *for* $\kappa_i \geq 0$, $i = 1, ..., d$. *Their tensor product is defined by*

$$(g_1 \otimes \cdots \otimes g_d)(x_1, ..., x_d) := g_1(x_1) \cdot ... \cdot g_d(x_d) \quad , \quad x = (x_1, ..., x_d) \in \mathbb{R}^d \,,$$

and satisfies

$$\|g_1 \otimes \cdots \otimes g_d | S_2^{\bar{\kappa}} H(\mathbb{R}^d)\| = \|g_1 | H_2^{\kappa_1}(\mathbb{R})\| \cdot ... \cdot \|g_d | H_2^{\kappa_d}(\mathbb{R})\| \tag{1.14}$$

and therefore belongs to $S_2^{\bar{\kappa}} H(\mathbb{R}^d)$. *Because of property (1.14) we call* $\| \cdot | S_2^{\bar{\kappa}} H(\mathbb{R}^d)\|$ *a cross-norm. The spaces defined below also provide this property. See also Remark 1.8 .*

Lemma 1.4 *Let* $0 < p \leq 2$ *and* $\bar{m} = (m_1, ..., m_d) \geq 0 \in \mathbb{R}^d$. *Let further* $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ *satisfy*

$$\bar{r} > \bar{m} + \frac{1}{p} - \frac{1}{2} \,.$$

Then a positive constant c exists such that

$$\|(1+\xi_1^2)^{m_1/2}(1+\xi_2^2)^{m_2/2} \cdots (1+\xi_d^2)^{m_d/2} \mathcal{F}f(\xi) | L_p(\mathbb{R}^d)\| \leq c \|f | S_2^{\bar{r}} H(\mathbb{R}^d)\| \,. \tag{1.15}$$

Proof We follow [32, 1.8.3], where the proof is given in case $d = 2$. Let us start with the left-hand side of (1.15) and use Hölder's inequality twice with $p/2 + (2-p)/2 = 1$. Recall the notation given in Paragraph 1.2.1 concerning vector-valued quantities. In addition, we shall use (1.47) in Paragraph 1.6.1 to decompose the integration over $\mathbb{R}^d$ into a sum of integrals over dyadic rectangles. This gives

$$\begin{aligned}
\Big(\int_{\mathbb{R}^d} \prod_{k=1}^{d} (1+|\xi_k|^2)^{p\,m_k/2} |\mathcal{F}f(\xi)|^p \, d\xi \Big)^{1/p} &= \Big(\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{\bar{\ell}\cdot\bar{m}p} \int_{Q_{\bar{\ell}}^{\Delta}} |\mathcal{F}f(\xi)|^p \Big)^{1/p} \\
&\leq \Big[\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{\bar{\ell}(\bar{m}+1/p-1/2)p} \Big(\int_{Q_{\bar{\ell}}^{\Delta}} |\mathcal{F}f(\xi)|^2 \, d\xi \Big)^{p/2} \Big]^{1/p} \\
&= \Big[\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{\bar{\ell}(\bar{m}+1/p-1/2-\bar{r})p} 2^{\bar{r}\bar{\ell}p} \Big(\int_{Q_{\bar{\ell}}^{\Delta}} |\mathcal{F}f(\xi)|^2 \, d\xi \Big)^{p/2} \Big]^{1/p} \\
&\leq c_1 \Big(\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{2\bar{r}\bar{\ell}} \int_{Q_{\bar{\ell}}^{\Delta}} |\mathcal{F}f(\xi)|^2 \, d\xi \Big)^{1/2} \\
&\leq c_2 \, \|f | S_2^{\bar{r}} H(\mathbb{R}^d)\| \,.
\end{aligned}$$

□

Let us present the main result of this section.

Proposition 1.6 *Let $\bar{b} > 0$ and $\Lambda \subset \mathbb{Z}^d$ be a finite subset of $\mathbb{Z}^d$ satisfying $\Lambda \subset Q_{\bar{b}} = \{x \in \mathbb{R}^d : |x_i| \leq b_i \ , \ i = 1, ..., d\}$. Let further $0 < p \leq \infty$ and $\bar{r} \in \mathbb{R}^d$ satisfy*

$$\bar{r} > \frac{1}{\min(1,p)} - \frac{1}{2} .$$

Then a constant $c > 0$ exists (independent of $\bar{b}$) such that

$$\Big\| \sum_{k \in \Lambda} M(k) c_k(t) e^{ikx} | L_p(\mathbb{T}^d) \Big\| \leq c \| M(b_1 \cdot, ..., b_d \cdot) | S_2^{\bar{r}} H(\mathbb{R}^d) \| \cdot \| t | L_p(\mathbb{T}^d) \|$$

holds for all $M \in S_2^{\bar{r}} H(\mathbb{R}^d)$ and all $t \in T^{\Lambda}$.

Remark 1.2 *Because of $M \in S_2^{\bar{r}} H(\mathbb{R}^d)$ the previous lemma, applied with $\bar{m} = (0, ..., 0)$, implies $\mathcal{F} M \in L_1(\mathbb{R}^d)$. This gives immediately the continuity of M (with interpretation). Therefore $M(k)$ is well-defined.*

Proof Let $\psi(\xi)$ be a function belonging to $S(\mathbb{R}^d)$ satisfying $\operatorname{supp} \psi \subset [-2, 2]^d$ and

$$\psi(\xi) = 1 \quad , \quad x \in [-1, 1]^d .$$

Of course, we have $\psi M \in S_2^{\bar{r}} H(\mathbb{R}^d)$ and

$$\| \psi M | S_2^{\bar{r}} H \| \leq c_{\psi} \| M | S_2^{\bar{r}} H \|. \tag{1.16}$$

By $\tilde{\psi}(x)$ we denote the function

$$\tilde{\psi}(x) = \psi(b_1^{-1} x_1, ..., b_d^{-1} x_d).$$

This implies immediately $\tilde{\psi}(\xi) = 1$ for $x \in Q_{\bar{b}}$. Lemma 1.4 used with $\bar{m} = (0, ..., 0)$ again gives $\mathcal{F}(\tilde{\psi} M) \in L_1(\mathbb{R}^d)$. With the help of Lemma 1.1 we derive

$$\begin{aligned} \Big| \sum_{k \in \Lambda} M(k) c_k(t) e^{ikx} \Big| &= \Big| \sum_{k \in \Lambda} (M(k) \tilde{\psi}(k)) c_k(t) e^{ikx} \Big| \\ &\leq c \int_{\mathbb{R}^d} | \mathcal{F}^{-1}(\tilde{\psi} M)(y) t(x-y) | dy \end{aligned}$$

and therefore

$$\Big| \sum_{k \in \Lambda} M(k) c_k(t) e^{ikx} \Big| \leq c \| \mathcal{F}^{-1}(\tilde{\psi} M)(y) t(x-y) | L_1(\mathbb{R}^d, y) \| .$$

For fixed $x \in \mathbb{T}^d$ the function

$$g(y) = \mathcal{F}^{-1}(\tilde{\psi} M)(y) t(x-y)$$

has the Fourier transform

$$\begin{aligned}\mathcal{F}g(\xi) =& \frac{1}{\sqrt{2\pi}^d} \sum_{k\in\Lambda} c_k(t)e^{ikx} \int\limits_{\mathbb{R}^d} \mathcal{F}^{-1}(\tilde{\psi}M)(y)e^{-iy(k+\xi)}dy \\ =& \sum_{k\in\Lambda} c_k(t)e^{ikx}(\tilde{\psi}M)(k+\xi)\,.\end{aligned}$$

Consequently, $\operatorname{supp}\mathcal{F}g$ is included in $\{y : |y_i| \leq 3b_i\}$. Let us apply Proposition 1.1 (Nikol'skij's inequality) to the function $g(y)$ by putting $\tilde{p} = \min(p,1)$ and $\alpha = (0,...,0)$. This yields

$$\begin{aligned}\Big|\sum_{k\in\Lambda} M(k)c_k(t)e^{ikx}\Big| \leq& c\|\mathcal{F}^{-1}(\tilde{\psi}M)(y)t(x-y)|L_1(\mathbb{R}^d,y)\| \\ \leq& c_1 b_1^{1/\tilde{p}-1}\cdots b_d^{1/\tilde{p}-1}\|\mathcal{F}^{-1}(\tilde{\psi}M)(y)t(x-y)|L_{\tilde{p}}(\mathbb{R}^d,y)\|.\end{aligned}$$

Taking the $L_p(\mathbb{T}^d)$-(quasi-)norm we obtain

$$\begin{aligned}\Big\|\sum_{k\in\Lambda} M(k)c_k(t)e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| \leq& c_1 b_1^{1/\tilde{p}-1}\cdots b_d^{1/\tilde{p}-1}\times \\ &\times\Big[\int\limits_{\mathbb{T}^d}\Big(\int\limits_{\mathbb{R}^d}|\mathcal{F}^{-1}(\tilde{\psi}M)(y)t(x-y)|^{\tilde{p}}dy\Big)^{p/\tilde{p}}dx\Big]^{1/p}.\end{aligned}$$

In the case $0 < p \leq 1$ we have $\tilde{p} = p$. Fubini's theorem implies

$$\Big\|\sum_{k\in\Lambda} M(k)c_k(t)e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| \leq c_1 b_1^{1/\tilde{p}-1}\cdots b_d^{1/\tilde{p}-1}\|t|L_p(\mathbb{T}^d)\|\cdot\|\mathcal{F}^{-1}(\tilde{\psi}M)(y)|L_p(\mathbb{R}^d)\|.$$

Minkowski's inequality gives

$$\Big\|\sum_{k\in\Lambda} M(k)c_k(t)e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| \leq c_1 b_1^{1/\tilde{p}-1}\cdots b_d^{1/\tilde{p}-1}\|\mathcal{F}^{-1}(\tilde{\psi}M)|L_1(\mathbb{R}^d)\|\cdot\|t|L_p(\mathbb{T}^d)\|$$

in the case $p > 1$ ($\tilde{p} = 1$). Therefore, both cases finish in the inequality

$$\begin{aligned}\Big\|\sum_{k\in\Lambda} M(k)c_k(t)e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| \leq& c_1 b_1^{1/\tilde{p}-1}\cdots b_d^{1/\tilde{p}-1}\|\mathcal{F}^{-1}(\tilde{\psi}M)|L_{\tilde{p}}(\mathbb{R}^d)\|\cdot\|t|L_p(\mathbb{T}^d)\| \\ =& c_1\|\mathcal{F}^{-1}(\psi M(b_1\cdot,...,b_d\cdot))|L_{\tilde{p}}(\mathbb{R}^d)\|\cdot\|t|L_p(\mathbb{T}^d)\|\,.\end{aligned}$$

Because of $\bar{r} > \frac{1}{\tilde{p}} - \frac{1}{2}$ we can apply Lemma 1.4 on $\mathcal{F}^{-1}(\psi M(b_1\cdot,...,b_d\cdot))$ with $\bar{m} = (0,...,0)$. This yields

$$\|\mathcal{F}^{-1}(\psi M(b_1\cdot,...,b_d\cdot))|L_{\tilde{p}}(\mathbb{R}^d)\| \leq c_2\|\psi M(b_1\cdot,...,b_d\cdot)|S_2^{\bar{r}}H(\mathbb{R}^d)\|.$$

The right-hand side can be estimated using (1.16) by a constant c_ψ. Together with (1.17) we finally get

$$\Big\|\sum_{k\in\Lambda} M(k)c_k(t)e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| \leq c_3\|M(b_1\cdot,...,b_d\cdot)|S_2^{\bar{r}}H(\mathbb{R}^d)\|\cdot\|t|L_p(\mathbb{T}^d)\|.$$

□

1.3.4 Fourier Multipliers for Vector-Valued Spaces

Let us now present a Fourier multiplier assertion for the spaces $L_p(\mathbb{T}^d, \ell_q)$, defined in Paragraph 1.2.4. We combine [32, 1.10.3] and [32, 3.4.1] to obtain the following result.

Proposition 1.7 *Let $0 < p < \infty$ and $0 < q \leq \infty$. Let further $\{\bar{b}^{\bar{\ell}}\}_{\bar{\ell}\in\mathbb{N}_0^d} \subset (0,\infty)^d$ and $\Lambda = \{\Lambda_{\bar{\ell}}\}_{\bar{\ell}\in\mathbb{N}_0^d}$ be a sequence of finite subsets of $\mathbb{Z}^d$ satisfying*

$$\Lambda_{\bar{\ell}} \subset \{x \in \mathbb{R}^d : |x_i| \leq b_i^{\bar{\ell}}, i = 1, ..., d\}.$$

The vector $\bar{\kappa} = (\kappa_1, ..., \kappa_d)$ is supposed to fulfill

$$\bar{\kappa} > \frac{1}{\min(p,q)} + \frac{1}{2}\,. \tag{1.17}$$

Then a constant $c > 0$ exists (independent of $\{\bar{b}^{\bar{\ell}}\}_{\bar{\ell}\in\mathbb{N}_0^d}, \Lambda$ and $M_{\bar{\ell}}$) such that

$$\Big\| \sum_{k\in\mathbb{Z}^d} M_{\bar{\ell}}(k) c_k(t_{\bar{\ell}}) e^{ikx} \Big| L_p(\mathbb{T}^d, \ell_q) \Big\| \leq c \sup_{\bar{\ell}\in\mathbb{N}_0^d} \|M_{\bar{\ell}}(b_1^{\bar{\ell}}\cdot, ..., b_d^{\bar{\ell}}\cdot)|S_2^{\bar{\kappa}}H(\mathbb{R}^d)\| \cdot \|t_{\bar{\ell}}|L_p(\mathbb{T}^d, \ell_q)\|$$

holds for all systems $t = \{t_{\bar{\ell}}\}_{\bar{\ell}\in\mathbb{N}_0^d} \in L_p(\mathbb{T}^d, \ell_q)$ and all sequences $\{M_{\bar{\ell}}\}_{\bar{\ell}\in\mathbb{N}_0^d} \subset S_2^{\bar{\kappa}}H(\mathbb{R}^d)$.

Proof We fix $\bar{\ell} \in \mathbb{N}_0^d$ and put

$$g_{\bar{\ell}}(x) := \sum_{k\in\mathbb{Z}^d} M_{\bar{\ell}}(k) c_k(t_{\bar{\ell}}) e^{ikx}.$$

Analogously to the scalar case we estimate

$$|g_{\bar{\ell}}(x)| \leq c \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}M_{\bar{\ell}})(y)| \cdot |t_{\bar{\ell}}(x-y)|\, dy.$$

In order to use the maximal function technique we consider

$$\begin{aligned}
&|g_{\bar{\ell}}(x-z)| \\
&\leq c \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}M_{\bar{\ell}})(x-z-y)| \cdot |t_{\bar{\ell}}(y)|\, dy \\
&= c \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}M_{\bar{\ell}})(x-z-y)| \cdot \frac{|t_{\bar{\ell}}(x-(x-y))|}{\prod_{i=1}^d (1 + |b_i^{\bar{\ell}}(x_i - y_i)|^{m_i})} \times \\
&\quad \times (1 + |b_1^{\bar{\ell}}(x_1-y_1)|^{m_1}) \cdot ... \cdot (1 + |b_d^{\bar{\ell}}(x_d - y_d)|^{m_d})\, dy \\
&\leq c\, P_{\bar{b}^{\bar{\ell}},\bar{m}}\, t_{\bar{\ell}}(x) \int_{\mathbb{R}^d} |(\mathcal{F}^{-1}M_{\bar{\ell}})(x-z-y)| \times \\
&\quad \times (1 + |b_1^{\bar{\ell}}(x_1-y_1)|^{m_1}) \cdot ... \cdot (1 + |b_d^{\bar{\ell}}(x_d - y_d)|^{m_d})\, dy.
\end{aligned} \tag{1.18}$$

The vector $\bar{m} = (m_1, ..., m_d)$ is at our disposal. By the change of variable

$$u_i = b_i^{\bar{\ell}}(x_i - z_i - y_i) \quad , \quad i = 1, ..., d,$$

we continue with (1.18) and obtain

$$|g_{\bar{\ell}}(x - z)| \le c\, P_{\bar{b}^{\bar{\ell}},\bar{m}}\, t_{\bar{\ell}}(x) \cdot \frac{1}{b_1^{\bar{\ell}} \cdot ... \cdot b_d^{\bar{\ell}}} \int\limits_{\mathbb{R}^d} \Big| (\mathcal{F}^{-1} M_{\bar{\ell}}) \Big(\frac{u_1}{b_1^{\bar{\ell}}}, ..., \frac{u_d}{b_d^{\bar{\ell}}} \Big) \Big| \times$$
$$(1 + |u_1 + b_1^{\bar{\ell}} z_1|^{m_1}) \cdot ... \cdot (1 + |u_d + b_d^{\bar{\ell}} z_d|^{m_d})\, du.$$

Because of $(1 + |a + b|^s) \le c_s(1 + |a|^s)(1 + |b|^s)$ $(a, b \in \mathbb{R}$ and $s > 0)$ and

$$\frac{1}{b_1^{\bar{\ell}} \cdot ... \cdot b_d^{\bar{\ell}}} (\mathcal{F}^{-1} M_{\bar{\ell}}) \left(\frac{u_1}{b_1^{\bar{\ell}}}, ..., \frac{u_d}{b_d^{\bar{\ell}}} \right) = (\mathcal{F}^{-1} M_{\bar{\ell}}(b_1^{\bar{\ell}}\cdot, ..., b_d^{\bar{\ell}}\cdot))(u)$$

we obtain

$$P_{\bar{b}^{\bar{\ell}},\bar{m}} g_{\bar{\ell}}(x) \le c\, P_{\bar{b}^{\bar{\ell}},\bar{m}} t_{\bar{\ell}}(x) \times$$
$$\times \, \|(1 + |u_1|^{m_1}) \cdot ... \cdot (1 + |u_d|^{m_d})(\mathcal{F}^{-1} M_{\bar{\ell}}(b_1^{\bar{\ell}}\cdot, ..., b_d^{\bar{\ell}}\cdot))(u) | L_1(\mathbb{R}^d)\|.$$

Let us now employ Lemma 1.4 with $p = 1$, where the quantities m_i are chosen such that $m_i > 1/\min(p, q)$ and $\kappa_i > m_i + 1/2$, $i = 1, ..., d$. We can proceed like this because of (1.17). The first condition is required later in order to employ Proposition 1.5. Now Lemma 1.4 gives

$$\|(1 + |u_1|^{m_1}) \cdot ... \cdot (1 + |u_d|^{m_d})(\mathcal{F}^{-1} M_{\bar{\ell}}(b_1^{\bar{\ell}}\cdot, ..., b_d^{\bar{\ell}}\cdot))(u) | L_1(\mathbb{R}^d)\|$$
$$\le c_1 \|M_{\bar{\ell}}(b_1^{\bar{\ell}}\cdot, ..., b_1^{\bar{\ell}}\cdot) | S_2^{\bar{\kappa}} H(\mathbb{R}^d)\|\,.$$

This means

$$|g_{\bar{\ell}}(x)| \le P_{\bar{b}^{\bar{\ell}},\bar{m}} g_{\bar{\ell}}(x) \le c_2 P_{\bar{b}^{\bar{\ell}},\bar{m}} t_{\bar{\ell}}(x) \times \|M_{\bar{\ell}}(b_1^{\bar{\ell}}\cdot, ..., b_1^{\bar{\ell}}\cdot) | S_2^{\bar{\kappa}} H(\mathbb{R}^d)\|\,.$$

Taking the $L_p(\mathbb{T}^d)$-(quasi-)norm, Proposition 1.5 guarantees

$$\|g_{\bar{\ell}}\, | L_p(\mathbb{T}^d, \ell_q)\| \le c_3 \sup_{\bar{\ell} \in \mathbb{N}_0^d} \|M_{\bar{\ell}}(b_1^{\bar{\ell}}\cdot, ..., b_1^{\bar{\ell}}\cdot) | S_2^{\bar{\kappa}} H(\mathbb{R}^d)\| \cdot \|t_{\bar{\ell}}\, | L_p(\mathbb{T}^d, \ell_q)\|.$$

□

1.3.5 Lizorkin's Multiplier Theorem

Let us recall the definition of signed and complex measures. For details concerning the following we mainly refer to [23, A.6]. Here $(X, \mathcal{F})$ denotes a measurable space. A mapping $\mu : \mathcal{F} \to \mathbb{R} \cup \{-\infty, \infty\}$ is called a signed measure if and only if the following two conditions are satisfied

(i) $\mu(\emptyset) = 0\,,$

(ii)$\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n) \quad , \quad$ for $A_i, A_j \in \mathcal{F}$ with $A_i \cap A_j = \emptyset$, $i \neq j$.

Complex measures are defined analogously. The variation of a signed (complex) measure is given by the positive measure

$$|\mu|(E) = \sup\Big\{\sum_{k=1}^{n} |\mu(A_k)| : A_k \in \mathcal{F} \text{ pairwise disjoint }, \bigcup_{k=1}^{n} A_k = E\Big\}.$$

The number $|\mu|(X)$ is called total variation of the signed (complex) measure μ on the space X. Now we are able to state Lizorkin's multiplier theorem. As measurable space we consider now $(\mathbb{R}^d, \mathcal{R}^d)$, where $\mathcal{R}^d$ denotes the usual Borel σ-algebra.

Proposition 1.8 *Let $1 < p, q < \infty$. Let further $M = \{M_j(x)\}_{j=0}^{\infty} \subset L_\infty(\mathbb{R}^d)$ be a sequence of functions satisfying*

(i) There are finite complex measures μ_j, $j = 0, 1, ...,$ on $(\mathbb{R}^d, \mathcal{R}^d)$ such that

$$M_j(x_1, ..., x_d) = \mu_j((-\infty, x_1] \times ... \times (-\infty, x_d]) \ .$$

(ii) The measures μ_j, $j = 0, 1, 2...$ provide uniformly bounded total variation on $\mathbb{R}^d$, i.e.

$$|\mu_j|(\mathbb{R}^d) \leq C_M \quad , \quad j = 0, 1, 2... \quad .$$

(iii) The functions $M_j(x)$, $j = 0, 1, 2...$ are continuous in all points $k \in \mathbb{Z}^d$.

Under these conditions we have the existence of a positive constant $C(p, q, d)$ such that

$$\Big\| \sum_{k \in \mathbb{Z}^d} M_j(k) c_k(f_j) e^{ik\cdot x} \Big| L_p(\mathbb{T}^d, \ell_q)\Big\| \leq C \cdot C_M \|f_j | L_p(\mathbb{T}^d, \ell_q)\|$$

holds for all $\{f_j\}_{j=0}^{\infty} \in L_p(\mathbb{T}^d, \ell_q)$.

Proof The nonperiodic counterpart is due to Lizorkin, cf. [21]. The lemma follows by applying Theorem 3.4.2 in [32]. □

Remark 1.3 *Consider a compactly supported piecewise linear continuous function $\gamma : \mathbb{R} \to \mathbb{C}$ uniquely defined by the nodes $\{(k, \gamma(k)) : k \in \mathbb{Z}\}$ and its weak derivative denoted by $\gamma'(x)$. The latter exists as a piecewise constant step function. Consequently, the corresponding complex measure μ, which satisfies $\mu((-\infty, x]) = \gamma(x)$, $x \in \mathbb{R}$, is given by*

$$\mu(A) = \int\limits_A \gamma'(x)\, dx \quad , \quad A \in \mathcal{R}\,.$$

Then we observe

$$|\mu|(\mathbb{R}) = \sum_{k\in\mathbb{Z}} |\gamma(k)-\gamma(k-1)|\,. \tag{1.19}$$

Let us now consider a sequence $\gamma = \{\gamma_j(x)\}_{j=0}^{\infty}$ *of such functions. The sequence* $\{\mu_j\}_{j=0}^{\infty}$ *denotes the corresponding measures. Additionally, we assume the uniform boundedness of (1.19), i.e.*

$$\sup_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{Z}} |\gamma_j(k)-\gamma_j(k-1)| =: C_\gamma < \infty\,.$$

Their tensor product

$$M_u(x_1,...,x_d) := \gamma_{u_1}(x_1)\cdot ...\cdot \gamma_{u_d}(x_d) \quad , \quad x=(x_1,...,x_d)\in\mathbb{R}^d \quad ,$$

can be written as

$$\begin{aligned} M_u(x_1,...,x_d) &= \mu_{u_1}((-\infty,x_1])\cdot ...\cdot \mu_{u_d}((-\infty,x_d]) \\ &= (\mu_{u_1}\otimes ...\otimes \mu_{u_d})((-\infty,x_1]\times ...\times(-\infty,x_d])\,. \end{aligned}$$

Here $\mu_{u_1}\otimes ...\otimes\mu_{u_d}$ *denotes the product measure of* $\mu_{u_1},...,\mu_{u_d}$ *. An easy calculation using the Jordan decomposition of a signed measure yields*

$$|\mu_{u_1}\otimes ...\otimes \mu_{u_d}|(\mathbb{R}^d) \le (4C_\gamma)^d\,.$$

See for instance [23, A.6] and (1.19).
Finally, Proposition 1.8 implies

$$\Big\|\sum_{k\in\mathbb{Z}^d} c_k(f_u)M_u(k)e^{ikx}\Big|L_p(\mathbb{T}^d,\ell_q)\Big\| \le C\cdot(4C_\gamma)^d\|f_u|L_p(\mathbb{T}^d,\ell_q)\|\,,$$

for all systems $f=\{f_u\}_{u\in\mathbb{N}_0^d}\in L_p(\mathbb{T}^d,\ell_q)$*, where* C *just depends on* p,q *and* d*.*

1.4 The Spaces $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ and $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$

This section is devoted to the definition of Besov and Triebel-Lizorkin as well as Sobolev type spaces with dominating mixed smoothness properties. We follow the approach in [32, Chapt. 2], where the nonperiodic bivariate case is treated. As we have seen in the previous paragraph all the necessary techniques carry over to the periodic setting.

1.4.1 Decomposition of Unity

First of all we introduce the classical concept of a smooth dyadic decomposition of unity. It is used to decompose a distribution into Fourier analytical building blocks to decide whether this distribution belongs to a certain function space of Besov and Triebel-Lizorkin type.

Definition 1.1 *Let $\Phi(\mathbb{R})$ be the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \subset S(\mathbb{R})$ satisfying*

(i) $\operatorname{supp} \varphi_0 \subset \{x : |x| \le 2\}$,

(ii) $\operatorname{supp} \varphi_j \subset \{x : 2^{j-1} \le |x| \le 2^{j+1}\} \quad , \quad j = 1, 2, ...,$

(iii) $\forall\, \ell \in \mathbb{N}_0$ *we have* $\sup\limits_{x,j} 2^{j\ell}\, |D^\ell \varphi_j(x)| \le c_\ell < \infty$,

(iv) $\sum\limits_{j=0}^{\infty} \varphi_j(x) = 1$ *for all* $x \in \mathbb{R}$.

Remark 1.4 *The class $\Phi(\mathbb{R})$ is not empty. Consider the following example. Let $\varphi_0(x) \in S(\mathbb{R})$ be smooth function with $\varphi_0(x) = 1$ on $[-1, 1]$ and $\varphi_0(x) = 0$ if $|x| > 2$. For $j > 0$ we define*

$$\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x).$$

Now it is easy to verify that the system $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty}$ satisfies (i) - (iv).

Let us modify this definition in the following way and define the class $\Psi(\mathbb{R})$.

Definition 1.2 *A system $\varphi = \{\varphi_j\}_{j=0}^{\infty} \subset S(\mathbb{R})$ belongs to the class $\Psi(\mathbb{R})$ if and only if*

(i) A positive constant A exists such that $\operatorname{supp} \varphi_0 \subset [-A, A]$.

(ii) There are constants $0 < B < C$ with $\operatorname{supp} \varphi_j \subset \{x \in \mathbb{R} : B2^j \le |x| \le C2^j\}$.

(iii) For all $\ell \in \mathbb{N}_0$ holds

$$\sup_{x \in \mathbb{R}, j \in \mathbb{N}_0} 2^{j\ell} |D^\ell \varphi_j(x)| \le c_\ell < \infty \quad .$$

(iv) The identity $\sum\limits_{j=0}^{\infty} \varphi_j(x) = 1$ *is valid for all* $x \in \mathbb{R}$.

Remark 1.5 *The inclusion $\Phi(\mathbb{R}) \subset \Psi(\mathbb{R})$ follows immediately.*

Based on this we construct decompositions of unity in $\mathbb{R}^d$ via tensor products. This requires the following notation. Assume $\varphi^i = \{\varphi^i_j(x)\}_{j=0}^{\infty} \subset S(\mathbb{R})$, $i = 1, ..., d$, and let $\bar{\ell} = (\ell_1, ..., \ell_d) \in \mathbb{N}_0^d$. We put

$$(\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(x) := \varphi^1_{\ell_1}(x_1) \cdot ... \cdot \varphi^d_{\ell_d}(x_d) \quad , \quad x \in \mathbb{R}^d .$$

If $\varphi = \varphi^1 = ... = \varphi^d$ we simply write $\varphi_{\bar{\ell}}(x)$ instead of $(\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(x)$, $\bar{\ell} \in \mathbb{N}_0^d$. Suppose $(\varphi^1, ..., \varphi^d) \in \Psi(\mathbb{R})^d$. Then one can decompose $f \in D'(\mathbb{T}^d)$ into the sum

$$f = \sum_{\bar{\ell} \in \mathbb{N}_0^d} f_{\bar{\ell}} \quad , \tag{1.20}$$

where

$$f_{\bar{\ell}}(x) = \sum_{k \in \mathbb{Z}^d} (\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(k) c_k(f) e^{ik \cdot x} \quad , \quad x \in \mathbb{T}^d . \tag{1.21}$$

The convergence is considered in $D'(\mathbb{T}^d)$, see Paragraph 1.2.3 .

1.4.2 Definition and Basic Properties

Let us introduce the function spaces $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ and $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ of Besov and Triebel-Lizorkin spaces with dominating mixed smoothness. We make use of the building blocks given in (1.21) .

Definition 1.3 *Let* $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ *and* $\bar{\varphi} = (\varphi^1, ..., \varphi^d) \in \Psi(\mathbb{R})^d$.

(i) Let $0 < p \le \infty$ *and* $0 < q < \infty$. *Then* $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ *is the collection of all* $f \in D'(\mathbb{T}^d)$ *such that*

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{\bar{\varphi}} = \Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\cdot\bar{\ell}q}\Big\|\sum_{k\in\mathbb{Z}^d}(\varphi^1\otimes...\otimes\varphi^d)_{\bar{\ell}}(k)c_k(f)e^{ik\cdot x}\Big|L_p(\mathbb{T}^d)\Big\|^q\Big)^{1/q} \quad (1.22)$$

is finite. In case $q = \infty$ *we replace (1.22) by*

$$\|f|S^{\bar{r}}_{p,\infty}B(\mathbb{T}^d)\|^{\bar{\varphi}} = \sup_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\cdot\bar{\ell}}\Big\|\sum_{k\in\mathbb{Z}^d}(\varphi^1\otimes...\otimes\varphi^d)_{\bar{\ell}}(k)c_k(f)e^{ik\cdot x}\Big|L_p(\mathbb{T}^d)\Big\|.$$

(ii) Let $0 < p < \infty$ *and* $0 < q < \infty$. *Then* $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ *is the collection of all* $f \in D'(\mathbb{T}^d)$ *such that*

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{\bar{\varphi}} = \Big\|\Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\cdot\bar{\ell}q}\Big|\sum_{k\in\mathbb{Z}^d}(\varphi^1\otimes...\otimes\varphi^d)_{\bar{\ell}}(k)c_k(f)e^{ik\cdot x}\Big|^q\Big)^{1/q}\Big|L_p(\mathbb{T}^d)\Big\| \quad (1.23)$$

is finite. In case $q = \infty$ *we replace (1.23) by*

$$\|f|S^{\bar{r}}_{p,\infty}F(\mathbb{T}^d)\|^{\bar{\varphi}} = \Big\|\sup_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\bar{r}\cdot\bar{\ell}}\Big|\sum_{k\in\mathbb{Z}^d}(\varphi^1\otimes...\otimes\varphi^d)_{\bar{\ell}}(k)c_k(f)e^{ik\cdot x}\Big|\Big|L_p(\mathbb{T}^d)\Big\|.$$

Remark 1.6 (i) *In the special case* $\bar{r} = (r, ..., r)$, $r \in \mathbb{R}$, *we simply write* $S^{r}_{p,q}F(\mathbb{T}^d)$ *and* $S^{r}_{p,q}B(\mathbb{T}^d)$ *instead of* $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ *and* $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$.
(ii) *In case* $d = 1$ *the spaces defined degenerate into the usual isotropic Besov and Triebel-Lizorkin spaces on the torus. They are denoted by* $B^r_{p,q}(\mathbb{T})$ *and* $F^r_{p,q}(\mathbb{T})$.
See also [32, Chapt. 3].
(iii) *The spaces* $S^{\bar{r}}_{p,p}B(\mathbb{T}^d)$ *and* $S^{\bar{r}}_{p,p}F(\mathbb{T}^d)$ *coincide. In this situation even the (quasi-)norms (1.22) and (1.23) are equal.*

Remark 1.7 *We observe that all these classes are (quasi-)normed spaces (normed spaces in case* $\min(p,q) \ge 1$*). Although indicated the spaces do not depend on the chosen system* $\bar{\varphi} \in \Psi(\mathbb{R})^d$. *The corresponding (quasi-)norms are even equivalent, see Theorem 1.2 below. The analogous definition of* $S^{\bar{r}}_{\infty,q}F(\mathbb{T}^d)$ *does not work (the spaces would depend on the system* $\bar{\varphi}$*), cf. also [32, 3.5.2].*

Remark 1.8 *Recall Remark 1.1. The (quasi-)norms* $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{\bar{\varphi}}$ *and* $\|\cdot|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{\bar{\varphi}}$ *are cross-norms in the sense*

$$\|(g_1\otimes\cdots\otimes g_d)|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{\bar{\varphi}} = \|g_1|B^{r_1}_{p,q}(\mathbb{T})\|^{\varphi^1}\cdot\ldots\cdot\|g_d|B^{r_d}_{p,q}(\mathbb{T})\|^{\varphi^d}$$

and

$$\|(g_1\otimes\cdots\otimes g_d)|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{\bar{\varphi}} = \|g_1|F^{r_1}_{p,q}(\mathbb{T})\|^{\varphi^1}\cdot\ldots\cdot\|g_d|F^{r_d}_{p,q}(\mathbb{T})\|^{\varphi^d}\,.$$

Let us state the independence of the spaces from the chosen system $\bar{\varphi}\in\Psi(\mathbb{R})^d$. For sake of simplicity we will prove it only in case $\bar{\varphi}\in\Phi(\mathbb{R})^d$ by following [32, 2.2.3].

Theorem 1.2 *Let* $\bar{\varphi},\bar{\psi}\in\Psi(\mathbb{R})^d$, $\bar{r}=(r_1,...,r_d)\in\mathbb{R}^d$ *and* $0<q\leq\infty$. *Then the following assertions hold true.*

(i) If $0<p\leq\infty$ *the (quasi-)norms* $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{\bar{\varphi}}$ *and* $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{\bar{\psi}}$ *are equivalent.*

(ii) If $0<p<\infty$ *the (quasi-)norms* $\|\cdot|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{\bar{\varphi}}$ *and* $\|\cdot|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{\bar{\psi}}$ *are equivalent.*

Proof We show the existence of a constant c with $\|\cdot|S^{\bar{r}}_{p,q}F\|^{\bar{\varphi}}\leq c\|\cdot|S^{\bar{r}}_{p,q}F\|^{\bar{\psi}}$. In the B-case the proof is similar. We follow the proof of [48, Prop. 2.3.2/1] and [32, Prop. 2.2.3]. The properties (ii) and (iv) in Definition 1.1 imply

$$\varphi^i_j(t) = (\psi^i_{j-1}(j)+\psi^i_j(t)+\psi^i_{j+1}(t))\varphi^i_j(t)\quad,\quad t\in\mathbb{R}\quad,\quad i=1,...,d.$$

Putting always $\varphi^i_{-1},\psi^i_{-1}\equiv 0$ we have for any $\bar{\ell}\in\mathbb{N}_0^d$

$$\begin{aligned}&(\varphi^1\otimes\ldots\otimes\varphi^d)_{\bar{\ell}}(\xi)\\&=\Big(\sum_{\bar{j}\in\{-1,0,1\}^d}(\psi^1\otimes\ldots\otimes\psi^d)_{\bar{\ell}+\bar{j}}(\xi)\Big)(\varphi^1\otimes\ldots\otimes\varphi^d)_{\bar{\ell}}(\xi)\quad,\quad\xi\in\mathbb{R}^d\,.\end{aligned}$$

This immediately yields

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d}(\varphi^1\otimes\ldots\otimes\varphi^d)_{\bar{\ell}}(k)c_k(f)e^{ikx}\\&=\sum_{\bar{j}\in\{-1,0,1\}^d}\Big(\sum_{k\in\mathbb{Z}^d}(\varphi^1\otimes\ldots\otimes\varphi^d)_{\bar{\ell}}(k)\cdot(\psi^1\otimes\ldots\otimes\psi^d)_{\bar{\ell}+\bar{j}}(k)c_k(f)e^{ikx}\Big)\quad,\quad x\in\mathbb{T}^d\,,\end{aligned}$$

for a fixed $f\in D'(\mathbb{T}^d)$. Taking the $L_p(\mathbb{T}^d,\ell_q)$-(quasi-)norm and applying the triangle inequality this leads to

$$\begin{aligned}&\Big\|2^{\bar{r}\bar{\ell}}\sum_{k\in\mathbb{Z}^d}(\varphi^1\otimes\ldots\otimes\varphi^d)_{\bar{\ell}}(k)c_k(f)e^{ikx}\Big|L_p(\mathbb{T}^d,\ell_q)\Big\|\\&\leq c\sum_{\bar{j}\in\{-1,0,1\}^d}\Big\|2^{\bar{r}\bar{\ell}}\sum_{k\in\mathbb{Z}^d}(\varphi^1\otimes\ldots\otimes\varphi^d)_{\bar{\ell}}(k)\cdot(\psi^1\otimes\ldots\otimes\psi^d)_{\bar{\ell}+\bar{j}}(k)c_k(f)e^{ikx}\Big|L_p(\mathbb{T}^d,\ell_q)\Big\|\,.\end{aligned}$$

Using again Definition 1.1/(ii) together with Proposition 1.7 we derive

$$\begin{aligned}&\Big\|2^{\bar r\bar\ell}\sum_{k\in\mathbb{Z}^d}(\varphi^1\otimes\ldots\otimes\varphi^d)_{\bar\ell}(k)c_k(f)e^{ikx}\Big|L_p(\mathbb{T}^d,\ell_q)\Big\|\\ &\le c_1\cdot C(\bar\varphi)\cdot\sum_{\bar j\in\{-1,0,1\}^d}\Big\|2^{\bar r\bar\ell}\sum_{k\in\mathbb{Z}^d}(\psi^1\otimes\ldots\otimes\psi^d)_{\bar\ell+\bar j}(k)c_k(f)e^{ikx}\Big|L_p(\mathbb{T}^d,\ell_q)\Big\|,\end{aligned}$$

where

$$\begin{aligned}C(\bar\varphi)&:=\sup_{\bar\ell\in\mathbb{N}_0^d}\|\varphi^1_{\ell_1}(2^{\ell_1+2}x_1)\cdot\ldots\cdot\varphi^d_{\ell_d}(2^{\ell_d+2}x_d)|S^{\bar\kappa}_2H(\mathbb{R}^d)\|\\ &=\sup_{\ell\in\mathbb{N}_0}\|\varphi^1_{\ell_1}(2^{\ell_1+2}\cdot)|H^{\kappa_1}_2(\mathbb{R})\|\cdot\ldots\cdot\sup_{\ell_d\in\mathbb{N}_0}\|\varphi^d_{\ell_d}(2^{\ell_d+2}\cdot)|H^{\kappa_d}_2(\mathbb{R})\|\end{aligned}$$

with $\bar\kappa$ large enough. Elementary properties of the Fourier transform together with (ii) and (iii) in Definition 1.1 imply the uniform boundedness of $\|\varphi^i_n(2^{n+2}\cdot)|H^{\kappa_d}_2(\mathbb{R})\|$, $i=1,...,d$, for $n\in\mathbb{N}_0$. Together with this we receive

$$C(\bar\varphi)<\infty\,.$$

Finally, we obtain

$$\|f|S^{\bar r}_{p,q}F(\mathbb{T}^d)\|^{\bar\varphi}\le c_3\|f|S^{\bar r}_{p,q}F(\mathbb{T}^d)\|^{\bar\psi}\,.$$

Of course, the converse inequality holds as well. This proves (ii). □

Remark 1.9 *The modifications in the case $\bar\varphi,\bar\psi\in\Psi(\mathbb{R})^d$ are obvious.*

Remark 1.10 *Let us give a remark on Lizorkin representations of Besov as well as Triebel-Lizorkin spaces. We need a special covering of $\mathbb{R}^d$. Let*

$$\begin{aligned}P_0 &:= [-1,1]\,, \qquad P_k:=\{x\in\mathbb{R}^d:\ 2^{k-1}<|x|\le 2^k\}\,,\quad k\in\mathbb{N}\,,\\ \mathcal{P}_{\bar k} &:= P_{k_1}\times\ldots\times P_{k_d}\,,\qquad \bar k\in\mathbb{N}_0^d\,.\end{aligned}\tag{1.24}$$

Then

$$\mathbb{R}^d=\bigcup_{\bar k\in\mathbb{N}_0^d}\mathcal{P}_{\bar k}\qquad\text{and}\qquad\mathcal{P}_{\bar k}\cap\mathcal{P}_{\bar\ell}=\emptyset\quad\text{if}\quad\bar k\neq\bar\ell\,.$$

Hence, with

$$\widetilde{f}_{\bar\ell}(x):=\sum_{k\in\mathcal{P}_{\bar\ell}}c_k(f)\,e^{ikx}\,,\qquad x\in\mathbb{T}^d,\ \bar\ell\in\mathbb{N}_0^d\,,\tag{1.25}$$

we find

$$f=\sum_{\bar\ell\in\mathbb{N}_0^d}\widetilde{f}_{\bar\ell}$$

(convergence in $D'(\mathbb{T}^d)$, compare with (1.20)).

Lemma 1.5 *Let $1 < p < \infty$ and $\bar{r} \in \mathbb{R}^d$. Then, if $0 < q \leq \infty$*

$$\| f \,| S^{\bar{r}}_{p,q} B(\mathbb{T}^d) \| \asymp \Big(\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q} \, \| \widetilde{f}_{\bar{\ell}} \,| L_p(\mathbb{T}^d) \|^q \Big)^{1/q}, \tag{1.26}$$

and if $1 < q < \infty$

$$\| f \,| S^{\bar{r}}_{p,q} F(\mathbb{T}^d) \| \asymp \Big\| \Big(\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q} \, | \widetilde{f}_{\bar{\ell}}(x) |^q \Big)^{1/q} \Big| L_p(\mathbb{T}^d) \Big\| \tag{1.27}$$

holds for all $f \in D'(\mathbb{T}^d)$ (modification in case $q = \infty$).

1.4.3 Elementary Embeddings

Let $S^{\bar{r}}_{p,q} A(\mathbb{T}^d)$ be either $S^{\bar{r}}_{p,q} B(\mathbb{T}^d)$ or $S^{\bar{r}}_{p,q} F(\mathbb{T}^d)$ (with $p < \infty$ in the F-case). We observe the following elementary continuous embeddings.

Lemma 1.6 *Let $(r_1, ..., r_d) \in \mathbb{R}^d$ and $0 < p \leq \infty$.*

(i) Let $0 < q \leq v \leq \infty$. Then we have

$$S^{\bar{r}}_{p,q} A(\mathbb{T}^d) \hookrightarrow S^{\bar{r}}_{p,v} A(\mathbb{T}^d).$$

(ii) Let $0 < q, v \leq \infty$ and $\bar{\varepsilon} = (\varepsilon_1, ..., \varepsilon_d)$ with $\bar{\varepsilon} \geq 0$. Then

$$S^{\bar{r}+\bar{\varepsilon}}_{p,q} A(\mathbb{T}^d) \hookrightarrow S^{\bar{r}}_{p,q} A(\mathbb{T}^d).$$

If $\bar{\varepsilon}$ satisfies $\bar{\varepsilon} > 0$, even the embedding

$$S^{\bar{r}+\bar{\varepsilon}}_{p,q} A(\mathbb{T}^d) \hookrightarrow S^{\bar{r}}_{p,v} A(\mathbb{T}^d)$$

holds.

(iii) Let $0 < q \leq \infty$ and $0 < p < \infty$. The chain of embeddings

$$S^{\bar{r}}_{p,\min(p,q)} B(\mathbb{T}^d) \hookrightarrow S^{\bar{r}}_{p,q} F(\mathbb{T}^d) \hookrightarrow S^{\bar{r}}_{p,\max(p,q)} B(\mathbb{T}^d) \tag{1.28}$$

holds true.

Proof In [32, 2.2.3] the nonperiodic case $d = 2$ is treated in detail. In the periodic case all arguments carry over. Let us also refer to Temlyakov [45, pp. 20/21] . □

Let $S^{\bar{r}}_{p,q} A(\mathbb{T}^d)$ be either $S^{\bar{r}}_{p,q} B(\mathbb{T}^d)$ or $S^{\bar{r}}_{p,q} F(\mathbb{T}^d)$ (with $p < \infty$ in the F-case).

Lemma 1.7 *Let $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $0 < p, q \le \infty$. In this case the embedding*

$$D(\mathbb{T}^d) \hookrightarrow S^r_{p,q}A(\mathbb{T}^d) \hookrightarrow D'(\mathbb{T}^d)$$

is valid.

Proof *Step 1.* Because of Lemma 1.6 it suffices to show

$$D(\mathbb{T}^d) \hookrightarrow S^{\bar{r}}_{\infty,\infty}B(\mathbb{T}^d) \tag{1.29}$$

for every $\bar{r} \in \mathbb{R}^d$.
It holds

$$\|f|S^{\bar{r}}_{\infty,\infty}B(\mathbb{T}^d)\| \le C(\bar{\alpha}, \bar{r}) \sum_{\bar{\gamma} \le \bar{\alpha}} \|D^{\bar{\gamma}} f|L_\infty(\mathbb{T}^d)\| \tag{1.30}$$

for $f \in D(\mathbb{T}^d)$ and $\bar{r} \le \bar{\alpha} \in \mathbb{N}_0^d$, which implies immediately (1.29). See Paragraph 1.2.3. Let us now prove (1.30). We make use of the partition $\mathbb{N}_0^d = \bigcup_{\bar{\beta} \in \{0,1\}^d} I_{\bar{\beta}}$, see Paragraph 1.6.1 below. Recall the definition $\bar{\alpha} * \bar{\beta}$ in Paragraph 1.2.1. Assume that $\varphi \in \Phi(\mathbb{R})$. Then we obtain

$$\begin{aligned} \|f|S^{\bar{r}}_{\infty,\infty}B(\mathbb{T}^d)\| &= \sup_{\bar{\ell} \in \mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}} \Big\| \sum_{k \in \mathbb{Z}^d} \varphi_{\bar{\ell}}(k) c_k(f) e^{ik\cdot x} \Big| L_\infty(\mathbb{T}^d) \Big\| \\ &\le \sum_{\bar{\beta} \in \{0,1\}^d} \sup_{\bar{\ell} \in I_{\bar{\beta}}} 2^{\bar{r}\bar{\ell}} \Big\| \sum_{k \in \mathbb{Z}^d} \varphi_{\bar{\ell}}(k) M_{\bar{\alpha}*\bar{\beta}}(k) \psi_{\bar{\ell}}(k) (ik)^{\bar{\alpha}*\bar{\beta}} c_k(f) e^{ik\cdot x} \Big| L_\infty(\mathbb{T}^d) \Big\|, \end{aligned}$$

with $\bar{\alpha} \ge \bar{r}$. Furthermore, the function $M_{\bar{\alpha}}$ is given by

$$M_{\bar{\alpha}}(x_1, ..., x_d) = M_{\alpha_1}(x_1) \cdot ... \cdot M_{\alpha_d}(x_d)\,.$$

Additionally we assume for $\alpha \in \mathbb{N}_0$

$$|M_\alpha(x)| \le c_\alpha (1 + |x|^2)^{-\alpha/2} \quad , \quad x \in \mathbb{R} \quad ,$$

and $M_\alpha(k) = (ik)^{-\alpha}$ for $k \in \mathbb{Z} \setminus \{0\}$. Finally, $\psi \in S(\mathbb{R}^d)$ is supposed to satisfy

$$\psi(x) = \begin{cases} 1 & : \ |x| \le 2 \\ 0 & : \ |x| > 3 \end{cases}$$

and $\psi_{\bar{\ell}}(x) = \psi(2^{-\ell_1}x_1) \cdot ... \cdot \psi(2^{-\ell_d}x_d)$ for $x \in \mathbb{R}^d$ and $\bar{\ell} \in \mathbb{N}_0^d$. Now we obtain by Proposition 1.6

$$\begin{aligned} &\sup_{\bar{\ell} \in I_{\bar{\beta}}} 2^{\bar{r}\bar{\ell}} \Big\| \sum_{k \in \mathbb{Z}^d} \varphi_{\bar{\ell}}(k) M_{\bar{\alpha}*\bar{\beta}}(k) \psi_{\bar{\ell}}(k) (ik)^{\bar{\alpha}*\bar{\beta}} c_k(f) e^{ik\cdot x} \Big| L_\infty(\mathbb{T}^d) \Big\| \\ &\le c_1 \sup_{\bar{\ell} \in I_{\bar{\beta}}} 2^{\bar{r}\bar{\ell}} \|\varphi_{\bar{\ell}}(2^{\ell_1}\cdot, ..., 2^{\ell_d}\cdot) M_{\bar{\alpha}*\bar{\beta}}(2^{\ell_1}\cdot, ..., 2^{\ell_d}\cdot)|S^{\bar{1}}_2 H(\mathbb{R}^d)\| \\ &\quad \times \Big\| \sum_{k \in \mathbb{Z}^d} \psi_{\bar{\ell}}(k) (ik)^{\bar{\alpha}*\bar{\beta}} c_k(f) e^{ik\cdot x} \Big| L_\infty(\mathbb{T}^d) \Big\|. \end{aligned} \tag{1.31}$$

Of course, it also holds

$$\begin{aligned}
&\sup_{\bar{\ell}\in I_{\bar{\beta}}} 2^{\bar{r}\bar{\ell}} \|\varphi_{\bar{\ell}}(2^{\ell_1}\cdot,...,2^{\ell_d}\cdot)M_{\bar{\alpha}}(2^{\ell_1}\cdot,...,2^{\ell_d}\cdot)|S^{\bar{1}}_2H(\mathbb{R}^d)\| \\
&\leq \sup_{\bar{\ell}\in I_{\bar{\beta}}} \prod_{i=1}^d 2^{r_i\ell_i\beta_i}\|\varphi_{\ell_i}(2^{\ell_i}\cdot)M_{\alpha_i\beta_i}(2^{\ell_i}\cdot)|H^1_2(\mathbb{R})\|\,.
\end{aligned} \tag{1.32}$$

Now, for $\ell > 0$ we have

$$2^{\ell r_i}\|\varphi_\ell(2^\ell\cdot)M_{\alpha_i}(2^\ell\cdot)|H^1_2(\mathbb{R})\| \leq 2^{\ell(r_i-\alpha_i)}\Big\|\frac{\varphi_\ell(2^\ell x_i)}{|x_i|^{\alpha_i}}\Big|H^1_2(\mathbb{R})\Big\|. \tag{1.33}$$

The uniform boundedness of

$$\Big\|\frac{\varphi_\ell(2^\ell x_i)}{|x_i|^{\alpha_i}}\Big|H^1_2(\mathbb{R})\Big\|$$

(with respect to ℓ) directly follows from Definition 1.1 . Consequently, (1.31), (1.32), (1.33) and $\bar{\alpha} \geq \bar{r}$ imply

$$\begin{aligned}
\|f|S^{\bar{r}}_{\infty,\infty}B(\mathbb{T}^d)\| &\leq c_2 \sum_{\bar{\beta}\in\{0,1\}^d} \sup_{\bar{\ell}\in I_{\bar{\beta}}} \Big\|\sum_{k\in\mathbb{Z}^d} \psi_{\bar{\ell}}(k)(ik)^{\bar{\alpha}*\bar{\beta}}c_k(f)e^{ik\cdot x}\Big|L_\infty(\mathbb{T}^d)\Big\| \\
&= c_2 \sum_{\bar{\beta}\in\{0,1\}^d} \sup_{\bar{\ell}\in I_{\bar{\beta}}} \Big\|\sum_{k\in\mathbb{Z}^d} \psi_{\bar{\ell}}(k)c_k(D^{\bar{\alpha}*\bar{\beta}}f)e^{ik\cdot x}\Big|L_\infty(\mathbb{T}^d)\Big\| \\
&\leq c_3 \sum_{\bar{\beta}\in\{0,1\}^d} \|D^{\bar{\alpha}*\bar{\beta}}f|L_\infty(\mathbb{T}^d)\| \\
&\leq c_3 \sum_{\bar{\gamma}\leq\bar{\alpha}} \|D^{\bar{\gamma}}f|L_\infty(\mathbb{T}^d)\|.
\end{aligned}$$

The last step is a consequence of Lemma 1.1, where we used the uniform boundedness of $\|\mathcal{F}^{-1}\psi_{\bar{\ell}}|L_1(\mathbb{R}^d)\|$.

Step 2. We follow [32, 2.2.4]. It suffices to prove

$$S^{\bar{r}}_{p,\infty}B(\mathbb{T}^d) \hookrightarrow D'(\mathbb{T}^d)\,.$$

Let $f \in S^{\bar{r}}_{p,\infty}B(\mathbb{T}^d)$, $\varphi = \{\varphi_j\}_j \in \Phi(\mathbb{R})$ and $\psi \in D(\mathbb{T}^d)$. We put $\bar{\varphi} = (\varphi,...,\varphi)$. Let us further define

$$\chi_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}\,,$$

where again $\varphi_{-1} \equiv 0$. Therefore, we can estimate.

$$\begin{aligned}
|f(\psi)| &= \Big|\sum_{\bar{\ell}\in\mathbb{N}_0^d}\sum_{k\in\mathbb{Z}^d} \varphi_{\bar{\ell}}(k)\cdot\chi_{\bar{\ell}}(k)c_k(f)\int_{\mathbb{T}^d} e^{ikx}\psi(x)\,dx\Big| \\
&= \Big|\sum_{\bar{\ell}\in\mathbb{N}_0^d}\int_{\mathbb{T}^d}\Big(\sum_{k\in\mathbb{Z}^d}\varphi_{\bar{\ell}}(k)c_k(f)e^{ikx}\Big)\cdot\Big(\sum_{m\in\mathbb{Z}^d}\chi_{\bar{\ell}}(-m)c_m(\psi)e^{imx}\Big)\,dx\Big| \\
&\leq c\sum_{\ell\in\mathbb{N}_0^d}\Big\|\sum_{k\in\mathbb{Z}^d}\varphi_{\bar{\ell}}(k)c_k(f)e^{ikx}\Big|L_\infty(\mathbb{T}^d)\Big\| \times \Big\|\sum_{m\in\mathbb{Z}^d}\chi_{\bar{\ell}}(-m)c_m(\psi)e^{imx}\Big|L_1(\mathbb{T}^d)\Big\|\,.
\end{aligned}$$

Nikol'skij's inequality, see Theorem 1.1, gives

$$|f(\psi)| \le c_1 \sum_{\ell\in\mathbb{N}_0^d} 2^{\ell_1/p+\ldots+\ell_d/p} \Big\| \sum_{k\in\mathbb{Z}^d} \varphi_{\bar{\ell}}(k) c_k(f) e^{ikx} \Big| L_p(\mathbb{T}^d) \Big\| \times$$
$$\times \Big\| \sum_{m\in\mathbb{Z}^d} \chi_{\bar{\ell}}(-m) c_m(\psi) e^{imx} \Big| L_1(\mathbb{T}^d) \Big\| ,$$

which implies immediately

$$|f(\psi)| \le c_2 \|f|S^{\bar{r}}_{p,\infty}B(\mathbb{T}^d)\|^{\bar{\varphi}} \cdot \|\psi|S^{\bar{\kappa}}_{1,1}B(\mathbb{T}^d)\|^{\bar{\varphi}} ,$$

where $\bar{\kappa} = (1/p - r_1, ..., 1/p - r_d)$. With Step 1 we obtain for $N \in \mathbb{N}$ large enough

$$|f(\psi)| \le c_3 \|f|S^{\bar{r}}_{p,\infty}B(\mathbb{T}^d)\| \cdot \|\psi\|_N . \tag{1.34}$$

This proves that $S^{\bar{r}}_{p,\infty}B(\mathbb{T}^d)$ is continuously embedded in $D'(\mathbb{T}^d)$.

□

Theorem 1.3 *Let $0 < p, q \le \infty$ and $\bar{r} \in \mathbb{R}^d$. Then the spaces $S^{\bar{r}}_{p,q}A(\mathbb{T}^d)$ are (quasi-)Banach spaces .*

Proof This is a consequence of Remark 1.7, Lemma 1.7 and the completeness of $D'(\mathbb{T}^d)$, cf. Paragraph 1.2.3. □

We proceed with the following important embedding.

Proposition 1.9 *Let $0 < p, q \le \infty$. The continuous embedding*

$$S^{\bar{r}}_{p,q}A(\mathbb{T}^d) \hookrightarrow L_p(\mathbb{T}^d) \cap L_{\max(p,1)}(\mathbb{T}^d)$$

holds for $\bar{r} > \sigma_p$.

Proof *Step 1.* Following [32, Prop. 2.2.3/3] we show the embedding $S^{r}_{p,\infty}B(\mathbb{T}^d) \hookrightarrow L_{\max(p,1)}(\mathbb{T}^d)$. The right-hand side space is a Banach space. Thus, we have the usual triangle inequality available. Recall the decomposition (1.20) of $f \in S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$. Although it is not yet clear whether f belongs to $L_{\max(p,1)}(\mathbb{T}^d)$ we estimate using Nikol'skij's inequality (Theorem 1.1)

$$\begin{aligned} \|f|L_{\max(p,1)}(\mathbb{T}^d)\| &\le \sum_{\ell\in\mathbb{N}_0^d} \|f_\ell|L_{\max(1,p)}(\mathbb{T}^d)\| \\ &\le c_1 \sum_{\ell\in\mathbb{N}_0^d} 2^{\ell_1(1/\min(1,p)-1)+\ldots+\ell_d(1/\min(1,p)-1)} \|f_\ell|L_p(\mathbb{T}^d)\| \\ &\le c_1 \sum_{\ell\in\mathbb{N}_0^d} 2^{\ell_1(1/\min(1,p)-1-r_1)+\ldots+\ell_d(1/\min(1,p)-1-r_d)} 2^{\bar{r}\ell} \|f_\ell|L_p(\mathbb{T}^d)\| \\ &\le c_2 \|f|S^{\bar{r}}_{p,\infty}B(\mathbb{T}^d)\| . \end{aligned} \tag{1.35}$$

Because of its absolute convergence the sum of course exists in $L_{\max(p,1)}(\mathbb{T}^d)$. As we have $f = \sum_{\ell\in\mathbb{N}_0^d} f_\ell$ in $D'(\mathbb{T}^d)$ the limit element must be f. The embedding $L_1(\mathbb{T}^d) \hookrightarrow L_p(\mathbb{T}^d)$ gives us also $f \in L_p(\mathbb{T}^d)$ if $p < 1$. Here the situation is much simpler than in the non-periodic case, compare with [32, 2.2.3] . Having this, we conclude

$$\begin{aligned}\|f|L_p(\mathbb{T}^d)\| &= \Big\|\sum_{\ell\in\mathbb{N}_0^d} f_\ell\Big|L_p(\mathbb{T}^d)\Big\| \le \Big\|\sum_{\ell\in\mathbb{N}_0^d} |f_\ell|\Big|L_p(\mathbb{T}^d)\Big\| \\ &= \|f|S^{\bar{0}}_{p,1}F(\mathbb{T}^d)\| \le \|f|S^{\bar{r}}_{p,\infty}B(\mathbb{T}^d)\|\,.\end{aligned}$$

□

Remark 1.11 *The previous embedding is sharp in the following way. If $\bar{r} = \sigma_p$ then the corresponding space $S^{\sigma_p}_{p,q}A(\mathbb{T}^d)$ contains distributions, which are not regular (see Paragraph 1.2.3). Recall the δ-distribution from Paragraph 1.2.3. We show that δ belongs to $S^{1/p-1}_{p,\infty}B(\mathbb{T}^d)$ if $p \le 1$. Let us choose a system $\varphi \in \Phi(\mathbb{R})$. With view on (1.4) we have to estimate the quantities*

$$\Big\|\sum_{k\in\mathbb{Z}^d} \varphi_{\bar{\ell}}(k)e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| \quad , \quad \bar{\ell} \in \mathbb{N}_0^d\,.$$

By using Lemma 1.2 we derive for fixed $\bar{\ell} \in \mathbb{N}_0$ the following. The first estimate will be a consequence of the inequality $(\sum |a_k|)^p \le \sum |a_k|^p$ for $p \le 1$. We obtain

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}^d} \varphi_{\bar{\ell}}(k)e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| &= c_1\Big(\int_{\mathbb{T}^d} \Big|\sum_{\bar{\ell}\in\mathbb{Z}^d} \mathcal{F}^{-1}\varphi_{\bar{\ell}}\,(x+2\pi\bar{\ell})\Big|^p\,dx\Big)^{1/p} \\ &\le c_2\Big(\int_{\mathbb{T}^d} \sum_{\bar{\ell}\in\mathbb{Z}^d} |\mathcal{F}^{-1}\varphi_{\bar{\ell}}\,(x+2\pi\bar{\ell})|^p\,dx\Big)^{1/p} \\ &= c_2\Big(\sum_{\bar{\ell}\in\mathbb{Z}^d} \int_{\mathbb{T}^d} |\mathcal{F}^{-1}\varphi_{\bar{\ell}}\,(x+2\pi\bar{\ell})|^p\,dx\Big)^{1/p} \\ &= c_2\Big(\int_{\mathbb{R}^d} |\mathcal{F}^{-1}\varphi_{\bar{\ell}}\,(y)|^p\,dy\Big)^{1/p}\,.\end{aligned}$$

Now well-known homogeneity properties of the Fourier transform yield finally

$$\Big\|\sum_{k\in\mathbb{Z}^d} \varphi_{\bar{\ell}}(k)e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| \le c_3 2^{|\bar{\ell}|_1(1-1/p)}\,. \tag{1.36}$$

We finish with

$$\begin{aligned}\|f|S^{1/p-1}_{p,\infty}B(\mathbb{T}^d)\| &= \sup_{\bar{\ell}\in\mathbb{N}_0^d} 2^{|\bar{\ell}|_1(1/p-1)}\Big\|\sum_{k\in\mathbb{Z}^d} \varphi_{\bar{\ell}}(k)e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| \\ &\le c_3 2^{|\bar{\ell}|_1(1/p-1)} 2^{|\bar{\ell}|_1(1-1/p)} < \infty\,.\end{aligned} \tag{1.37}$$

It is important for us to know which conditions force the considered function spaces to be continuously embedded into the space $C(\mathbb{T}^d)$. This space has to be considered as the closed subspace of $L_\infty(\mathbb{T}^d)$ containing only continuous functions. We restrict to the case where we have the same smoothness in each direction. See Remark 1.6 .

Theorem 1.4 *Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $r \in \mathbb{R}$. Then the following assertions are equivalent.*
(i) $S^r_{p,q}B(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$,
(ii) $S^r_{p,q}B(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d)$ *and*
(iii) $r > 1/p$ *or* $r = 1/p$ *and* $q \leq 1$.

Proof *Step 1.* Let us prove the implications $(iii) \to (i)$ and $(iii) \to (ii)$ simultaneously. We start with an $f \in S^r_{p,q}B(\mathbb{T}^d)$. Let us estimate $\|f|L_\infty(\mathbb{T}^d)\|$ with the help of Theorem 1.1

$$\begin{aligned}\|f|L_\infty(\mathbb{T}^d)\| &\leq \sum_{\bar{\ell}\in\mathbb{N}_0^d} \|f_{\bar{\ell}}|L_\infty(\mathbb{T}^d)\| \\ &\leq c_1 \sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{\ell_1/p+\ldots+\ell_d/p}\|f_{\bar{\ell}}|L_p(\mathbb{T}^d)\| \\ &\leq c_2\|f|S^{1/p}_{p,1}B(\mathbb{T}^d)\| \,.\end{aligned}$$

Precisely, we argue as follows. Reading the estimates backwards we observe that $\sum_{\bar{\ell}} f_{\bar{\ell}}$, where $f_{\bar{\ell}}$ is continuous, converges absolutely in $L_\infty(\mathbb{T}^d)$. This implies the convergence itself and because of (1.20) the limit must be f.
Step 2. We prove $\neg(iii) \to \neg(i)$ and $\neg(iii) \to \neg(ii)$ simultaneously. We assume $r < 1/p$ or ($r = 1/p$ and $q > 1$). We construct a family $\{\tilde{f}_n\}_n \subset C(\mathbb{T}^d)$ of functions satisfying

$$\sup_n \|\tilde{f}_n|L_\infty(\mathbb{T}^d)\| = \infty \quad \text{and} \quad \sup_n \|\tilde{f}_n|S^r_{p,q}B(\mathbb{T}^d)\| < \infty \,. \tag{1.38}$$

Let us define the building blocks $g_\ell \in C(\mathbb{T})$, $\ell \in \mathbb{N}$, by

$$g_\ell(x) = \frac{1}{\ell 2^\ell}\sum_{k\in\mathbb{Z}} \psi_\ell(k)e^{ikx} \,,$$

where $\psi_\ell = \psi_0(2^{-\ell}x)$, $\psi_0(x) = \psi(10/9x) - \psi(4/3x)$ and $0 \leq \psi \leq 1$ satisfies

$$\psi(x) = \begin{cases} 1 & : \quad |x| \leq 1 \\ 0 & : \quad |x| > 10/9 \,. \end{cases}$$

Of course, $\operatorname{supp}\psi_0 \subset [3/4, 1]$ and $\psi_0(x) = 1$ for all $x \in [5/6, 9/10]$. The functions f_n are defined by

$$f_n = \sum_{\ell=1}^n g_\ell \quad , \quad n \in \mathbb{N} \,.$$

Further, we need a special element $\varphi \in \Phi(\mathbb{R})$. We refer to Remark 1.4 and make use of almost the same function $\varphi_0(x)$ we employed in this context but with the modification $\varphi(x) = 0$ for $|x| > 3/2$. Obviously $\varphi_\ell(x) = 1$ if $x \in \operatorname{supp} \psi_\ell$. Then for fixed $n \in \mathbb{N}$ we have the equation

$$\Big\| \sum_{k\in\mathbb{Z}} \varphi_\ell(k) c_k(f_n) e^{ikx} \Big| L_p(\mathbb{T}) \Big\| = \frac{1}{\ell 2^\ell} \Big\| \sum_{k\in\mathbb{Z}} \psi_\ell(k) e^{ikx} \Big| L_p(\mathbb{T}) \Big\| \quad , \quad \ell \in \mathbb{N}\,.$$

Let us distinguish the cases $p \le 1$ and $p > 1$. The first case can be treated as in Remark 1.11. Using Lemma 1.2 we obtain in the case $p \le 1$

$$\Big\| \sum_{k\in\mathbb{Z}} \varphi_\ell(k) c_k(f_n) e^{ikx} \Big| L_p(\mathbb{T}) \Big\| \le \frac{c}{\ell 2^\ell} 2^{\ell(1-1/p)} = \frac{c}{\ell 2^{\ell/p}} \quad , \quad \ell \in \mathbb{N}\,.$$

The same holds in the case $p > 1$. We employ Lemma 1.1, Minkowski's inequality and use assertions concerning the $L_p(\mathbb{T})$-norm of the Dirichlet kernel. See for instance [45, Sect. I.1]. Recall that $r < 1/p$ or $r = 1/p$ and $q > 1$. Hence, for $n \in \mathbb{N}$

$$\begin{aligned} \|f_n|B^r_{p,q}(\mathbb{T})\|^\varphi &= \Big(\sum_{\ell\in\mathbb{N}_0} 2^{r\ell q} \Big\| \sum_{k\in\mathbb{Z}} \varphi_\ell(k) c_k(f_n) e^{ikx} \Big| L_p(\mathbb{T}) \Big\|^q \Big)^{1/q} \\ &\le c \Big(\sum_{\ell\in\mathbb{N}} \frac{2^{\ell(r-1/p)q}}{\ell^q} \Big)^{1/q} \le C(r,p,q) < \infty \end{aligned}$$

holds (modification in case $q = \infty$). So we have uniform boundedness of $\|f_n|B^r_{p,q}(\mathbb{T})\|$ which implies immediately the same for $\|\tilde{f}_n|S^r_{p,q}B(\mathbb{T}^d)$, where

$$\tilde{f}_n(x_1, ..., x_d) = f_n(x_1) \quad , \quad n \in \mathbb{N}\,.$$

It remains to prove the first equation in (1.38). Obviously, we have

$$\begin{aligned} \|\tilde{f}_n|L_\infty(\mathbb{T}^d)\| = \|f_n|L_\infty(\mathbb{T})\| \ge |f_n(0)| &= \sum_{\ell=1}^n g_\ell(0) \\ &= \sum_{\ell=1}^n \frac{1}{\ell 2^\ell} \sum_{k\in\mathbb{Z}} \psi_\ell(k) \ge c_1 \sum_{\ell=1}^n \frac{1}{\ell} \xrightarrow[n\to\infty]{} \infty \end{aligned}$$

with some positive constant $c_1 > 0$. □

For the F-scale we have a similar result.

Theorem 1.5 *Let $0 < p < \infty$, $0 < q \le \infty$ and $r \in \mathbb{R}$. Then the following assertions are equivalent.*
(i) $S^r_{p,q}F(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$,
(ii) $S^r_{p,q}F(\mathbb{T}^d) \hookrightarrow L_\infty(\mathbb{T}^d)$ *and*
(iii) $r > 1/p$ *or* $r = 1/p$ *and* $p \le 1$.

Proof We apply the strategy used in [20, Lem. 4.7] (real interpolation of vector-valued L_p-spaces) to prove the embedding

$$S^{1/p}_{p,\infty}F(\mathbb{T}^d) \hookrightarrow S^0_{\infty,p}B(\mathbb{T}^d)\,.$$

All arguments used there carry over to the periodic d-dimensional case. Theorem 1.4 then implies

$$S^{1/p}_{p,\infty}F(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$$

if $p \leq 1$. What remains follows as a consequence of a corresponding result for isotropic F-spaces (see for instance [38]) and Remark 1.8 (cross-norm). □

Corollary 1.1 *Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $\bar{r} \in \mathbb{R}^d$ satisfying $\bar{r} > 1/p$. Then we have both*

$$S^{\bar{r}}_{p,q}B(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d) \quad \text{and} \quad S^{\bar{r}}_{p,q}F(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)\,,$$

with $p < \infty$ in the F-case.

1.4.4 Littlewood-Paley Theory

We also state a theorem of Littlewood-Paley type for spaces with dominating mixed smoothness. Let us first define Sobolev spaces of this type. This is the direct generalization of the spaces Nikol'skij considered (recall the introduction).

If $\bar{r} \in \mathbb{N}_0^d$ and $1 \leq p \leq \infty$ the Sobolev space $S^{\bar{r}}_pW(\mathbb{T}^d)$ of dominating mixed smoothness of order $\bar{r}$ is defined as the collection of all $f \in L_p(\mathbb{T}^d)$ such that

$$D^\alpha f \in L_p(\mathbb{T}^d)\,, \qquad \alpha = (\alpha_1, \ldots, \alpha_d)\,, \quad 0 \leq \alpha \leq \bar{r}.$$

Derivatives have to be understood in the weak sense. We endow these classes with the norm

$$\|\, f\,|S^{\bar{r}}_pW(\mathbb{T}^d)\| := \sum_{\alpha \leq \bar{r}} \|\, D^\alpha f\,|L_p(\mathbb{T}^d)\|\,. \tag{1.39}$$

For general $\bar{r} \geq 0$ and $1 < p < \infty$ one may use

$$\sum_{k \in \mathbb{Z}^d} c_k(f)(1+|k_1|^2)^{r_1/2} \ldots (1+|k_d|^2)^{r_d/2}\, e^{ikx} \in L_p(\mathbb{T}^d)$$

as well as

$$\|\, f\,|S^{\bar{r}}_pW(\mathbb{T}^d)\| := \Big\| \sum_{k \in \mathbb{Z}^d} c_k(f)(1+|k_1|^2)^{r_1/2} \ldots (1+|k_d|^2)^{r_d/2}\, e^{ikx} \Big| L_p(\mathbb{T}^d)\Big\|\,. \tag{1.40}$$

We will also write $S^r_p\,W(\mathbb{T}^d)$ instead of $S^{(r,\ldots,r)}_pW(\mathbb{T}^d)$. Clearly we have $S^{(0,\ldots,0)}_pW(\mathbb{T}^d) = L_p(\mathbb{T}^d)$. Also the following assertion of Littlewood-Paley type holds true.

Theorem 1.6 *Let $1 < p < \infty$ and $\bar{r} \geq 0$. Let $f_{\bar{\ell}}$ be defined as in (1.20). Then*

$$\| f \,| S^{\bar{r}}_p W(\mathbb{T}^d)\| \asymp \Big\| \Big(\sum_{\bar{\ell}\in\mathbb{N}_0^d} 2^{2\bar{r}\bar{\ell}} |f_{\bar{\ell}}(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^d) \Big\|$$

holds for all $f \in L_p(\mathbb{T}^d)$.

Proof For $\bar{r} = (0, ..., 0)$ this can be found in Nikol'skij [27, 1.5.2/(13)]. In the general case one has to use a lifting property. We refer to [32, 2.2.6] and [32, 2.3.1] for the nonperiodic counterpart. □

Now the following corollary is obvious.

Corollary 1.2 *Let $1 < p < \infty$ and $\bar{r} \geq 0$. Then the identity*

$$S^{\bar{r}}_p W(\mathbb{T}^d) = S^{\bar{r}}_{p,2} F(\mathbb{T}^d)$$

is valid in the sense of equivalent norms.

1.5 Complex Interpolation

We briefly describe the complex interpolation method following [47]. Let A_0, A_1 be Banach spaces. If a linear Hausdorff space $\mathcal{A}$ exists such that $A_0, A_1 \hookrightarrow \mathcal{A}$ then (A_0, A_1) is said to be an interpolation couple. For two interpolation couples (A_0, A_1) and (B_0, B_1) we denote by $\mathcal{L}((A_0, A_1), (B_0, B_1))$ the collection of all linear operators $T : A_0 + A_1 \to B_0 + B_1$ such that the restrictions $T|_{A_i} : A_i \to B_i$, $i = 0, 1$ are continuous.
The class $\mathcal{F} = \mathcal{F}(A_0, A_1)$ is the collection of all vector valued continuous functions $f : \bar{S} \to A_0 + A_1$, which are additionally analytic on $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\} \subset \mathbb{C}$. This class equipped with the norm

$$\|f|\mathcal{F}\| = \max\Big\{ \sup_{t\in\mathbb{R}} \|f(it)|A_0\| \;,\; \sup_{t\in\mathbb{R}} \|f(1+it)|A_1\| \Big\}$$

is a Banach space. Finally, the interpolation space $[A_0, A_1]_\vartheta$ is defined via

$$[A_0, A_1]_\vartheta := \{a \in A_0 + A_1 : \text{ it exists } f \in \mathcal{F} \text{ such that } a = f(\vartheta)\}$$

and equipped with the norm

$$\|a|[A_0, A_1]_\vartheta\| := \inf_{f\in\mathcal{F}\,,\, f(\vartheta)=a} \|f|\mathcal{F}\| \,. \tag{1.41}$$

Let T belong to $\mathcal{L}((A_0, A_1), (B_0, B_1))$ then it turns out that
(i)

$$T|_{[A_0,A_1]_\vartheta} : [A_0, A_1]_\vartheta \to [B_0, B_1]_\vartheta$$

is continuous and moreover
(ii)

$$\|T : [A_0, A_1]_\vartheta \to [B_0, B_1]_\vartheta\| \le \|T : A_0 \to B_0\|^{1-\vartheta} \|T : A_1 \to B_1\|^{\vartheta}. \tag{1.42}$$

In order to apply this method to function spaces, let us mention the retraction and coretraction concept. For two Banach spaces A and B we call a linear continuous mapping $R \in \mathcal{L}(A, B)$ retraction if an operator $S \in \mathcal{L}(B, A)$ exists such that $R \circ S = id_B$. We call S the corresponding coretraction. Let $R \in \mathcal{L}((A_0, A_1), (B_0, B_1))$ and $S \in \mathcal{L}((B_0, B_1), (A_0, A_1))$ be retraction and coreatraction in $\mathcal{L}(A_0, B_0)$ as well as in $\mathcal{L}(A_1, B_1)$. Then

$$\|f |\, [B_0, B_1]_\vartheta\| \asymp \|Sf |\, [A_0, A_1]_\vartheta\| \quad , \quad f \in [B_0, B_1]_\vartheta \,, \tag{1.43}$$

holds true.

1.5.1 Basic Tools

Consider the interpolation method applied to weighted sequence spaces of type $\ell_p^\sigma(A_j)$ (defined comparably with [47, 1.18.1/2]), where $A = \{A_j\}_j$ is a sequence of Banach spaces. In the case $1 \le q_0, q_1 < \infty$, $\sigma_0, \sigma_1 \in \mathbb{R}$ this method yields the formula

$$[\ell_{q_0}^{\sigma_0}(A_j), \ell_{q_1}^{\sigma_1}(B_j)]_\vartheta = \ell_q^\sigma([A_j, B_j]_\vartheta)\,, \tag{1.44}$$

where $0 < \vartheta < 1$ and also

$$(1/q, \sigma) = (1-\vartheta)(1/q_0, \sigma_0) + \vartheta(1/q_1, \sigma_1)\,.$$

Secondly, we need a result concerning $L_p(A)$-spaces of A-valued functions, where A is a Banach space (see [47, 1.18.4]) .

Lemma 1.8 *Let $1 \le p_0, p_1 < \infty$, $0 < \vartheta < 1$ and $\frac{1}{p} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}$. Then the interpolation formula*

$$[L_{p_0}(A), L_{p_1}(B)]_\vartheta = L_p([A, B]_\vartheta)$$

is valid.

Proof A proof can be found in [47, 1.18.4]. □

Comparable with [31] for the case $d = 2$ we introduce the sequence spaces

$$\ell_p^\sigma(\mathbb{N}_0^d) = \left\{ a = \{a_{\bar\ell}\}_{\bar\ell \in \mathbb{N}_0^d} \,\middle|\, a_{\bar\ell} \in \mathbb{C} \text{ and } \|a|\ell_p^\sigma\| = \Big(\sum_{\bar\ell \in \mathbb{N}_0^d} 2^{\sigma|\bar\ell|_1 p} |a_{\bar\ell}|^p \Big)^{1/p} < \infty \right\},$$

and obtain the following interpolation result.

Lemma 1.9 *Let $1 \le q_0, q_1 < \infty$, $\sigma, \nu \in \mathbb{R}$ and $0 < \vartheta < 1$ satisfy the equation*

$$(1/p, \mu) = (1-\vartheta)(1/p_0, \sigma) + \vartheta(1/p_1, \nu).$$

Then the formula

$$[\ell_{p_0}^\sigma(\mathbb{N}_0^d), \ell_{p_1}^\nu(\mathbb{N}_0^d)]_\vartheta = \ell_p^\mu(\mathbb{N}_0^d)$$

holds true.

Proof We prove this formula by iterating (1.44) .

1.5.2 Interpolation within the Scale $S^r_{p,q}F(\mathbb{T}^d)$

We construct proper retractions and coretractions (see (1.43)) to apply the results from the previous paragraph. Recall that

$$\|f|S^r_{p,q}F(\mathbb{T}^d)\| = \|f_{\bar{\ell}}|L_p(\mathbb{T}^d,\ell^r_q)\|\,,$$

where $f_{\bar{\ell}}$, $\bar{\ell} \in \mathbb{N}_0^d$, comes from (1.20). Let us fix a system $\varphi = \{\varphi_j(x)\}_{j=0}^\infty \in \Phi(\mathbb{R})$. According to this we define once more the system $\psi = \{\psi_j\}_j \subset S(\mathbb{R})$ by

$$\psi_j(x) = \varphi_{j-1}(x) + \varphi_j(x) + \varphi_{j+1}(x) \quad , \quad x \in \mathbb{R}\,.$$

Let us start by defining the mapping S_φ through

$$S_\varphi\, f(x) = \Big\{\sum_{k\in\mathbb{Z}^d} \varphi_{\bar{\ell}}(k)c_k(f)e^{ik\cdot x}\Big\}_{\bar{\ell}\in\mathbb{N}_0^d} \quad , \quad f \in D'(\mathbb{T}^d)\,.$$

Moreover, the mapping R_ψ is defined by

$$R_\psi\, g = R_\psi(g_{\bar{\ell}})_{\bar{\ell}\in\mathbb{N}_0^d} = \sum_{\bar{\ell}\in\mathbb{N}_0^d}\Big(\sum_{k\in\mathbb{Z}^d}\psi_{\bar{\ell}}(k)c_k(g_{\bar{\ell}})e^{ikx}\Big) \quad , \quad g = (g_{\bar{\ell}})_{\bar{\ell}} \subset D'(\mathbb{T}^d)\,,$$

providing that the right-hand side makes sense. A simple consequence is the following. Let f belong to $D'(\mathbb{T}^d)$. Then $R_\psi(S_\varphi\, f)$ is well-defined and we obtain

$$\begin{aligned} R_\psi(S_\varphi f) &= \sum_{\bar{\ell}\in\mathbb{N}_0^d}\psi_{\bar{\ell}}(k)\left(\sum_{k\in\mathbb{Z}^d}\varphi_{\bar{\ell}}(k)\hat{f}(k)e^{ik\cdot x}\right)\\ &= \sum_{\bar{\ell}\in\mathbb{N}_0^d}\left(\sum_{k\in\mathbb{Z}^d}\varphi_{\bar{\ell}}(k)\hat{f}(k)e^{ik\cdot x}\right)\\ &= f. \end{aligned}$$

Consequently, we can prove the following result.

Lemma 1.10 *If we assume $1 \le p < \infty$, $1 \le q \le \infty$ and $r > 0$ then both $S_\varphi : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d,\ell^r_q)$ and $R_\psi : L_p(\mathbb{T}^d,\ell^r_q) \to S^r_{p,q}F(\mathbb{T}^d)$ are bounded linear operators.*

Proof Obviously, S_φ belongs to $\mathcal{L}(S^r_{p,q}F(\mathbb{T}^d), L_p(\mathbb{T}^d,\ell^r_q))$ because of

$$\|S_\varphi f|L_p(\mathbb{T}^d,\ell^r_q)\| = \|f|S^r_{p,q}F(\mathbb{T}^d)\|\,.$$

Let $g = (g_{\bar\ell})_{\bar\ell}$ be an element of the space $L_p(\mathbb{T}^d, \ell_q^r)$. Of course, for fixed $\bar\ell \in \mathbb{N}_0^d$, $g_{\bar\ell}$ belongs to $L_p(\mathbb{T}^d)$. This leads to

$$\begin{aligned}
\|R_\psi g | S^r_{p,q}F(\mathbb{T}^d)\|^\varphi &= \|R_\psi \{g_{\bar\ell}\}_{\bar\ell} | S^r_{p,q}F(\mathbb{T}^d)\|^\varphi \\
&= \Big\| \sum_{\bar\ell \in \mathbb{N}_0^d} \Big(\sum_{k\in\mathbb{Z}^d} \varphi_{\bar u}(k)\psi_{\bar\ell}(k) c_k(g_{\bar\ell}) e^{ik\cdot x}\Big) \Big| L_p(\mathbb{T}^d, \ell_q^r)\Big\| \\
&= \Big\| \sum_{\substack{|u_j-\ell_j|\le 2\\ j=1,\ldots,d}} \Big(\sum_{k\in\mathbb{Z}^d} \varphi_{\bar u}(k)\psi_{\bar\ell}(k) c_k(g_{\bar\ell}) e^{ik\cdot x}\Big) \Big| L_p(\mathbb{T}^d, \ell_q^r)\Big\| \\
&\le \sum_{\bar\ell \in ([-2,2]\cap\mathbb{Z})^d} \Big\| \sum_{k\in\mathbb{Z}^d} \underbrace{(\varphi_{\bar u}\cdot \psi_{\bar u+\bar\ell})}_{=:\Phi_{\bar u,\bar\ell}}(k) c_k(g_{\bar u+\bar\ell}) e^{ikx} \Big| L_p(\mathbb{T}^d, \ell_q^r)\Big\| .
\end{aligned}$$

Lemma 1.1 gives

$$\sum_{k\in\mathbb{Z}^d} \Phi_{\bar u,\bar\ell}(k) c_k(g_{\bar u+\bar\ell}) e^{ikx} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathcal{F}^{-1}\Phi_{\bar u,\bar\ell}(y) g_{\bar u+\bar\ell}(x-y)\, dy \quad , \quad x \in \mathbb{T}^d .$$

Using Minkowski's inequality for the Banach space $L_p(\ell_q)$ and the uniform boundedness of

$$\int_{\mathbb{R}^d} |\mathcal{F}^{-1}\Phi_{\bar u,\bar\ell}(y)|\, dy$$

we obtain by a standard argumentation

$$\|R_\psi g | S^r_{p,q}F(\mathbb{T}^d)\| \le c_3 \|g_{\bar u} | L_p(\mathbb{T}^d), \ell_q^r\| .$$

□

The final result of this subsection reads as follows.

Theorem 1.7 *Let $1 \le p_0, p_1 < \infty$, $1 \le q_0, q_1 < \infty$ and $r_0, r_1 \in \mathbb{R}$. Let further $0 < \vartheta < 1$ and*

$$(1/p, 1/q, r) = (1-\vartheta)\cdot(1/p_0, 1/q_0, r_0) + \vartheta\cdot(1/p_1, 1/q_1, r_1).$$

Under these assumptions we get the complex interpolation formula

$$[S^{r_0}_{p_0,q_0}F(\mathbb{T}^d), S^{r_1}_{p_1,q_1}F(\mathbb{T}^d)]_\vartheta = S^r_{p,q}F(\mathbb{T}^d) .$$

Proof We make use of both Lemma 1.8 and the previously defined mappings S_φ and R_ψ together with (1.43). We consider the interpolation couple $(A_0, A_1) = \big(L_{p_0}(\mathbb{T}^d, \ell^{r_0}_{q_0}), L_{p_1}(\mathbb{T}^d, \ell^{r_1}_{q_1})\big)$ as well as $(B_0, B_1) = \big(S^{r_0}_{p_0,q_0}F(\mathbb{T}^d), S^{r_1}_{p_1,q_1}F(\mathbb{T}^d)\big)$. This leads to

$$\begin{aligned}
\|f \,|\, [S^{r_0}_{p_0,q_0}F(\mathbb{T}^d),\, S^{r_1}_{p_1,q_1}F(\mathbb{T}^d)]_\vartheta\,\| &\asymp \|S_\varphi f \,|\, [L_{p_0}(\mathbb{T}^d, \ell^{r_0}_{q_0}),\, L_{p_1}(\mathbb{T}^d, \ell^{r_1}_{q_1})]_\vartheta\,\| \\
&\asymp \|S_\varphi f \,|\, L_p(\mathbb{T}^d, [\ell^{r_0}_{q_0}, \ell^{r_1}_{q_1}]_\vartheta)\| \\
&\asymp \|S_\varphi f \,|\, L_p(\mathbb{T}^d, \ell^r_q)\| \\
&\asymp \|f | S^r_{p,q}F(\mathbb{T}^d)\|^\varphi .
\end{aligned}$$

□

1.6 Characterization by Differences

This section deals with the problem of characterizing the spaces, defined in Section 1.4, by quantities involving first (or even higher) order differences of type $\Delta_h f(x) = f(x+h)-f(x)$. This field has a long tradition, especially in the former Soviet Union (see [1], [45], [27] and others). Here we present results for the characterization of the spaces $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ and $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ by integral means of differences for arbitrary d, see Paragraph 1.6.4. We mainly aim to give a counterpart of what Triebel did in [48, 2.5.11] for the isotropic case, i.e. $F^s_{p,q}(\mathbb{R}^d)$. Based on this we additionally obtain further difference characterizations, which use for instance classical moduli of smoothness. See Paragraph 1.6.6, Section 1.6.7 and confer also [32, 2.3.3, 2.3.4] for the bivariate case. Finally, we concentrate on the question, whether one can replace $\int_0^\infty$ by $\int_0^1$ in our characterizations. This problem investigated in Paragraph 1.6.5 turned out to be rather difficult in the dominating mixed case. However, we are able to give a partial answer.

1.6.1 Notation

Let us introduce some further notation. To shorten some formulas (integrals) in the sequel we shall often use $d\bar{h} = (dh_1, ..., dh_n)$, $\frac{d\bar{t}}{\bar{t}} = \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}$ and $\frac{\lambda}{\bar{t}} = \left(\frac{\lambda}{t_1}, ..., \frac{\lambda}{t_n}\right)$, where $\lambda \in \mathbb{R}$. Also the following convention will be helpful. If we have a tuple $\bar{\beta} \in \{0,1\}^d$ such that $|\bar{\beta}|_1 = n \geq 1$, we assign a tuple $\bar{\delta} = (\delta_1, ..., \delta_n)$ to $\bar{\beta}$ with the property

$$1 \leq \delta_1 < \delta_2 < \cdots < \delta_n \leq d \quad \text{and} \quad \beta_{\delta_i} = 1 \quad , \quad i = 1, ..., n. \tag{1.45}$$

Let us here fix some often used sets and index-sets. For $k \in \mathbb{Z}$ we put

$$I_k = [-2^k, 2^k] \quad \text{and} \quad I_k^+ = [0, 2^k].$$

Additionally, we put for $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}$

$$I_j^\Delta := \begin{cases} I_j \setminus I_{j-1} & : \ j \geq 1 \\ I_0 & : \ j = 0 \end{cases} \quad \text{and} \quad I_k^{+\Delta} := I_k^+ \setminus I_{k-1}^+ \ .$$

Assuming $\bar{\beta} \in \{0,1\}^d$ with $|\bar{\beta}|_1 = n \geq 1$ and $\bar{\delta}$ according to $\bar{\beta}$ we define for $\bar{\mu} \in \mathbb{Z}^d$

$$\begin{aligned} Q_{\bar{\mu},\bar{\beta}} &:= I_{\mu_{\delta_1}} \times ... \times I_{\mu_{\delta_n}} \ , \\ Q^+_{\bar{\mu},\bar{\beta}} &:= I^+_{\mu_{\delta_1}} \times ... \times I^+_{\mu_{\delta_n}} \ , \\ Q^{+\Delta}_{\bar{\mu},\bar{\beta}} &:= I^{+\Delta}_{\mu_{\delta_1}} \times ... \times I^{+\Delta}_{\mu_{\delta_n}} \end{aligned} \tag{1.46}$$

and for $\bar{\mu} \in \mathbb{N}_0^d$

$$Q^\Delta_{\bar{\mu},\bar{\beta}} := I^\Delta_{\mu_{\delta_1}} \times ... \times I^\Delta_{\mu_{\delta_n}} \ . \tag{1.47}$$

If $\bar{\beta} = (1, ..., 1)$ we put $Q_{\bar{\mu}} := Q_{\bar{\mu},\bar{\beta}}$, $Q_{\bar{\mu}}^{+} := Q_{\bar{\mu},\bar{\beta}}^{+}$, $Q_{\bar{\mu}}^{+\,\Delta} := Q_{\bar{\mu},\bar{\beta}}^{+\,\Delta}$ and $Q_{\bar{\mu}}^{\Delta} := Q_{\bar{\mu},\bar{\beta}}^{\Delta}$.
The mentioned index-sets are given by

$$
\begin{aligned}
I_{\bar{\beta}} &= \{\bar{\ell} = (\ell_1, ..., \ell_d) \in \mathbb{N}_0^d : \ell_i = 0 \Longleftrightarrow \beta_i = 0\,,\ i = 1, ..., d\}\ , \\
Z_{\bar{\beta}} &= \left\{(k_1, ..., k_d) \in \mathbb{Z}^d : \beta_i = 0 \implies k_i = 0,\ i = 1, ..., d\right\}\ , \\
\bar{Z}_{\bar{\beta}} &= Z_{\bar{1}-\bar{\beta}}\ , \\
N_{\bar{\beta}} &= Z_{\bar{\beta}} \cap \mathbb{N}_0^d\ , \\
\bar{N}_{\bar{\beta}} &= N_{\bar{1}-\bar{\beta}}\ , \\
E_{\bar{\beta}} &= Z_{\bar{\beta}} \cap \{0,1\}^d\ , \\
\bar{E}_{\bar{\beta}} &= E_{\bar{1}-\bar{\beta}}\ .
\end{aligned}
\tag{1.48}
$$

And according to $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ we define the sets $M_1^i = \{0, ..., m_i\}$ and $M_0^i = \{0\}$, $i = 1, ..., d$. Finally, for $\bar{\beta} \in \{0,1\}^d$ we put

$$M_{\bar{\beta}} = M_{\beta_1}^1 \times ... \times M_{\beta_d}^d. \tag{1.49}$$

1.6.2 Preliminaries

For several reasons we will need a modification of the famous Paley-Wiener-Schwartz theorem.

Theorem 1.8 *Let $\bar{b} > 0$. The following assertions are equivalent.*

(i) f belongs to $S'(\mathbb{R}^d)$ and satisfies $\operatorname{supp} \mathcal{F}f \subset \{y : |y_i| \le b_i, i = 1, ..., d\}$.

(ii) f is a regular distribution which can be holomorphically extended to $\mathbb{C}^d$. Furthermore, we have the growth condition: For an appropriate real number $\lambda > 0$ and any $\varepsilon > 0$ there exists a constant c_ε such that

$$|f(z)| \le c_\varepsilon (1 + |x|)^\lambda e^{(b_1+\varepsilon)|y_1|} \cdot ... \cdot e^{(b_d+\varepsilon)|y_d|} \quad , \quad z = x + iy \quad , \quad x, y \in \mathbb{R}^d.$$

Proof *With some obvious modifications in the proof of the classical theorem, see for example [58, VI.4], one obtains the version above.* □

Remark 1.12 *For us an important consequence is the following. We consider $f \in S'(\mathbb{R}^d)$ with* $\operatorname{supp} \mathcal{F}f \subset \{y : |y_i| \le b_i,\ i = 1, ..., d\}$. *One direction of Theorem 1.8 tells us that f is representable as an entire analytic function $f(z)$ with some growth condition. If we fix one variable, say z_k, in f (to a real number) the outcome is also an entire analytic function on $\mathbb{C}^{d-1}$ with a corresponding growth condition, but without the term $e^{(b_k+\varepsilon)|y_k|}$. Applying the second direction of the equivalence one can interpret this trace function as a tempered distribution with support of the Fourier transform in $[-b_1, b_1] \times ... \times [-b_{k-1}, b_{k-1}] \times [-b_{k+1}, b_{k+1}] \times ... \times [-b_d, b_d]$.*

Recall Paragraph 1.3.2. It is necessary to introduce a modification of the Peetre-Feffermann-Stein maximal operator, defined in (1.12). Let $\bar{b}, \bar{s}$ and f be defined like in this context. Additionally, we need a parameter $\bar{\alpha} \in \{0,1\}^d \backslash \{(0,...,0)\}$. The modified maximal function $P_{\bar{b},\bar{s},\bar{\alpha}} f$ is given by

$$P_{\bar{b},\bar{s},\bar{\alpha}} f(x) := \sup_{z \in \mathbb{R}^d} \frac{|f(x - \bar{\alpha} * z)|}{(1 + |b_1 \alpha_1 z_1|^{s_1}) \cdot ... \cdot (1 + |b_d \alpha_d z_d|^{s_d})} \quad . \tag{1.50}$$

Obviously, the maximal operator defined in (1.12) is applied to the directions where the corresponding component of $\bar{\alpha}$ equals one. With the same arguments used for the maximal operator M and M_i, respectively, we obtain the following generalization of Proposition 1.5 .

Theorem 1.9 *Assume $p, q, \bar{b}^\ell, \bar{s}$ and $\Lambda = \{\Lambda_\ell\}_{\ell \in \mathcal{I}}$ as in Proposition 1.5. Let further be $\bar{\alpha} \in \{0,1\}^d$. Then a constant $c > 0$ exists (independent of f and Λ) such that*

$$\big\| P_{\bar{b}^\ell,\bar{s},\bar{\alpha}} f_\ell | L_p(\mathbb{T}^d, \ell_q) \big\| \leq c \big\| f_\ell | L_p(\mathbb{T}^d, \ell_q) \big\|$$

holds for all systems $f = \{f_\ell\}_{\ell \in \mathcal{I}} \subset L_p^\Lambda(\mathbb{T}^d, \ell_q)$.

Proof We argue analogously to Proposition 1.4. Doing so the assertion follows immediately from Proposition 1.5, Theorem 1.8 and Remark 1.12. □

1.6.3 Differences versus Maximal Functions

Definitions

We define differences of order M as well as corresponding mixed differences. Essentially the same notation will be used as in [32, 2.3.3]. Fix $h \in \mathbb{R}$. Under a first order difference with step-length h of a function $f : \mathbb{R} \to \mathbb{C}$ we want to understand the function $\triangle_h f$ which is defined by

$$\triangle_h f(x) = f(x+h) - f(x) \quad , \quad x \in \mathbb{R}.$$

Iteration leads to Mth order differences, given by

$$\triangle_h^M f(x) = \triangle_h(\triangle_h^{M-1} f)(x) \quad , \quad M \in \mathbb{N} \quad , \quad \triangle_h^0 = I. \tag{1.51}$$

Using mathematical induction one can show the explicit formula

$$\triangle_h^M f(x) = \sum_{j=0}^{M} (-1)^j \binom{M}{j} f(x + (M-j)h). \tag{1.52}$$

For our special purpose we need differences with respect to a certain component of f as well as mixed differences. Let us first define the operator $\triangle_{h,i}^m f$ applied to a function $f : \mathbb{R}^d \to \mathbb{C}$. Having (1.52) in mind we set

$$\triangle_{h,i}^m f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x_1, ..., x_i + (m-j)h, x_{i+1}, ..., x_d) \quad , \tag{1.53}$$

where $m \in \mathbb{N}_0,\ h \in \mathbb{R},\ i = 1, ..., d$ and $x = (x_1, ..., x_d) \in \mathbb{R}^d$. Hence, one obtains similar to (1.51) for $m \in \mathbb{N}$ the recursion formula

$$\triangle_{h,i}^m f(x) = \triangle_{h,i}^1(\triangle_{h,i}^{m-1} f)(x) \quad , \quad i = 1, ..., d.$$

The combination of this kind of operators (acting on different components) is called mixed difference. For later use we need a suitable abbreviating symbol. Let $\bar{h} \in \mathbb{R}^d$, $\bar{\alpha} \in \{0,1\}^d$ and $\bar{\delta}$ be assigned to $|\bar{\alpha}|_1 = n \geq 1$ in the sense of (1.45). Let us further define the operator

$$\triangle_{\bar{h},\bar{\alpha}}^{\bar{m}} := \Big(\prod_{i=1}^{n} \triangle_{h_{\delta_i},\delta_i}^{m_{\delta_i}}\Big) := \begin{cases} \triangle_{h_{\delta_1},\delta_1}^{m_{\delta_1}} \circ \cdots \circ \triangle_{h_{\delta_n},\delta_n}^{m_{\delta_n}} & : \ |\bar{\alpha}|_1 = n \\ I & : \ |\bar{\alpha}|_1 = 0 \end{cases} . \tag{1.54}$$

We try to avoid the product-symbol. Nevertheless, in few cases it is useful and therefore we will refer to (1.54) to recall the exact definition.

Differences and Maximal Functions

We want to develop some tools to estimate differences by maximal functions.
The first inequality is obvious but nevertheless essential. Let $m \in \mathbb{N}_0$ and f be a locally integrable function. Then

$$\int_{-1}^{1} |f(x + mh)|\, dh \leq 2 \cdot Mf(x) \tag{1.55}$$

holds for almost all $x \in \mathbb{R}$. In the following we concentrate on estimating differences by maximal functions, defined in (1.12) and (1.50). See also [32, Lem. 2.3.3].

Lemma 1.11 *Let $a, b > 0$, $m \in \mathbb{N}$, $h \in \mathbb{R}$ and $f \in S'(\mathbb{R})$ with $\operatorname{supp} \mathcal{F}f \subset [-b, b]$. Then there exists a constant $c > 0$ independent of f, b and h such that*

$$|\triangle_h^m f(x)| \leq c \max(1, |bh|^a) \min(1, |bh|^m) P_{b,a} f(x)$$

holds for all $x \in \mathbb{R}$.

Proof The main instruments of the proof are the left-hand inequality in Lemma 1.3 and the mean-value theorem of calculus. Because of Theorem 1.8 the distribution f is an entire analytic function and hence the restriction to $\mathbb{R}$ is C^∞. So the mean-value theorem provides us ξ with $|\xi - x| \leq h$ such that

$$|\triangle_h^1 f(x)| = f'(\xi) \cdot h.$$

Iteration of this argument leads to

$$\begin{aligned} |\triangle_h^m f(x)| &= |\triangle_h^1 (\triangle_h^{m-1} f)(x)| \\ &\leq \sup_{|y| \leq h} |\triangle_h^{m-1} f'(x - y)| \cdot |h| \\ &\ \vdots \\ &\leq |h|^m \sup_{|y| \leq mh} \frac{|f^{(m)}(x - y)|}{1 + |by|^a}(1 + |by|^a). \end{aligned}$$

Hence, we obtain

$$\begin{aligned}|\triangle_h^m f(x)| &\le |h|^m \sup_{y\in\mathbb{R}} \frac{|f^{(m)}(x-y)|}{1+|by|^a} \cdot \sup_{|y|\le mh}(1+|by|^a)\\ &= |h|^m \sup_{y\in\mathbb{R}} \frac{|f^{(m)}(x-y)|}{1+|by|^a} \cdot (1+m^a|bh|^a) \\ &\le \underbrace{2m^a}_{=:c_{m,a}} \cdot |h|^m \max(1,|bh|^a) \sup_{y\in\mathbb{R}} \frac{|f^{(m)}(x-y)|}{1+|by|^a}.\end{aligned} \tag{1.56}$$

What remains is a consequence of a homogeneity argument. To understand this let $g \in S'(\mathbb{R})$ such that $\operatorname{supp} \mathcal{F}g \subset [-1,1]$. Then

$$|\triangle_h^m g(x)| \le c_{m,a}|h|^m \max(1,|h|^a) \cdot \sup_{y\in\mathbb{R}} \frac{|g^{(m)}(x-y)|}{1+|y|^a} \tag{1.57}$$

follows by (1.56). We apply (1.57) to the function $g(x) = f(x/b)$. It is easy to see, that $g \in S'(\mathbb{R})$ and $\operatorname{supp} \mathcal{F}g \subset [-1,1]$. Furthermore, we have

$$\triangle_h^m f(x) = \triangle_{bh}^m g(bx).$$

Hence, (1.57) gives us

$$\begin{aligned}|\triangle_h^m f(x)| &= |\triangle_{bh}^m g(bx)|\\ &\le c_{m,a}|bh|^m \max(1,|bh|^a) \cdot \sup_{y\in\mathbb{R}} \frac{|g^{(m)}(bx-y)|}{1+|y|^a}.\end{aligned}$$

At next we use the left hand side of Lemma 1.3 and obtain

$$\begin{aligned}|\triangle_h^m f(x)| &\le c'_{m,a}|bh|^m \max(1,|bh|^a) \sup_{y\in\mathbb{R}} \frac{|g(bx-y)|}{1+|y|^a}\\ &= c'_{m,a}|bh|^m \max(1,|bh|^a) \sup_{y\in\mathbb{R}} \frac{|g(b(x-y))|}{1+|by|^a} \\ &= c'_{m,a}|bh|^m \max(1,|bh|^a) P_{b,a}f(x).\end{aligned} \tag{1.58}$$

On the other hand we can directly estimate using (1.52). This yields

$$\begin{aligned}|\triangle_h^m f(x)| &= \left|\sum_{j=0}^m (-1)^j \binom{m}{j} f(x+(m-j)h)\right|\\ &\le c_m \sup_{|y|\le mh} \frac{|f(x-y)|}{1+|by|^a}(1+|by|^a) \\ &\le 2c''_m \sup_{y\in\mathbb{R}} \frac{|f(x-y)|}{1+|by|^a} \cdot \max(1,|mhb|^a)\\ &\le c''_{m,a} \max(1,|hb|^a) P_{b,a}f(x).\end{aligned} \tag{1.59}$$

And finally, (1.58) together with (1.59) completes the proof. $\square$

The following lemma generalizes the previous one to d dimensions. For sake of brevity we define

$$A(s,t) := \max(1,s)\min(1,t) \quad , \quad \text{for } s,t \geq 0.$$

A simple property consists in the following. For fixed $v,w > 0$ there is a constant $c_{v,w} > 0$ such that

$$A(v \cdot s, w \cdot t) \leq c_{v,w} A(s,t) \tag{1.60}$$

for all $s,t \geq 0$.

Lemma 1.12 *Let $\bar{a} = (a_1,...,a_d), \bar{b} = (b_1,...,b_d) \in \mathbb{R}^d$ satisfy $\bar{a}, \bar{b} > 0$. Let further be $\bar{m} = (m_1,...,m_d) \in \mathbb{N}^d$, $\bar{h} = (h_1,...,h_d) \in \mathbb{R}^d$ and $f \in S'(\mathbb{R}^d)$ with $\operatorname{supp}\mathcal{F}f \subset Q_{\bar{b}}$, where*

$$Q_{\bar{b}} := [-b_1,b_1] \times ... \times [-b_d,b_d].$$

Then there exists a constant $c > 0$ (independent of f, $\bar{b}$ and $\bar{h}$) such that

$$\begin{aligned}&|(\triangle^{m_1}_{h_1,1} \circ \triangle^{m_2}_{h_2,2} \circ \cdots \circ \triangle^{m_d}_{h_d,d} f)(x)| \\ &\leq c \cdot A(|b_1h_1|^{a_1}, |b_1h_1|^{m_1}) \cdot ... \cdot A(|b_dh_d|^{a_d}, |b_dh_d|^{m_d}) \cdot P_{\bar{b},\bar{a}}f(x)\end{aligned}$$

holds for all $x \in \mathbb{R}^d$.

Proof The idea is to iterate the previous lemma. We define the function g by

$$g = (\triangle^{m_2}_{h_2,2} \circ \cdots \circ \triangle^{m_d}_{h_d,d} f)(x).$$

Because of

$$\mathcal{F}g = (e^{i\xi_2 h_2} - 1)^{m_2} \cdot ... \cdot (e^{i\xi_d h_d} - 1)^{m_d} \mathcal{F}f$$

the inclusion $\operatorname{supp}\mathcal{F}g \subset \operatorname{supp}\mathcal{F}f \subset Q_{\bar{b}}$ holds. Let us now fix the components $x_2,...,x_d$ and consider the function

$$\tilde{g} := g(\cdot, x_2, ..., x_d).$$

Remark 1.12 after Theorem 1.8 gives us $\tilde{g} \in S'(R)$ and $\operatorname{supp}\mathcal{F}\tilde{g} \subset [-b_1,b_1]$. Finally, we use Lemma 1.11 with $\triangle^{m_1}_{h_1}\tilde{g}(x_1) = \triangle^{m_1}_{h_1,1} g(x_1,...,x_d)$ and obtain

$$\begin{aligned}|(\triangle^{m_1}_{h_1,1} \circ \triangle^{m_2}_{h_2,2} \circ \cdots \circ \triangle^{m_d}_{h_d,d} f)(x)| &= \triangle^{m_1}_{h_1}\tilde{g}(x_1) \\ &\leq c_{a_1,m_1} A(|b_1h_1|^{a_1}, |b_1h_1|^{m_1}) \cdot P_{b_1,a_1}\tilde{g}(x_1) \\ &= c_{a_1,m_1} A(|b_1h_1|^{a_1}, |b_1h_1|^{m_1}) \cdot \sup_{y_1} \frac{|g(x_1-y_1,x_2,...,x_d)|}{(1+|b_1y_1|^{a_1})}.\end{aligned}$$

We continue by estimating $|g(x_1-y_1,x_2,...,x_d)|$ in the same manner. Iteration of this procedure finishes the proof. $\square$

Remark 1.13 *With exactly the same arguments one proves a version of the previous lemma to estimate mixed differences of type $\triangle^{\bar{m}}_{\bar{h},\bar{\alpha}} f(x)$ by the maximal function $P_{\bar{b},\bar{a},\bar{\alpha}}f$.* $\square$

1.6.4 Integral Means of Differences

The main result in this section is a characterization of the spaces $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$, $0 < p < \infty$, $0 < q \le \infty$ and $\bar{r} > \sigma_{p,q}$, using integral means (rectangle means) of differences. In some sense it is the counterpart of [48, Th. 2.5.11], where the isotropic scale is treated in terms of ball means. It will be also used as starting point for further difference characterization later in this section.

Theorem 1.10 *Let $0 < p < \infty$, $0 < q \le \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}^d$ with $\bar{m} > \bar{r} > \sigma_{p,q}$. Under these conditions the space $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ is the collection of all functions $f \in L_p(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)$, such that*

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^R = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\bar{\beta}|_1 \ge 1} S^R_{\bar{\beta}}(f) < \infty \quad . \tag{1.61}$$

For $|\bar{\beta}|_1 = n \ge 1$ we put

$$S^R_{\bar{\beta}}(f) = \Big\|\Big[\int\limits_{(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big)\Big(\int\limits_{[-1,1]^n} \big|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_nh_n,\delta_n} f)(x)\big| d\bar{h}\Big)^q \frac{d\bar{t}}{\bar{t}}\Big]^{1/q}\Big| L_p(\mathbb{T}^d)\Big\| \tag{1.62}$$

with $\bar{\delta}$ assigned to $\bar{\beta}$ in the sense of (1.45). In case $q = \infty$ one has to replace (1.62) by

$$S^R_{\bar{\beta}}(f) = \Big\|\sup_{\bar{t}\in(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i}}\big) \int\limits_{[-1,1]^n} \big|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_nh_n,\delta_n} f)(x)\big|\, d\bar{h}\Big| L_p(\mathbb{T}^d)\Big\|.$$

Moreover, (1.61) is an equivalent (quasi-)norm in $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$.

Proof The proof will be divided into 2 steps.

Step 1. We fix $\bar{\varphi} = (\varphi^1, ..., \varphi^d) \in \Phi(\mathbb{R})^d$. We want to show that a constant $c > 0$ exists such that

$$\|f|S^{\bar{r}}_{p,q}F\|^R \le c\|f|S^{\bar{r}}_{p,q}F\|^{\bar{\varphi}}$$

holds for every $f \in L_1(\mathbb{T}^d) \cap L_p(\mathbb{T}^d)$. The basic idea is to use the decomposition (1.20) and proceed by estimating the appearing differences by maximal functions via (1.55), Lemma 1.11 and Lemma 1.12. Afterwards we shall exploit the maximal inequalities, Proposition 1.4 and Proposition 1.9. We follow [48, 2.5.11] and apply the isotropic strategy in a certain sense to every direction. Let us make some further preparation. We choose a tuple $\bar{a} = (a_1, ..., a_d) > \frac{1}{\min(p,q)}$ and a number $0 < \lambda < \min(p, q)$ such that $\bar{r} > (1 - \lambda)\bar{a}$. It is easy to see that this is possible: In case $\min(p, q) \le 1$ we have

$$r_i > \frac{1}{\min(p, q)}(1 - \min(p, q)) \quad , \quad i = 1, ..., d \quad ,$$

as a consequence of $\bar{r} > \sigma_{p,q}$. In case $\min(p,q) > 1$ we simply choose $\lambda = 1$. Assume firstly $q < \infty$. We start by considering the expression $A_{\bar\beta}(x)$ defined by

$$A_{\bar\beta}(x) = \int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\Big)\Big(\int\limits_{[-1,1]^n} \big|\big(\triangle^{m_{\delta_1}}_{t_1 h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n,\delta_n} f\big)(x)\big| d\bar{h}\Big)^q \frac{d\bar{t}}{\bar{t}} \quad , \tag{1.63}$$

where $\bar\beta$ with $|\bar\beta|_1 = n > 0$ (and corresponding $\bar\delta$) is fixed. In order to replace in (1.63) the integral by a sum we obtain with elementary calculations

$$\begin{aligned} A_{\bar\beta}(x) &\le c \sum_{\bar{k}\in Z_{\bar\beta}} \int\limits_{Q^{+\Delta}_{-\bar{k},\bar\beta}} 2^{\bar{r}\bar{k}q} \Big(2^{\bar{k}} \int\limits_{Q_{-\bar{k},\bar\beta}} \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big|\, d\bar{h}\Big)^q \frac{d\bar{t}}{2^{-\bar{k}}} \\ &\le c \sum_{\bar{k}\in Z_{\bar\beta}} 2^{\bar{r}\bar{k}q} \Big(2^{\bar{k}} \int\limits_{Q_{-\bar{k},\bar\beta}} \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big|\, d\bar{h}\Big)^q \\ &= c \sum_{\bar{k}\in Z_{\bar\beta}} 2^{\bar{r}\bar{k}q} \Big(\int\limits_{[-1,1]^n} \big|(\triangle^{m_{\delta_1}}_{2^{-k_{\delta_1}} h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{2^{-k_{\delta_n}} h_n,\delta_n} f)(x)\big|\, d\bar{h}\Big)^q \\ &\le c \sum_{\bar{k}\in Z_{\bar\beta}} 2^{\bar{r}\bar{k}q} \Big(\int\limits_{[-1,1]^d} \big| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar\beta} f(x)\big|\, d\bar{h}\Big)^q . \end{aligned}$$

For the used notation we refer to (1.46). Recall the Fourier analytical decomposition of f in

$$f_{\bar\ell}(x) = \sum_{k\in\mathbb{Z}^d} (\varphi^1 \otimes \ldots \otimes \varphi^d)_{\bar\ell}(k) c_k(f) e^{ik\cdot x} \quad , \quad x \in \mathbb{T}^d, \quad \bar\ell \in \mathbb{N}_0^d.$$

See (1.20) for details. Obviously, it holds for every $\bar{k} \in Z_{\bar\beta}$

$$f = \sum_{\bar\ell\in Z_{\bar\beta}} \sum_{\bar{u}\in \bar{N}_{\bar\beta}} f_{\bar{k}+\bar\ell+\bar{u}} \tag{1.64}$$

in $D'(\mathbb{T}^d)$, where we put $\varphi^i_\ell \equiv 0$, $i = 1, ..., d$ if $\ell < 0$. Hence, for $q \le 1$ we obtain

$$A_{\bar\beta}(x) \le c \sum_{\bar\ell\in Z_{\bar\beta}} \sum_{\bar{u}\in \bar{N}_{\bar\beta}} \sum_{\bar{k}\in Z_{\bar\beta}} 2^{\bar{r}\bar{k}q} \Big(\int\limits_{[-1,1]^d} \big| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar\beta} f_{\bar{k}+\bar\ell+\bar{u}}(x)\big|\, d\bar{h}\Big)^q . \tag{1.65}$$

Precisely, we used the unconditional $L_1([-1,1]^n)$-convergence (with respect to $\bar{h}$) of the sum

$$\sum_{\bar\ell\in Z_{\bar\beta}} \sum_{\bar{u}\in \bar{N}_{\bar\beta}} \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar\beta} f_{\bar{k}+\bar\ell+\bar{u}}(x)$$

for fixed x. This follows from its absolute L_1-convergence and (1.64). In some sense the arguments are justified if one reads the estimates backwards.

If $q > 1$ the triangle inequality in ℓ_q gives the estimate

$$A_{\bar\beta}(x)^{1/q} \le c \sum_{\bar\ell\in Z_{\bar\beta}} \sum_{\bar u \in \bar N_{\bar\beta}} \Big[\sum_{\bar k \in Z_{\bar\beta}} 2^{\bar r \bar k q} \Big(\int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\beta} f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h \Big)^q \Big]^{1/q} \tag{1.66}$$

instead of (1.65). We continue in estimating $A_{\bar\beta}(x)$ in case $q \le 1$. For the moment we postpone the case $q > 1$. We decompose RHS(1.65) into the following blocks with respect to the $\bar\ell$-sum

$$A^{\bar\alpha}_{\bar\beta}(x) = \sum_{\bar\ell\in Z^{\bar\alpha}_{\bar\beta}} \sum_{\bar u \in \bar N_{\bar\beta}} \sum_{\bar k \in Z_{\bar\beta}} 2^{\bar r \bar k q} \Big(\int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\beta} f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h \Big)^q \quad , \tag{1.67}$$

where $\bar\alpha \le \bar\beta$,

$$\begin{aligned} Z^{\bar\alpha}_{\bar\beta} = \big\{ (k_1, ..., k_d) \in \mathbb{Z}^d : (\beta_i = 0 \implies k_i = 0) \\ \wedge\ ((\alpha_i, \beta_i) = (0,1) \implies k_i \ge 0) \\ \wedge\ ((\alpha_i, \beta_i) = (1,1) \implies k_i \le 0)\ ,\ i = 1, ..., d \big\} \end{aligned}$$

and hence $Z_{\bar\beta} \subset \bigcup\limits_{\bar\alpha \le \bar\beta} Z^{\bar\alpha}_{\bar\beta}$. Consequently, we obtain

$$A_{\bar\beta}(x) \le c \sum_{\bar\alpha \le \bar\beta} A^{\bar\alpha}_{\bar\beta}(x). \tag{1.68}$$

We investigate the behaviour of the integral

$$\int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\beta} f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h = \int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\alpha} \circ \triangle^{\bar m}_{2^{-\bar k} * \bar h, (\bar\beta - \bar\alpha)} f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h \tag{1.69}$$

inside RHS(1.67) for a fixed $\bar\alpha \le \bar\beta$. To avoid technical difficulties we just consider the special situation $\bar\beta = (1, ..., 1, 0, ..., 0)$ and $\bar\alpha = (1, ..., 1, 0, ..., 0)$, where $|\bar\alpha|_1 \le |\bar\beta|_1$. All other cases can be treated analogously. One only has to change the order of difference operators appropriately. With the help of Lemma 1.12 and Remark 1.13, respectively, we lose the first part $\triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\alpha}$ of the mixed difference in (1.69). Estimation by a maximal function of type (1.50) using Lemma 1.12 and (1.60) yields

$$\begin{aligned} &\int\limits_{[-1,1]^d} \big| \triangle^{\bar m}_{2^{-\bar k} * \bar h, \bar\alpha} \circ \triangle^{\bar m}_{2^{-\bar k} * \bar h, (\bar\beta - \bar\alpha)} f_{\bar k + \bar\ell + \bar u}(x) \big| \, d\bar h \\ &\le c_1 A(|2^{\ell_1 + k_1} 2^{-k_1}|^{a_1}, |2^{\ell_1 + k_1} 2^{-k_1}|^{m_1}) \cdot ... \cdot A(|2^{\ell_{|\bar\alpha|_1} + k_{|\bar\alpha|_1}} 2^{-k_{|\bar\alpha|_1}}|^{a_{|\bar\alpha|_1}}, |2^{\ell_{|\bar\alpha|_1} + k_{|\bar\alpha|_1}} 2^{-k_{|\bar\alpha|_1}}|^{m_{|\bar\alpha|_1}}) \\ &\quad \cdot \int\limits_{[-1,1]^d} P_{\bar b, \bar a, \bar\alpha}(\triangle^{\bar m}_{2^{-\bar k} * \bar h, (\bar\beta - \bar\alpha)} f_{\bar k + \bar\ell + \bar u})(x) \, d\bar h \quad , \end{aligned} \tag{1.70}$$

where $\bar{a}$ was chosen together with λ right at the beginning. Additionally, we put $\bar{b} = (2^{k_1+\ell_1}, ..., 2^{k_d+\ell_d})$. Recall the fact $\ell_1, ..., \ell_{|\bar{\alpha}|_1} \leq 0$. In case $|\bar{\alpha}|_1 < |\bar{\beta}|_1 = n$ this simplifies estimate (1.70) to

$$\begin{aligned} &\int\limits_{[-1,1]^d} \left| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,\bar{\alpha}} \circ \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,(\bar{\beta}-\bar{\alpha})}\, f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\right| d\bar{h} \\ &\leq c_1 2^{\ell_1 m_1} \cdot ... \cdot 2^{\ell_{|\bar{\alpha}|_1} m_{|\bar{\alpha}|_1}} \int\limits_{[-1,1]^d} P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})(x)\, d\bar{h}\ . \end{aligned} \tag{1.71}$$

Whereas we otherwise (in case $|\bar{\alpha}|_1 = |\bar{\beta}|_1$) end up with

$$\int\limits_{[-1,1]^d} \left| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,\bar{\alpha}} \circ \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,(\bar{\beta}-\bar{\alpha})}\, f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\right| d\bar{h} \leq c_1 2^{\ell_1 m_1} \cdot ... \cdot 2^{\ell_n m_n} P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})(x)$$

instead of (1.71). To proceed with (1.71) we need a new strategy. The remaining difference operator inside (1.71) acts on the components of $f_{\bar{k}+\bar{\ell}+\bar{u}}(x)$, that correspond to the $\bar{\ell}$-components in $Z^{\bar{\alpha}}_{\bar{\beta}}$, which run over $\mathbb{N}_0$. Of course, it can be estimated in the same manner we did above. We obtain by Lemma 1.12

$$P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})(x) \leq c 2^{\ell_{|\bar{\alpha}|_1+1} a_{|\bar{\alpha}|_1+1}} \cdot ... \cdot 2^{\ell_n a_n} P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})(x). \tag{1.72}$$

Let us first split the integrand in RHS(1.71) into the product

$$\begin{aligned} &P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})(x) \\ &= |P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})|^{1-\lambda}(x) \cdot |P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})|^{\lambda}(x). \end{aligned} \tag{1.73}$$

In case $\min(p,q) > 1$ there is no need for such a splitting, as $\lambda = 1$ indicates. Moreover, the difference $\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}}$ can be expanded to a sum with the help of (1.53). Using an obvious subadditivity property of the operator $P_{\bar{b},\bar{a},\bar{\alpha}}$, we can tear the sum out of the argument. This yields

$$P_{\bar{b},\bar{a},\bar{\alpha}}(\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},(\bar{\beta}-\bar{\alpha})} f_{\bar{k}+\bar{\ell}+\bar{u}})(x) \leq c_2 \sum_{\bar{w}\in M_{\bar{\beta}-\bar{\alpha}}} c_{\bar{w}} P_{\bar{b},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{\ell}+\bar{u}})(x + \bar{w} * 2^{-\bar{k}} * \bar{h}) \quad , \tag{1.74}$$

where we refer to (1.49) for the index notation. Combining (1.72), (1.74) and (1.73) we obtain

$$\begin{aligned} &\int\limits_{[-1,1]^d} \left| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,\bar{\alpha}} \circ \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,(\bar{\beta}-\bar{\alpha})}\, f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\right| d\bar{h} \\ &\leq c_3 2^{\ell_1 m_1} \cdot ... \cdot 2^{\ell_{|\bar{\alpha}|_1} m_{|\bar{\alpha}|_1}} 2^{\ell_{|\bar{\alpha}|_1+1} a_{|\bar{\alpha}|_1+1}(1-\lambda)} \cdot ... \cdot 2^{\ell_n a_n (1-\lambda)} |P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{1-\lambda}(x) \times \\ &\times \sum_{\bar{w}\in M_{\bar{\beta}-\bar{\alpha}}} c_{\bar{w}} \int\limits_{[-1,1]^d} |P_{\bar{b},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{\lambda}(x + \bar{w} * 2^{-\bar{k}} * \bar{h})\, d\bar{h}. \end{aligned}$$

And finally (1.55) leads to

$$\begin{aligned}&\int\limits_{[-1,1]^d} \left| \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,\bar{\alpha}} \circ \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\,(\bar{\beta}-\bar{\alpha})}\, f_{\bar{k}+\bar{\ell}+\bar{u}}(x)\right| d\bar{h}\\ &\leq c_4 2^{\ell_1 m_1}\cdot\ldots\cdot 2^{\ell_{|\bar{\alpha}|_1} m_{|\bar{\alpha}|_1}} 2^{\ell_{|\bar{\alpha}|_1+1} a_{|\bar{\alpha}|_1+1}(1-\lambda)}\cdot\ldots\cdot 2^{\ell_n a_n(1-\lambda)} |P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{1-\lambda}(x)\\ &\quad\cdot M_{|\bar{\alpha}|_1+1}\circ\ldots\circ M_n(|P_{\bar{b},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{\lambda})(x).\end{aligned} \tag{1.75}$$

We return to $A^{\bar{\alpha}}_{\bar{\beta}}(x)$. Together with (1.75) we obtain in our special situation

$$\begin{aligned}A^{\bar{\alpha}}_{\bar{\beta}}(x) \leq & c_4 \sum_{\bar{\ell}\in Z^{\alpha}_{\bar{\beta}}} 2^{\ell_1(m_1-r_1)q}\cdot\ldots\cdot 2^{\ell_{|\bar{\alpha}|_1}(m_{|\bar{\alpha}|_1}-r_{|\bar{\alpha}|_1})q}\\ &\cdot 2^{\ell_{|\bar{\alpha}|_1+1}\left[a_{|\bar{\alpha}|_1+1}(1-\lambda)-r_{|\bar{\alpha}|_1+1}\right]q}\cdot\ldots\cdot 2^{\ell_n[a_n(1-\lambda)-r_n]q}\\ &\cdot \sum_{\bar{u}\in\bar{N}_{\bar{\beta}}}\sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}(\bar{k}+\bar{\ell})q} |P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{(1-\lambda)q}(x)\\ &\cdot \left|M_{|\bar{\alpha}|_1+1}\circ\ldots\circ M_n(|P_{\bar{b},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^{\lambda})\right|^q(x).\end{aligned} \tag{1.76}$$

After modifying the sum over $\bar{k}$ in order to make it independent of $\bar{\ell}$, the sum over $\bar{\ell}$ is nothing but a geometric series and breaks down to a constant. This is a consequence of $\bar{m} > \bar{r}$ and the second condition to λ (and $\bar{a}$). Hence, we arrive at

$$\begin{aligned}A^{\bar{\alpha}}_{\bar{\beta}}(x) \leq & c_5 \sum_{\bar{u}\in\bar{N}_{\bar{\beta}}}\sum_{\bar{k}\in N_{\bar{\beta}}} 2^{\bar{r}\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{u}})|^{(1-\lambda)q}(x)\cdot\left|M_{|\bar{\alpha}|_1+1}\circ\ldots\circ M_n(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}}(f_{\bar{k}+\bar{u}})|^{\lambda})\right|^q(x)\\ = & c_5 \sum_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}} f_{\bar{k}}|^{(1-\lambda)q}(x)\cdot\left|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}|^{\lambda})\right|^q(x)\quad ,\end{aligned}$$

where $\mathbf{M}_{\bar{\gamma}} = M_{\delta_1}\circ\ldots\circ M_{\delta_{|\bar{\gamma}|}}$ (recall (1.11)) and $\bar{\delta}$ belongs to $\bar{\gamma}\in\{0,1\}^d$ as usual. Altogether we obtain the following estimates for the cases (C_1) $|\bar{\alpha}|_1 < |\bar{\beta}|_1$ and for (C_2) $|\bar{\alpha}|_1 = |\bar{\beta}|_1$, respectively,

$$A^{\bar{\alpha}}_{\bar{\beta}}(x) \leq c_5 \begin{cases} \sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}} f_{\bar{k}}|^{(1-\lambda)q}(x)\cdot\left|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}|^{\lambda})\right|^q(x) & : C_1\\ \sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{(\bar{r}*\bar{\beta})\cdot\bar{k}q} |P_{2^{\bar{k}},\bar{a},\bar{\beta}}(f_{\bar{k}})|^q(x) & : C_2\end{cases} . \tag{1.77}$$

The case $\bar{\alpha} = (0,...,0)$, i.e. $|\bar{\alpha}|_1 = 0$, fits into (C_1). One only has to replace $P_{2^{\bar{k}},\bar{a},\bar{\alpha}} f_{\bar{k}}$ by $f_{\bar{k}}$. With obvious modifications we obtain (1.77) for arbitrary $\bar{\beta}$ and $\bar{\alpha}\leq\bar{\beta}$. Inequality (1.77) invites us to exploit the Theorems 1.4 and 1.9 for estimating $\|A^{\bar{\alpha}}_{\bar{\beta}}(x)^{1/q}|L_p(\mathbb{T}^d)\|$. Hence, we obtain in case (C_2)

$$\begin{aligned}\|A^{\bar\alpha}_{\bar\beta}(x)^{1/q}|L_p(\mathbb{T}^d)\| &\le c_5 \left\|\left(\sum_{\bar k\in\mathbb{N}_0^d} 2^{(\bar r*\bar\beta)\cdot\bar k q}|P_{2^{\bar k},\bar a,\bar\beta}(f_{\bar k})|^q(x)\right)^{1/q}\middle|L_p(\mathbb{T}^d)\right\| \\ &= c_5 \left\|P_{2^{\bar k},\bar a,\bar\beta}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})\middle|L_p(\ell_q)\right\| \\ &\le c_6 \left\|2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k}\middle|L_p(\ell_q)\right\| \\ &\le c_7 \,\|f|S^{\bar r}_{p,q}F\|^{\bar\varphi}.\end{aligned} \tag{1.78}$$

Let us now consider the case (C_1). We begin using Hölder's inequality for sums with the exponents $\frac{1}{\lambda}$ and $\frac{1}{1-\lambda}$ and obtain

$$A^{\bar\alpha}_{\bar\beta}(x) \le c_5\left(\sum_{\bar k\in\mathbb{N}_0^d} 2^{(\bar r*\bar\beta)\cdot\bar k q}\left|\mathbf{M}_{\bar\beta-\bar\alpha}(|P_{2^{\bar k},\bar a,\bar\alpha}f_{\bar k}|^\lambda)\right|^{q/\lambda}(x)\right)^\lambda\left(\sum_{\bar k\in\mathbb{N}_0^d} 2^{(\bar r*\bar\beta)\cdot\bar k q}|P_{2^{\bar k},\bar a,\bar\beta}f_{\bar k}|^q(x)\right)^{1-\lambda}.$$

Applying the $L_p(\mathbb{T}^d)$-(quasi-)norm to the previous inequality this leads to

$$\begin{aligned}&\|A^{\bar\alpha}_{\bar\beta}(x)^{1/q}|L_p(\mathbb{T}^d)\| \\ &\le c_5\left\|\left(\sum_{\bar k\in\mathbb{N}_0^d} 2^{(\bar r*\bar\beta)\cdot\bar k q}\left|\mathbf{M}_{\bar\beta-\bar\alpha}(|P_{2^{\bar k},\bar a,\bar\alpha}f_{\bar k}|^\lambda)\right|^{q/\lambda}(x)\right)^{\frac{\lambda p}{q}} \times \left(\sum_{\bar k\in\mathbb{N}_0^d} 2^{(\bar r*\bar\beta)\cdot\bar k q}|P_{2^{\bar k},\bar a,\bar\beta}f_{\bar k}|^q(x)\right)^{\frac{(1-\lambda)p}{q}}\middle|L_1(\mathbb{T}^d)\right\|^{1/p} \\ &\le c_5\left\|\left(\sum_{\bar k\in\mathbb{N}_0^d} 2^{(\bar r*\bar\beta)\cdot\bar k q}\left|\mathbf{M}_{\bar\beta-\bar\alpha}(|P_{2^{\bar k},\bar a,\bar\alpha}f_{\bar k}|^\lambda)\right|^{q/\lambda}(x)\right)^{\frac{p}{q}}\middle|L_1(\mathbb{T}^d)\right\|^{\lambda/p} \times \\ &\quad\times\left\|\left(\sum_{\bar k\in\mathbb{N}_0^d} 2^{(\bar r*\bar\beta)\cdot\bar k q}|P_{2^{\bar k},\bar a,\bar\beta}f_{\bar k}|^q(x)\right)^{\frac{p}{q}}\middle|L_1(\mathbb{T}^d)\right\|^{(1-\lambda)/p}.\end{aligned}$$

Again we used Hölder's inequality with the same exponents, but this time for integrals. Rewriting of the last inequality shows

$$\begin{aligned}\|A^{\bar\alpha}_{\bar\beta}(x)^{1/q}|L_p(\mathbb{T}^d)\| \le & c_5\left\|\mathbf{M}_{\bar\beta-\bar\alpha}(|P_{2^{\bar k},\bar a,\bar\alpha}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|^\lambda)(x)\middle|L_{p/\lambda}(\mathbb{T}^d,\ell_{q/\lambda})\right\|\times \\ & \times\left\|P_{2^{\bar k},\bar a,\bar\beta}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})(x)\middle|L_p(\mathbb{T}^d,\ell_q)\right\|^{1-\lambda}.\end{aligned} \tag{1.79}$$

The second factor can be estimated analogously to (1.78), i.e

$$\begin{aligned}\left\|P_{2^{\bar k},\bar a,\bar\beta}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})(x)\middle|L_p(\mathbb{T}^d,\ell_q)\right\|^{1-\lambda} &\le c_5\|2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k}|L_p(\mathbb{T}^d,\ell_q)\|^{1-\lambda} \\ &\le c_7\|f|S^{\bar r}_{p,q}F\|^{1-\lambda}.\end{aligned} \tag{1.80}$$

Because of $p/\lambda, q/\lambda > 1$ and Theorem 1.4 the first part of the product (1.79) can be reduced to

$$\begin{aligned}\Big\|\mathbf{M}_{\bar\beta-\bar\alpha}(|P_{2^{\bar k},\bar a,\bar\alpha}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|^{\lambda})(x)\Big|\, L_{p/\lambda}(\mathbb{T}^d,\ell_{q/\lambda})\Big\| &\le c_8\,\|\,|P_{2^{\bar k},\bar a,\bar\alpha}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|^{\lambda}|L_{p/\lambda}(\mathbb{T}^d,\ell_{q/\lambda})\|\\ &= c_8\,\|P_{2^{\bar k},\bar a,\bar\alpha}(2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|L_p(\mathbb{T}^d,\ell_q)\|^{\lambda}\\ &\le c_9\,\|2^{(\bar r*\bar\beta)\cdot\bar k}f_{\bar k})|L_p(\mathbb{T}^d,\ell_q)\|^{\lambda}\\ &\le c_{10}\,\|f|S^{\bar r}_{p,q}F\|^{\lambda}.\end{aligned}$$

This fact combined with (1.79) and (1.80) yields

$$\|A^{\bar\alpha}_{\bar\beta}(x)^{1/q}|L_p(\mathbb{T}^d)\| \le c_{11}\|f|S^{\bar r}_{p,q}F\|^{\bar\varphi}.$$

Hence, (1.68) gives

$$S^R_{\bar\beta}(f) = \|A^{1/q}_{\bar\beta}(x)|L_p(\mathbb{T}^d)\| \le c_{11}\|f|S^{\bar r}_{p,q}F\|^{\bar\varphi}.$$

What remains follows by Proposition 1.9. We now finish the case $q \le 1$. To complete Step 1 it remains to describe the necessary modifications in the case $q > 1$. Let us start with (1.66).
The condition $\bar r > 0$ allows the following estimation

$$\begin{aligned}A^{1/q}_{\bar\beta}(x) &\le c\sum_{\bar\ell\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}}2^{-\bar r\cdot\bar u}2^{\bar r\cdot\bar u}[\cdots]^{1/q} \le c\sum_{\bar\ell\in Z_{\bar\beta}}\sup_{\bar u}2^{\bar r\cdot\bar u}[\cdots]^{1/q}\sum_{\bar u\in\bar N_{\bar\beta}}2^{-\bar r\cdot\bar u}\\ &\le c'\sum_{\bar\ell\in Z_{\bar\beta}}\Bigg(\sum_{\bar u\in\bar N_{\bar\beta}}2^{\bar r\cdot\bar u q}[\cdots]\Bigg)^{1/q}\\ &= c'\sum_{\bar\ell\in Z_{\bar\beta}}\Bigg(\sum_{\bar k\in Z_{\bar\beta}}\sum_{\bar u\in\bar N_{\bar\beta}}2^{\bar r\cdot\bar k q}2^{\bar r\cdot\bar u q}\Bigg(\int\limits_{[-1,1]^d}\big|\,\triangle^{\bar m}_{2^{-\bar k}*\bar h,\,\bar\beta}\,f_{\bar k+\bar\ell+\bar u}(x)\big|\,d\bar h\Bigg)^{q}\Bigg)^{1/q}.\end{aligned}$$

At next, we decompose the sum over $\bar\ell$ in the same way we did in (1.68) and put

$$A^{\bar\alpha}_{\bar\beta}(x)^{1/q} = \sum_{\bar\ell\,\in Z^{\bar\alpha}_{\bar\beta}}\Bigg(\sum_{\bar u\in\bar N_{\bar\beta}}\sum_{\bar k\in Z_{\bar\beta}}\Bigg(\int\limits_{[-1,1]^d}\big|\,\triangle^{\bar m}_{2^{-\bar k}*\bar h,\,\bar\beta}\,f_{\bar k+\bar\ell+\bar u}(x)\big|\,d\bar h\Bigg)^{q}\Bigg)^{1/q}.$$

From now on we can carry over the estimates given in case $q \le 1$ almost word by word. The consequence are three cases for an upper bound concerning the expression $A^{\bar\alpha}_{\bar\beta}(x)^{1/q}$, depending on $|\bar\alpha|_1$ and $\min(p,q)$ (i.e. λ). We distinguish the case (C_1) $|\bar\alpha|_1 < |\bar\beta|_1, \lambda < 1$ from (C_2) $|\bar\alpha|_1 < |\bar\beta|_1, \lambda = 1$ and (C_3) $|\bar\alpha|_1 = |\bar\beta|_1$. Additionally, because of $q > 1$ the case

$\min(p,q)>1$, i.e $\lambda=1$, is possible. See also (1.77). Altogether we obtain

$$A^{\bar{\alpha}}_{\bar{\beta}}(x)^{1/q} \le c'' \left\{ \begin{array}{lr} \left[\sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{k}q}|P_{2^{\bar{k}},\bar{a},\bar{\beta}}f_{\bar{k}}|^{(1-\lambda)q}(x)\cdot\left|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}(|P_{2^{\bar{k}},\bar{a},\bar{\alpha}}f_{\bar{k}}|^{\lambda})\right|^q(x)\right]^{1/q} & : \; C_1 \\ \left[\sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{k}q}\left|\mathbf{M}_{\bar{\beta}-\bar{\alpha}}\circ P_{2^{\bar{k}},\bar{a},\bar{\alpha}}(f_{\bar{k}})\right|^q(x)\right]^{1/q} & : \; C_2 \\ \left[\sum\limits_{\bar{k}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{k}q}|P_{2^{\bar{k}},\bar{a},\bar{\beta}}(f_{\bar{k}})|^q(x)\right]^{1/q} & : \; C_3 \end{array} \right. .$$

What remains is similar as above. The case $q=\infty$ can be treated with obvious modifications and much more simpler arguments.

Step 2. We show the converse inequality using a classical construction by S. M. Nikol'skij, cf. [27, 5.2.1]. The basic idea is to prove

$$\|f|S^{\bar{r}}_{p,q}F\|^{\bar{\varphi}} \le c\|f|S^{\bar{r}}_{p,q}F\|^R$$

for every $f\in L_1(\mathbb{T}^d)\cap L_p(\mathbb{T}^d)$, where $\bar{\varphi}=(\varphi^1,...,\varphi^d)$ denotes an appropriate tuple from $\Psi(\mathbb{R})^d$. These decompositions of unity are adapted to the order of differences used to compute $\|f|S^{\bar{r}}_{p,q}F\|^R$. For every $i\in\{1,...,d\}$ we put

$$\varphi^i_0(x) = (-1)^{m_i+1}\sum_{\mu=0}^{m_i-1}\binom{m_i}{\mu}(-1)^{\mu}\psi((m_i-\mu)x) \quad , \tag{1.81}$$

where $\psi\in S(\mathbb{R})$ such that

$$\psi(x) = \left\{ \begin{array}{ccl} 1 & : & |x|\le 1 \\ 0 & : & |x|>3/2 \end{array} \right. .$$

Consequently, $\varphi^i_0\in S(\mathbb{R})$ is compactly supported and moreover

$$\varphi^i_0(x) = \left\{ \begin{array}{ccl} 1 & : & |x|\le 1/m_i \\ 0 & : & |x|>3/2 \end{array} \right. . \tag{1.82}$$

Equation (1.82) is clear in case $|x|>3/2$. In the case $|x|\le 1/m_i$ we have

$$\begin{aligned} \varphi^i_0(x) &= (-1)^{m_i+1}\left(\sum_{\mu=0}^{m_i}\binom{m_i}{\mu}(-1)^{\mu}-(-1)^{m_i}\right) \\ &= (-1)^{m_i+1}((1-1)^{m_i}-(-1)^{m_i}) \\ &= 1. \end{aligned}$$

Therefore, the function $\varphi^i_0(x)$ is in the sense of Definition 1.2 admissible to define via

$$\varphi^i_j(x) := \varphi^i_0(2^{-j}x)-\varphi^i_0(2^{-j+1}x) \quad , \quad j\ge 1\,,$$

a decomposition of unity $\varphi^i := \{\varphi^i_j(x)\}_{j=0}^\infty \in \Psi(\mathbb{R})$. Now formula (1.52) points out the connection to differences. Obviously, it holds

$$\varphi^i_0(x) = (-1)^{m_i+1}(\triangle^{m_i}_x \psi(0) - (-1)^{m_i})$$

and

$$\varphi^i_j(x) = (-1)^{m_i+1}(\triangle^{m_i}_{2^{-j}x} - \triangle^{m_i}_{2^{-j+1}x})\psi(0) \quad \text{for} \quad j > 0\ ,$$

respectively. The effect is the occurrence of differences on the Fourier side of f. We consider again the sequence $\{f_{\bar{\ell}}(x)\}_{\bar{\ell}\in\mathbb{N}_0^d}$, where

$$f_{\bar{\ell}}(x) := \sum_{k\in\mathbb{Z}^d} (\varphi^1 \otimes ... \otimes \varphi^d)_{\bar{\ell}}(k) c_k(f) e^{ik\cdot x} \quad , \quad x \in \mathbb{T}^d \quad , \quad \bar{\ell} \in \mathbb{N}_0^d\,,$$

see (1.21) . It is necessary to divide the index-set $\mathbb{N}_0^d$ into 2^d disjoint subsets in the following way

$$\mathbb{N}_0^d = \bigcup_{\bar{\beta}\in\{0,1\}^d} I_{\bar{\beta}} \quad ,$$

where we refer to (1.48) for the notation. To avoid technical difficulties we again discuss only the case $\bar{\beta} = (1, ..., 1, 0, ..., 0)$ with $|\bar{\beta}|_1 = n$. Hence for $\bar{\ell} \in I_{\bar{\beta}}$ we arrive at

$$\begin{aligned} |f_{\bar{\ell}}(x)| = \Big| \sum_{k\in\mathbb{Z}^d} \big[&(\triangle^{m_1}_{2^{-\ell_1}k_1} - \triangle^{m_1}_{2^{-\ell_1+1}k_1})\psi(0) \\ &\cdot (\triangle^{m_2}_{2^{-\ell_2}k_2} - \triangle^{m_2}_{2^{-\ell_2+1}k_2})\psi(0) \\ &\ \vdots \\ &\cdot (\triangle^{m_n}_{2^{-\ell_n}k_n} - \triangle^{m_n}_{2^{-\ell_n+1}k_n})\psi(0) \\ &\cdot \varphi^{n+1}_0(k_{n+1}) \cdot ... \cdot \varphi^d_0(k_d)\big] c_k(f) e^{ik\cdot x} \Big| \,. \end{aligned}$$

Together with (1.81) we obtain

$$\begin{aligned} |f_{\bar{\ell}}(x)| = \Big| &\sum_{\mu_{n+1}=0}^{m_{n+1}-1} \cdots \sum_{\mu_d=0}^{m_d-1} C^{m_{n+1}}_{\mu_{n+1}} \cdot ... \cdot C^{m_d}_{\mu_d} \times \\ &\times \sum_{k\in\mathbb{Z}^d} \big[(\triangle^{m_1}_{2^{-\ell_1}k_1} - \triangle^{m_1}_{2^{-\ell_1+1}k_1})\psi(0) \cdot ... \cdot (\triangle^{m_n}_{2^{-\ell_n}k_n} - \triangle^{m_n}_{2^{-\ell_n+1}k_n})\psi(0) \times \\ &\times\ \psi((m_{n+1} - \mu_{n+1})k_{n+1}) \cdot ... \cdot \psi((m_d - \mu_d)k_d)\big] c_k(f) e^{ik\cdot x} \Big| \quad , \end{aligned} \tag{1.83}$$

where $C^m_\mu := (-1)^\mu \binom{m}{\mu}$ for $0 \le \mu \le m$. Since $f \in L_1(\mathbb{T}^d)$ Lemma 1.1 gives the following

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^d} \psi(\eta_1 k_1) \cdot ... \cdot \psi(\eta_d k_d) c_k(f) e^{ik\cdot x} \\ &= -(2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathcal{F}\big[(\psi \otimes ... \otimes \psi)(\cdot)\big](\bar{h}) f(x_1 + \eta_1 h_1, ..., x_d + \eta_d h_d) d\bar{h}. \end{aligned}$$

The function $g(\bar{h}) := \mathcal{F}\big[(\psi \otimes ... \otimes \psi)(\cdot)\big](\bar{h})$ belongs to $S(\mathbb{R}^d)$. Consequently, for every $r > 0$ there exists constant $c_r > 0$ such that for every $\bar{h} \in \mathbb{R}^d$

$$(1 + |h_1|^2)^{r/2} \cdot ... \cdot (1 + |h_d|^2)^{r/2} |g(\bar{h})| \leq c_r \tag{1.84}$$

holds. See also (1.2). Applied to (1.83) the differences on the right-hand side carry over to f, precisely

$$|f_{\bar{\ell}}(x)| \leq \tag{1.85}$$
$$c \int\limits_{\mathbb{R}^d} |g(\bar{h})| \cdot \big|(\triangle^{m_1}_{2^{-\ell_1}h_{1,1}} - \triangle^{m_1}_{2^{-\ell_1+1}h_{1,1}}) \circ ... \circ (\triangle^{m_n}_{2^{-\ell_n}h_{n,n}} - \triangle^{m_n}_{2^{-\ell_n+1}h_{n,n}})(L_{\bar{h}} f)(x)\big| \, d\bar{h} \quad ,$$

where

$$\begin{aligned} L_{\bar{h}} f(x) \quad &= \quad \sum_{\mu_{n+1}=0}^{m_{n+1}-1} \cdots \sum_{\mu_d=0}^{m_d-1} C^{m_{n+1}}_{\mu_{n+1}} \cdot ... \cdot C^{m_d}_{\mu_d} \times \\ & \quad \times f(x_1, ..., x_n, x_{n+1} + (m_{n+1} - \mu_{n+1})h_{n+1}, ..., x_d + (m_d - \mu_d)h_d). \end{aligned} \tag{1.86}$$

Consulting (1.52) we notice that $L_{\bar{h}} f(x)$ is almost a mixed difference. Appropriate decomposition of RHS(1.86) gives us precisely a sum of mixed differences. Due to technical reasons we rewrite (1.86) in

$$\begin{aligned} L_{\bar{h}} f(x) = & \sum_{\mu_{n+1}=1}^{m_{n+1}} \cdots \sum_{\mu_d=1}^{m_d} C^{m_{n+1}}_{m_{n+1}-\mu_{n+1}} \cdot ... \cdot C^{m_d}_{m_d-\mu_d} \times \\ & \times f(x_1, ..., x_n, x_{n+1} + \mu_{n+1}h_{n+1}, ..., x_d + \mu_d h_d). \end{aligned}$$

See again (1.48) for our predefined index-sets. For abbreviation we define the quantity $C^{\bar{m}}_{\bar{\mu}} := \prod\limits_{i=1}^{d} C^{m_i}_{\mu_i}$. Now it follows

$$\begin{aligned} L_{\bar{h}} f(x) = & \sum_{\substack{\bar{\alpha} \in \bar{E}_{\bar{\beta}} \\ |\bar{\alpha}|_1 = d-n}} \sum_{\bar{\mu} \in M_{\bar{\alpha}}} C^{\bar{m}}_{\bar{m}-\bar{\mu}} \cdot f(x + \bar{\alpha} * \bar{\mu} * \bar{h}) \\ & - \sum_{\substack{\bar{\alpha} \in \bar{E}_{\bar{\beta}} \\ |\bar{\alpha}|_1 = d-n-1}} \sum_{\bar{\mu} \in M_{\bar{\alpha}}} C^{\bar{m}}_{\bar{m}-\bar{\mu}} \cdot f(x + \bar{\alpha} * \bar{\mu} * \bar{h}) \\ & + \sum_{\substack{\bar{\alpha} \in \bar{E}_{\bar{\beta}} \\ |\bar{\alpha}|_1 = d-n-2}} \sum_{\bar{\mu} \in M_{\bar{\alpha}}} C^{\bar{m}}_{\bar{m}-\bar{\mu}} \cdot f(x + \bar{\alpha} * \bar{\mu} * \bar{h}) \\ & \quad \vdots \\ & \pm f(x). \end{aligned} \tag{1.87}$$

Obviously

$$|\triangle^{\bar m}_{\bar h,\bar\alpha} f(x)| = \Big|\sum_{\bar\mu\in M_{\bar\alpha}} C^{\bar m}_{\bar m-\bar\mu}\cdot f(x+\bar\alpha*\bar\mu*\bar h)\Big|.$$

Together with (1.87) and (1.53) this implies

$$L_{\bar h}f(x) = \sum_{\bar\alpha\in\bar E_{\bar\beta}} \varepsilon_{\bar\alpha}\triangle^{\bar m}_{\bar h,\bar\alpha} f(x) \quad , \tag{1.88}$$

where the $\varepsilon_{\bar\alpha}\in\{-1,1\}$ were chosen suitable. Putting (1.88) into (1.85) and using the triangle-inequality we obtain the following

$$|f_{\bar\ell}(x)| \le c\sum_{\bar u\in E_{\bar\beta}}\sum_{\bar\alpha\in\bar E_{\bar\beta}}\int\limits_{\mathbb{R}^d}|g(\bar h)|\cdot\big|\triangle^{\bar m}_{2^{-(\bar\ell-\bar u)}*\bar h,\bar\beta}\circ\triangle^{\bar m}_{\bar h,\bar\alpha} f(x)\big|\,d\bar h.$$

We will write $\ell_q(\mathcal{I})$ instead of ℓ_q for some $\mathcal{I}\subset\mathbb{N}_0^d$ to indicate which index-set $\mathcal{I}$ belongs to ℓ_q. Altogether, this gives an estimate for $\|2^{\bar r\cdot\bar\ell}f_{\bar\ell}(x)|L_p(\mathbb{T}^d,\ell_q(\mathbb{N}_0^d))\|$, namely

$$\begin{aligned}
&\|2^{\bar r\cdot\bar\ell}f_{\bar\ell}(x)|L_p(\mathbb{T}^d,\ell_q(\mathbb{N}_0^d))\|\\
&\le c_1\sum_{\bar\beta\in\{0,1\}^d}\|2^{\bar r\cdot\bar\ell}f_{\bar\ell}(x)|L_p(\mathbb{T}^d,\ell_q(I_{\bar\beta}))\|\\
&\le c_2\sum_{\bar\beta\in\{0,1\}^d}\sum_{\bar u\in E_{\bar\beta}}\sum_{\bar\alpha\in\bar E_{\bar\beta}}\Big\|2^{\bar r\cdot\bar\ell}\int\limits_{\mathbb{R}^d}|g(\bar h)|\cdot\big|\triangle^{\bar m}_{2^{-(\bar\ell-\bar u)}*\bar h,\bar\beta}\circ\triangle^{\bar m}_{\bar h,\bar\alpha} f(x)\big|\,d\bar h\Big|L_p(\mathbb{T}^d,\ell_q(I_{\bar\beta}))\Big\|.
\end{aligned} \tag{1.89}$$

RHS(1.89) can be increased if we use the index-set $N_{\bar\beta}$ instead of $I_{\bar\beta}$ in the ℓ_q-norm and replace $\bar\ell-\bar u$ by $\bar\ell$. We get

$$\begin{aligned}
&\|2^{\bar r\cdot\bar\ell}f_{\bar\ell}(x)|L_p(\mathbb{T}^d,\ell_q(\mathbb{N}_0^d))\|\\
&\le c_2\sum_{\bar\beta\in\{0,1\}^d}\sum_{\bar u\in E_{\bar\beta}}\sum_{\bar\alpha\in\bar E_{\bar\beta}}\Big\|2^{\bar r\cdot\bar\ell}\int\limits_{\mathbb{R}^d}|g(\bar h)|\cdot\big|\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\beta}\circ\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\alpha} f(x)\big|\,d\bar h\Big|L_p(\mathbb{T}^d,\ell_q(N_{\bar\beta}))\Big\|\\
&= c_2\sum_{\bar\beta\in\{0,1\}^d}\sum_{\bar u\in E_{\bar\beta}}\sum_{\bar\alpha\in\bar E_{\bar\beta}}\Big\|2^{\bar r\cdot\bar\ell}\int\limits_{\mathbb{R}^d}|g(\bar h)|\cdot\big|\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\beta}\circ\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\alpha} f(x)\big|\,d\bar h\Big|L_p(\mathbb{T}^d,\ell_q(N_{\bar\alpha+\bar\beta}))\Big\|.
\end{aligned}$$

Because of

$$\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\beta}\circ\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\alpha}f(x) = \triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\alpha+\bar\beta}f(x)$$

for $\bar\alpha\in\bar E_{\bar\beta}$ it holds that

$$\begin{aligned}
\|f|S^{\bar r}_{p,q}F)\|^{\bar\varphi} &\le c_3\sum_{\bar\beta\in\{0,1\}^d}\sum_{\bar\alpha\in\bar E_{\bar\beta}}\Big\|2^{\bar r\cdot\bar\ell}\int\limits_{\mathbb{R}^d}|g(\bar h)|\cdot\big|\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\alpha+\bar\beta} f(x)\big|\,d\bar h\Big|L_p(\mathbb{T}^d,\ell_q(N_{\bar\alpha+\bar\beta}))\Big\|\\
&= c_3\sum_{\bar\beta\in\{0,1\}^d}\Big\|2^{\bar r\cdot\bar\ell}\int\limits_{\mathbb{R}^d}|g(\bar h)|\cdot\big|\triangle^{\bar m}_{2^{-\bar\ell}*\bar h,\bar\beta} f(x)\big|\,d\bar h\Big|L_p(\mathbb{T}^d,\ell_q(N_{\bar\beta}))\Big\|.
\end{aligned}$$

It remains to estimate the summand

$$\left\| 2^{\bar{r}\cdot\bar{\ell}} \int_{\mathbb{R}^d} |g(\bar{h})| \cdot \left| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x) \right| d\bar{h} \right| L_p(\mathbb{T}^d, \ell_q(N_{\bar{\beta}})) \Bigg\| \tag{1.90}$$

for every $\bar{\beta} \in \{0,1\}^d$. In the case $|\bar{\beta}|_1 = 0$ it degenerates to

$$\left\| |f(x)| \cdot \int_{\mathbb{R}^d} |g(\bar{h})| \, d\bar{h} \right| L_p(\mathbb{T}^d) \Bigg\| \quad ,$$

which equals $\|g|L_1(\mathbb{R}^d)\| \cdot \|f|L_p(\mathbb{T}^d)\|$. We will finish the proof by estimating (1.90) from above by $c \cdot S^R_{\bar{\beta}}(f)$ in case $|\bar{\beta}|_1 > 0$. For this purpose we want to discretisise the integral appearing in (1.90) similar to in Step 1. For the used notation we refer to (1.46) and (1.47). Obviously

$$\int_{\mathbb{R}^d} |g(\bar{h})| \cdot \left| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x) \right| d\bar{h} = \sum_{\bar{\mu}\in\mathbb{N}_0^d} \int_{Q^{\Delta}_{\bar{\mu}}} |g(\bar{h})| \cdot \left| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x) \right| d\bar{h}.$$

Together with (1.84) we obtain for any $s > 0$

$$\begin{aligned}
\int_{\mathbb{R}^d} |g(\bar{h})| \cdot \left| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x) \right| d\bar{h} &\le c_s \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{-|\bar{\mu}|\cdot s} \int_{Q^{\Delta}_{\bar{\mu}}} \left| \triangle^{\bar{m}}_{2^{-\bar{\ell}}*\bar{h},\bar{\beta}} f(x) \right| d\bar{h} \\
&\le c_s \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{-|\bar{\mu}|\cdot s} 2^{\bar{\ell}\cdot\bar{\beta}} \int_{Q_{\bar{\mu}-\bar{\beta}*\bar{\ell}}} \left| \triangle^{\bar{m}}_{\bar{h},\bar{\beta}} f(x) \right| d\bar{h} \\
&= c_s \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{|\bar{\mu}|(1-s)} 2^{-\bar{\beta}\cdot(\bar{\mu}-\bar{\ell})} 2^{-\bar{\mu}(\bar{1}-\bar{\beta})} \int_{Q_{\bar{\mu}-\bar{\beta}*\bar{\ell}}} \left| \triangle^{\bar{m}}_{\bar{h},\bar{\beta}} f(x) \right| d\bar{h}.
\end{aligned}$$

Consequently, we have for (1.90) in case $q \le 1$

$$(1.90) \le c_s \left\| \left[\sum_{\bar{\ell}\in N_{\bar{\beta}}} \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q} 2^{|\bar{\mu}|(1-s)q} 2^{-\bar{\beta}\cdot(\bar{\mu}-\bar{\ell})q} 2^{-\bar{\mu}(\bar{1}-\bar{\beta})q} \left(\int_{Q_{\bar{\mu}-\bar{\beta}*\bar{\ell}}} \left| \triangle^{\bar{m}}_{\bar{h},\bar{\beta}} f(x) \right| d\bar{h} \right)^q \right]^{1/q} \right| L_p(\mathbb{T}^d) \Bigg\|.$$

The next step is to estimate the sum over $\bar{\ell}$ by an integral with respect to $\bar{t}$. Having an arbitrary tuple $\bar{t} \in Q^{+\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}$ we can rewrite the previous estimate to

$$(1.90) \le c_1 \left\| \left[\sum_{\bar{\ell}\in N_{\bar{\beta}}} \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{\bar{r}\bar{\ell}q} 2^{|\bar{\mu}|(1-s)q} \left(\frac{1}{\bar{t}} \int_{[-t_1,t_1]\times\ldots\times[-t_d,t_d]} \left| \triangle^{\bar{m}}_{\bar{h},\bar{\beta}} f(x) \right| d\bar{h} \right)^q \right]^{1/q} \right| L_p(\mathbb{T}^d) \Bigg\| \; .$$

We use $2^{\bar{r}\bar{\ell}} = 2^{\bar{\beta}*(\bar{\ell}-\bar{\mu})\bar{r}} \cdot 2^{(\bar{\beta}*\bar{\mu})\cdot\bar{r}} \sim \bar{t}^{-\bar{r}*\bar{\beta}} \cdot 2^{(\bar{\beta}*\bar{\mu})\cdot\bar{r}}$ and obtain

$$
\begin{aligned}
(1.90) &\le c_2 \Big\| \Big[\sum_{\bar{\ell}\in N_{\bar{\beta}}} \sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{(\bar{\beta}*\bar{\mu})\cdot\bar{r}q} 2^{|\bar{\mu}|(1-s)q} \bar{t}^{-(\bar{r}*\bar{\beta})q} \Big(\int\limits_{[-1,1]^d} \big| \triangle^{\bar{m}}_{\bar{t}*\bar{h},\bar{\beta}} f(x) \big| \, d\bar{h} \Big)^q \Big]^{1/q} \Big| L_p(\mathbb{T}^d) \Big\| \\
&= c_2 \Big\| \Big[\sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{\bar{\mu}\cdot(1-s+\bar{\beta}*\bar{r})q} \sum_{\bar{\ell}\in N_{\bar{\beta}}} \bar{t}^{-(\bar{r}*\bar{\beta})q} \Big(\int\limits_{[-1,1]^d} \big| \triangle^{\bar{m}}_{\bar{t}*\bar{h},\bar{\beta}} f(x) \big| \, d\bar{h} \Big)^q \Big]^{1/q} \Big| L_p(\mathbb{T}^d) \Big\| .
\end{aligned}
\tag{1.91}
$$

It should be mentioned that the chosen $\bar{t}$ depends on the summation index $\bar{\ell}$. Because of measure theoretical reasons there must be a $\bar{t}(x,\bar{\ell}) \in Q^{+\,\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}$, such that the integral average of the function

$$
h_x(\bar{t}) := \bar{t}^{-(\bar{r}*\bar{\beta})q} \Big(\int\limits_{[-1,1]^d} \big| \triangle^{\bar{m}}_{\bar{t}*\bar{h},\bar{\beta}} f(x) \big| \, d\bar{h} \Big)^q
$$

with respect to the rectangle $Q^{+\,\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}$ is greater than or equal $h_x(\bar{t})$. It is also remarkable that h_x is invariant in the components t_i, which correspond to $\beta_i = 0$. Later this will be important later. Hence, we can replace $h_x(\bar{t})$ in (1.91) by

$$
\frac{1}{|Q^{+\,\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}|} \int\limits_{Q^{+\,\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}} h_x(\bar{t}) \, d\bar{t} \, .
$$

And this can be estimated from above by

$$
c \int\limits_{Q^{+\,\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}} h_x(\bar{t}) \, \frac{d\bar{t}}{\bar{t}} \quad ,
$$

where c is independent of x and $\bar{\ell}$. This yields

$$
(1.90) \le c_3 \Big\| \Big[\sum_{\bar{\mu}\in\mathbb{N}_0^d} 2^{\bar{\mu}\cdot(1-s+\bar{\beta}*\bar{r})q} \sum_{\bar{\ell}\in N_{\bar{\beta}}} \int\limits_{Q^{+\,\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}} h_x(\bar{t}) \, \frac{d\bar{t}}{\bar{t}} \Big]^{1/q} \Big| L_p(\mathbb{T}^d) \Big\| . \tag{1.92}
$$

Obviously the integration over the invariance-components of h_x breaks down to a constant. Finally it holds

$$
\sum_{\bar{\ell}\in N_{\bar{\beta}}} \int\limits_{Q^{+\,\Delta}_{\bar{\mu}-\bar{\beta}*\bar{\ell}+1}} h_x(\bar{t}) \, \frac{d\bar{t}}{\bar{t}} \le c \int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q} \Big) \Big(\int\limits_{[-1,1]^n} \big| \big(\triangle^{m_{\delta_1}}_{t_1 h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n,\delta_n} f \big)(x) \big| d\bar{h} \Big)^q \frac{d\bar{t}}{\bar{t}} \, .
$$

Thus we lost the $\bar{\mu}$-dependence. Putting this into (1.92), the $\bar{\mu}$-sum is nothing more than a convergent geometric series for sufficiently large s. This finishes the proof in case $q \le 1$. In case $1 < q < \infty$ we use the triangle inequality in ℓ_q to interchange $\bar{\mu}$- and $\bar{\ell}$-sum. The modifications in case $q = \infty$ are obvious. □

Remark 1.14 *Let us give a remark concerning Step 1 of the proof. If we would not exploit the integral structure of the rectangle means after (1.73) and continue similar as before, the method works only in case $\bar{r} > 1/\min(p,q)$. At exactly this point one gets an idea, why the integral means are more powerful than classical difference constructions (which will be discussed later, see Paragraph 1.6.6). Namely, the integral in RHS(1.71) allows the use of Hardy-Littlewood maximal functions. Here we need the chosen quantities λ and $\bar{a}$. On the one hand λ and $\bar{a}$ force the convergence of the $\bar{\ell}$-sum in (1.76) and on the other hand $\lambda < \min(p,q)$ is used to apply the corresponding maximal inequalities in (1.79).*

Remark 1.15 *It has been proved recently by Christ and Seeger, cf. [7], that at least in the isotropic case the condition $r > \sigma_{p,q}$ is necessary. We expect the same in our situation. See also [49, 1.11.9].*

1.6.5 Localization

As we have already seen, the philosophy of difference characterization is to test the smoothness of a function, checking the behavior of differences with small step lengths. Now the question arises, whether one can replace the $(0,\infty)$-integrals in (1.62) by $(0,1)$- or (what is essentially the same) by $(0,\varepsilon)$-integrals, where $\varepsilon > 0$. This would be a certain type of localization property, which holds unrestricted in the isotropic case, cf. [48, 2.5.11]. We present a partial result in the case of Banach spaces, except the constellation $1 = p < q \le \infty$, using complex interpolation.

Let $0 < p < \infty$, $0 < q \le \infty$ and $\bar{r} > 0$. For $f \in L_p(\mathbb{T}^d)$. Similar to (1.61) we define the quantity

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{R,L} := \|f|L_p(\mathbb{T}^d)\| + \sum_{|\bar{\beta}|_1 \ge 1} S^{R,L}_{\bar{\beta}}(f) \quad ,$$

where (modification in case $q = \infty$)

$$S^{R,L}_{\bar{\beta}}(f) = \Big\| \Big[\int_{(0,1)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\Big) \Big(\int_{[-1,1]^n} \big|(\triangle^{m_{\delta_1}}_{t_1 h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n,\delta_n} f)(x)\big| d\bar{h} \Big)^q \frac{d\bar{t}}{\bar{t}} \Big]^{1/q} \Big| L_p(\mathbb{T}^d) \Big\|$$

for $|\bar{\beta}|_1 \ge 1$.

Proposition 1.10 *Let $1 \le q \le p < \infty$ and $\bar{r} > 0$. Then there exists a constant $c > 0$ such that*

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^R \le c\, \|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{R,L}$$

for all $f \in L_p(\mathbb{T}^d)$.

Proof We use $p/q \geq 1$ and

$$\Big\| \int\limits_X g(x,y)\,dy|L_{p/q}(x)\Big\| \leq \int\limits_X \|g(x,y)|L_{p/q}(x)\|\,dy$$

in order to lose the $[1,\infty)$-integrals with respect to $\bar{t}$. For full details let us refer to [52, Prop. 3.5.1]. □

Proposition 1.11 *For $1 < p < \infty$ and $\bar{r} > 0$ there exists a constant $c > 0$ such that*

$$\|f|S^{\bar{r}}_{p,\infty}F(\mathbb{T}^d)\|^R \leq c\,\|f|S^{\bar{r}}_{p,\infty}F(\mathbb{T}^d)\|^{R,L}$$

for all $f \in L_p(\mathbb{T}^d)$.

Proof Fix $\bar{\beta} = (1,...,1,0,...,0)$. Because of $q = \infty$ we obtain the relation

$$S^R_{\bar{\beta}}(f) \leq c_1 \sum_{\bar{\alpha}\in E_{\bar{\beta}}} \Big\| \sup_{A_{\alpha_1}\times\cdots\times A_{\alpha_n}} \big(\prod_{i=1}^n t_i^{-r_i}\big)\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x)\Big|L_p(\mathbb{T}^d)\Big\|,$$

where $A_0 = (0,1)$, $A_1 = [1,\infty)$ and

$$\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x) = \int\limits_{-1}^{1}\cdots\int\limits_{-1}^{1} \big|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{t_nh_n,\delta_n}f)(x)\big|\,dh_n\cdots dh_1.$$

Now $\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x)$ can be partly rewritten as an iterated Hardy/Littlewood maximal operator, see (1.9) and (1.55). The classical scalar maximal inequality completes the proof. Again we refer to [52, Prop. 3.5.2] for the details. □

Finally we state the main theorem of this paragraph.

Theorem 1.11 *Let $p = q = 1$ or $1 < p < \infty$ and $1 \leq q \leq \infty$. Let further $\bar{r} = (r_1,...,r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1,...,m_d) \in \mathbb{N}^d$ with $\bar{m} > \bar{r} > \sigma_{p,q}$. Then $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ is the collection of all functions $f \in L_p(\mathbb{T}^d)$ satisfying*

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{R,L} < \infty.$$

Moreover $\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{R,L}$ is an equivalent (quasi-)norm in $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$.

Proof This is a consequence of Theorem 1.10, Proposition 1.10, 1.11 and complex interpolation. See also [52, Thm. 3.5.1]. □

1.6.6 Moduli of Smoothness and Differences

This paragraph deals with equivalent (quasi-)norms for $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$ using moduli of smoothness and differences itself. The price one has to pay are more restrictive conditions to $\bar{r}$, namely $\bar{r} > 1/\min(p,q)$, which cannot be essentially relaxed. The aim is to derive the results from [32, 2.3.3] for arbitrary d. Our proof is based on the characterization by integral means, cf. Theorem 1.10. The first step is a small modification of the integral means used in Theorem 1.10.

Proposition 1.12 *Let $p, q, \bar{r}$ be given as in Theorem 1.10. In the sense of equivalent (quasi-)norms the integral means*

$$\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x) = \int\limits_{[-1,1]^n} \left|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_nh_n,\delta_n} f)(x)\right| d\bar{h} \tag{1.93}$$

in (1.62) can be replaced by

$$\bar{\mathcal{R}}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x) = \int\limits_{\substack{1\geq|h_i|>1/2\\ i=1,\dots,n}} \left|(\triangle^{m_{\delta_1}}_{t_1h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_nh_n,\delta_n} f)(x)\right| d\bar{h}\ . \tag{1.94}$$

Proof Concerning the proof we refer to [52, Prop. 3.6.1]. □

The following theorem is actually the counterpart of [48, Thm. 2.5.10] and [32, Thm. 2.3.3], respectively.

Theorem 1.12 *Let $0 < p < \infty$, $0 < q \leq \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}_0^d$ with $\bar{m} > \bar{r} > \frac{1}{\min(p,q)}$. The quantities*

$$\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{M/\Delta} = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\bar{\beta}|_1\geq 1} S^{M/\Delta}_{\bar{\beta}}(f)$$

are equivalent (quasi-)norms in $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$. In case $|\bar{\beta}|_1 = n \geq 1$ we put

(i)

$$S^M_{\bar{\beta}}(f) := \left\|\left[\int\limits_{(0,\infty)^n} \Big(\prod_{i=1}^n t_i^{-r_{\delta_i}q}\Big) \sup_{\substack{|h_i|\leq t_i\\ i=1,\dots,n}} \left|(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\right|^q \frac{d\bar{t}}{\bar{t}}\right]^{1/q} \Big| L_p(\mathbb{T}^d)\right\| \quad \textit{and}$$

(ii)

$$S^{\Delta}_{\bar{\beta}}(f) := \left\|\left[\int\limits_{\mathbb{R}^n} \Big(\prod_{i=1}^n |h_i|^{-r_{\delta_i}q}\Big) \left|(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\right|^q \frac{d\bar{h}}{\tilde{h}}\right]^{1/q} \Big| L_p(\mathbb{T}^d)\right\|$$

with $\tilde{h} = (|h_1|, ..., |h_n|)$ *and* $\bar{\delta}$ *as above. In case* $q = \infty$ *one modifies*

$$S^{M/\Delta}_{\bar{\beta}}(f) = \Big\| \sup_{\substack{\bar{h}\in\mathbb{R}^n \\ h\neq 0}} \Big(\prod_{i=1}^{n} |h_i|^{-r_{\delta_i}}\Big) \big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f)(x)\big| \Big| L_p(\mathbb{T}^d) \Big\|.$$

Proof *Step 1.* We show $\|f|S^{\bar{r}}_{p,q}F\|^M \leq c\|f|S^{\bar{r}}_{p,q}F\|^{\bar{\varphi}}$ following Step 1 of the proof of Theorem 1.10. First we discretisize the $\bar{t}$-integration and obtain (with the same notation used in Theorem 1.10)

$$\begin{aligned} A_{\bar{\beta}}(x) = &\int\limits_{[0,\infty]^n} \Big(\prod_{i=1}^{n} t_i^{-r_{\delta_i} q}\Big) \sup_{\substack{|h_i|\leq t_i \\ i=1,...,n}} \big|\big(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f\big)(x)\big|^q \frac{d\bar{t}}{\bar{t}} \\ &\leq c_1 \sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}\bar{k}q} \sup_{\bar{h}\in[-1,1]^d} |\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar{\beta}} f(x)|^q. \end{aligned}$$

Now we use again (1.64) and obtain in case $q \leq 1$

$$A_{\bar{\beta}}(x) \leq c_2 \sum_{\bar{\ell}\in Z_{\bar{\beta}}} \sum_{\bar{u}\in \bar{N}_{\bar{\beta}}} \sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}\bar{k}q} \sup_{\bar{h}\in[-1,1]^d} |\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar{\beta}} f_{\bar{k}+\bar{\ell}+\bar{u}}(x)|^q.$$

In case $q > 1$ it holds an estimate similar to (1.66). What follows is much simpler on the one hand, but much more restrictive on the other hand. Having no integral means for estimating differences by Hardy-Littlewood maximal functions we argue as follows. $Z_{\bar{\beta}}$ is decomposed as done before, but now we directly combine the modified estimates (1.70) and (1.72), i.e.

$$\begin{aligned} \sup_{\bar{h}\in[-1,1]^d} |\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h}} f_{\bar{k}+\bar{\ell}+\bar{u}}(x)| &= \sup_{\bar{h}\in[-1,1]^d} |\triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar{\beta}} \circ \triangle^{\bar{m}}_{2^{-\bar{k}}*\bar{h},\bar{\beta}-\bar{\alpha}} f_{\bar{k}+\bar{\ell}+\bar{u}}(x)| \\ &\leq c_3 2^{\ell_1 m_1} \cdot ... \cdot 2^{\ell_{|\bar{\alpha}|_1} m_{|\bar{\alpha}|_1}} 2^{\ell_{|\bar{\alpha}|_1+1} a_{|\bar{\alpha}|_1+1}} \cdot ... \cdot 2^{\ell_n a_n} P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})(x). \end{aligned}$$

Recall that $\bar{\beta} = (1, ..., 1, 0, ..., 0)$ with $|\bar{\beta}|_1 = n$, $\bar{\alpha} = (1, ..., 1, 0, ..., 0)$ with $\bar{\alpha} \leq \bar{\beta}$ and $\bar{b} = (2^{k_1+\ell_1}, ..., 2^{k_d+\ell_d})$. Now the stronger condition $\bar{r} > 1/\min(p, q)$ is required. In this case it is possible to choose a tuple $\bar{a} = (a_1, ..., a_d)$ such that $\bar{r} > \bar{a} > 1/\min(p, q)$. Proceeding similar to Theorem 1.10 we get the estimate

$$\begin{aligned} A^{\bar{\alpha}}_{\bar{\beta}}(x) \leq & c_4 \sum_{\bar{\ell}\in Z^{\bar{\alpha}}_{\bar{\beta}}} 2^{\ell_1(m_1-r_1)q} \cdot ... \cdot 2^{\ell_{|\bar{\alpha}|_1}(m_{|\bar{\alpha}|_1}-r_{|\bar{\alpha}|_1})q} \cdot 2^{\ell_{|\bar{\alpha}|_1+1}(a_{|\bar{\alpha}|_1+1}-r_{|\bar{\alpha}|_1+1})q} \cdot ... \cdot 2^{\ell_n(a_n-r_n)q} \\ &\cdot \sum_{\bar{u}\in\bar{N}_{\bar{\beta}}} \sum_{\bar{k}\in Z_{\bar{\beta}}} 2^{\bar{r}(\bar{k}+\bar{\ell})q} |P_{\bar{b},\bar{a},\bar{\beta}}(f_{\bar{k}+\bar{\ell}+\bar{u}})|^q(x) \end{aligned}$$

instead of (1.76). What remains is straightforward and essentially a consequence of Theorem 1.9.

Step 2. It is sufficient to show $\|f|S^{\bar{r}}_{p,q}F\|^{R'} \leq c\|f|S^{\bar{r}}_{p,q}F\|^{\Delta}$ for all $f \in S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$.
The case $q = \infty$ is trivial. Let us first consider the case $1 \leq q < \infty$. Again we refer to (1.54). For fixed $x \in \mathbb{T}^d$, Fubini's theorem and $L_q \hookrightarrow L_1$ on compact domains give

$$\int\limits_{(0,\infty)^n} (\prod_{i=1}^n t_i^{-r_{\delta_i} q}) \Bigg(\int\limits_{\substack{1\geq |h_i|>1/2\\ i=1,\ldots,n}} \Bigg| \Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Bigg| d\bar{h} \Bigg)^q \frac{d\bar{t}}{\bar{t}}$$

$$\leq c_1 \int\limits_{\mathbb{R}^n} \Bigg| \Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i, \delta_i} \Big) f(x) \Bigg|^q \cdot \int\limits_{\substack{|h_i|\leq t_i \leq 2|h_i|\\ i=1,\ldots,n}} (\prod_{i=1}^n t_i^{-r_{\delta_i} q}) \frac{1}{\bar{t}} \frac{d\bar{t}}{\bar{t}} \; d\bar{h} \quad ,$$

where

$$\int\limits_{\substack{|h_i|\leq t_i \leq 2|h_i|\\ i=1,\ldots,n}} (\prod_{i=1}^n t_i^{-r_{\delta_i} q}) \frac{1}{\bar{t}} \frac{d\bar{t}}{\bar{t}} \; \leq \; c_2 \frac{\prod_{i=1}^n |h_i|^{-r_{\delta_i} q}}{\prod_{i=1}^n |h_i|} \, .$$

It remains to discuss the case $0 < q < 1$. We have

$$\Bigg(\int\limits_{\substack{1\geq |h_i|>1/2\\ i=1,\ldots,n}} \Bigg| \Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i, \delta_i} \Big) f(x) \Bigg| d\bar{h} \Bigg)^q \leq \sup_{\substack{|h_i|\leq t_i\\ i=1,\ldots,n}} \Bigg| \Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i, \delta_i} \Big) f(x) \Bigg|^{q(1-q)} \times$$

$$\times \Bigg(\frac{1}{\bar{t}} \int\limits_{\substack{t_i\geq |h_i|>t_i/2\\ i=1,\ldots,n}} \Bigg| \Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i, \delta_i} \Big) f(x) \Bigg|^q d\bar{h} \Bigg)^q .$$

Now we apply Hölder's inequality with the exponents $\frac{1}{q}$ and $\frac{1}{1-q}$ and achieve

$$\Bigg[\int\limits_{(0,\infty)^n} (\prod_{i=1}^n t_i^{-r_{\delta_i} q}) \Bigg(\int\limits_{\substack{1\geq |h_i|>1/2\\ i=1,\ldots,n}} \Bigg| \Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{t_i h_i, \delta_i} \Big) f(x) \Bigg| d\bar{h} \Bigg)^q \frac{d\bar{t}}{\bar{t}} \Bigg]^{1/q}$$

$$\leq \Bigg[\int\limits_{(0,\infty)^n} (\prod_{i=1}^n t_i^{-r_{\delta_i} q}) \frac{1}{\bar{t}} \int\limits_{\substack{t_i\geq |h_i|>t_i/2\\ i=1,\ldots,n}} \Bigg| \Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i, \delta_i} \Big) f(x) \Bigg|^q d\bar{h} \, \frac{d\bar{t}}{\bar{t}} \Bigg] \times$$

$$\times \Bigg[\int\limits_{(0,\infty)^n} (\prod_{i=1}^n t_i^{-r_{\delta_i} q}) \sup_{\substack{|h_i|\leq t_i\\ i=1,\ldots,n}} \Bigg| \Big(\prod_{i=1}^n \triangle^{m_{\delta_i}}_{h_i, \delta_i} \Big) f(x) \Bigg|^q \Bigg]^{\frac{1-q}{q}} .$$

Using Fubini and Hölder's inequality again, we obtain the estimate

$$S^{R'}_{\bar{\beta}}(f) \leq \big(\|f|S^{\bar{r}}_{p,q}F\|^{\Delta} \big)^q \cdot \big(\|f|S^{\bar{r}}_{p,q}F\|^{M} \big)^{1-q} .$$

Having additionally

$$\begin{aligned}\|f|L_p\| &\leq \|f|L_p\|^q \cdot \|f|L_p\|^{1-q} \\ &\leq \big(\|f|S^{\bar{r}}_{p,q}F\|^{\Delta}\big)^q \cdot \big(\|f|S^{\bar{r}}_{p,q}F\|^{M}\big)^{1-q}\end{aligned}$$

it turns out, that

$$\|f|S^{\bar{r}}_{p,q}F\|^{R'} \leq c_3\big(\|f|S^{\bar{r}}_{p,q}F\|^{\Delta}\big)^q \cdot \big(\|f|S^{\bar{r}}_{p,q}F\|^{M}\big)^{1-q}.$$

With the help of Step 1 and Proposition 1.12 we can use

$$\|f|S^{\bar{r}}_{p,q}F\|^{M} \leq c'\|f|S^{\bar{r}}_{p,q}F\|^{R'}$$

to finish the proof. □

Remark 1.16 *Step 2 proves* $\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{R'} \leq c\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{M}$ *on the basis of* $\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{R'} \leq c\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{\Delta}$. *Of course, this can also be proved directly by using the triangle-inequality for integrals starting with* $\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{R}$. *Hence, we got even more than stated in the theorem. Under the given conditions* $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ *is the collection of all* $f \in S'(\mathbb{R}^d) \cap L_p(\mathbb{T}^d)$ *such that* $\|f|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{M}$ *is finite.*

Remark 1.17 *With a strategy similar to the proof of Proposition 1.10 one can replace* $S^{\Delta}_{\bar{\beta}}(f)$, $|\bar{\beta}|_1 \geq 1$, *by*

$$S^{\Delta,L}_{\bar{\beta}}(f) := \Bigg\|\Bigg[\int\limits_{[-1,1]^n} \Big(\prod_{i=1}^n |h_i|^{-r_{\delta_i}q}\Big)\big|(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ\cdots\circ\triangle^{m_{\delta_n}}_{h_n,\delta_n}f)(x)\big|^q\,\frac{d\bar{h}}{\tilde{h}}\Bigg]^{1/q}\Bigg|L_p(\mathbb{T}^d)\Bigg\|$$

in the case $0 < q \leq p < \infty$. *Hence, the corresponding quantity* $\|\cdot|S^{\bar{r}}_{p,q}F(\mathbb{T}^d)\|^{\Delta,L}$ *is an equivalent (quasi-)norm in* $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ *for* $\bar{r} > \big(\frac{1}{q}-1\big)_+$.

1.6.7 Characterizations of $S^r_{p,q}B(\mathbb{T}^d)$

This paragraph deals with Besov spaces of dominating mixed smoothness property. We give a characterization of $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$, $0 < p \leq \infty$, $0 < q \leq \infty$ and $\bar{r} > \sigma_p$, using integral means of differences. Our main theorem is the counterpart of Theorem 1.10. In fact, it is not surprising, that our techniques work in B-case too. The situation here is much more simple. For the proofs we refer to [52, 3.7]. We employ scalar maximal inequalities instead of corresponding inequalities for the vector-valued case, cf. Section 1.3. Consequently, the condition to $\bar{r}$ gets independent of q and therefore it is possible to give a characterization for $\bar{r} > \sigma_p$. Our main result reads as follows.

Theorem 1.13 *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}_0^d$ with $\bar{m} > \bar{r} > \sigma_p$. Under these conditions the space $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ is the collection of all functions $f \in L_p(\mathbb{T}^d) \cap L_1(\mathbb{T}^d)$ such that*

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^R = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\beta|_1 \geq 1} S^R_{\bar{\beta}}(f) < \infty \quad .$$

For $|\bar{\beta}|_1 = n \geq 1$ we have

$$S^R_{\bar{\beta}}(f) = \Big[\int\limits_{(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big) \Big\| \int\limits_{[-1,1]^n} \big|\big(\triangle^{m_{\delta_1}}_{t_1 h_1, \delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n, \delta_n} f\big)(x)\big| d\bar{h} \Big| L_p \Big\|^q \frac{d\bar{t}}{\bar{t}} \Big]^{1/q} \qquad (1.95)$$

with $\bar{\delta}$ in the sense of (1.45). In case $q = \infty$ one has to replace (1.95) by

$$S^R_{\bar{\beta}}(f) = \sup_{\bar{t}\in(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i}}\big) \Big\| \int\limits_{[-1,1]^n} \big|(\triangle^{m_{\delta_1}}_{t_1 h_1, \delta_1} \circ \cdots \circ \triangle^{m_{\delta_n}}_{t_n h_n, \delta_n} f)(x)\big| \, d\bar{h} \Big| L_p(\mathbb{T}^d) \Big\|.$$

Moreover, $\| \cdot |S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^R$ is an equivalent (quasi-)norm in $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$.

Remark 1.18 *(Localization) As already done in the F-case we consider the question, whether one can replace the $(0,\infty)$-integrals in (1.95) by $(0,1)$-integrals. We obtain a positive answer for $1 \leq p \leq \infty$ and $0 < q \leq \infty$. The restriction $p \geq 1$ has only technical reasons (Minkowski's inequality) and does not seem to be a natural condition. In particular for $p < 1$ the problem is open.*

Let us state the results from [32, 2.3.4] for arbitrary d based on Theorem 1.13. See also [48, 2.5.12] for the isotropic case. We use again modified integral means, see (1.93), (1.94). Similar to Proposition 1.12 we can replace in (1.95) the quantity $\mathcal{R}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x)$ by $\bar{\mathcal{R}}^{\bar{m}}_{\bar{t},\bar{\beta}}f(x)$ in the sense of equivalent (quasi-)norms.
In what follows we need to compute moduli of smoothness with respect to a rectangle, given by

$$\big(\prod_{i=1}^n t_i^{-r_{\delta_i}}\big) \sup_{\substack{|h_i|\leq t_i \\ i=1,...,n}} \big\|\big(\triangle^{m_{\delta_1}}_{h_1,\delta_1} \circ ... \circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f\big)(x)\big|L_p(\mathbb{T}^d)\big\| \ , \ t_i > 0 \ , \ i = 1, ..., n \ .$$

The proof of the remaining two theorems are similar to the proof of Theorem 1.12. Details can be found in [52, 3.8]. Let us first treat the case $1 \leq p \leq \infty$. Having powerful techniques in this case (generalized Minkowski's inequality), we are able to cover all spaces with $\bar{r} > 0$.

Theorem 1.14 *Let $1 \leq p \leq \infty$, $0 < q \leq \infty$, $\bar{r} = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar{m} = (m_1, ..., m_d) \in \mathbb{N}_0^d$ with $\bar{m} > \bar{r} > 0$. The following quantities describe equivalent (quasi-)norms in the space $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$:*

$$\|f|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{M/\Delta} = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar{\beta}\in\{0,1\}^d, |\bar{\beta}|_1 \geq 1} S^{M/\Delta}_{\bar{\beta}}(f) \quad ,$$

where for $|\bar{\beta}|_1 = n \geq 1$

(i)

$$S^M_{\bar\beta}(f) = \Bigg(\int\limits_{(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big) \sup_{\substack{|h_i|\le t_i\\ i=1,...,n}} \big\|\big(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ ...\circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f\big)(x)\big|L_p(\mathbb{T}^d)\big\|^q \frac{d\bar t}{\bar t}\Bigg)^{1/q} \quad \text{and}$$

(ii)

$$S^\Delta_{\bar\beta}(f) = \Bigg[\int\limits_{\mathbb{R}^n} \big(\prod_{i=1}^n |h_i|^{-r_{\delta_i} q}\big) \big\|\big(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ \cdots\circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f\big)(x)\big|L_p(\mathbb{T}^d)\big\|^q \frac{d\bar h}{\tilde h}\Bigg]^{1/q} \quad ,$$

with $\tilde h = (|h_1|, ..., |h_n|)$ and $\bar\delta$ as usual (modification if $q=\infty$).

Remark 1.19 *Under the assumptions of the last theorem one can characterize the space $S^{\bar r}_{p,q}B(\mathbb{T}^d)$ by $\|\cdot|S^{\bar r}_{p,q}B(\mathbb{T}^d)\|^M$. See also Remark 1.16.*

The last theorem in this section deals with the case $0 < p < 1$. We are able to give a result for $\bar r > 1/p$, but not for $\bar r > \sigma_p = 1/p - 1$, like is possible in the isotropic case, cf. [48, 2.5.12]. This problem still remains open, cf. also [32, Remark 2.3.4/2].

Theorem 1.15 *Let $0 < p, q \le \infty$, $\bar r = (r_1, ..., r_d) \in \mathbb{R}^d$ and $\bar m = (m_1, ..., m_d) \in \mathbb{N}_0^d$ with $\bar m > \bar r > 1/p$. The following quantities describe equivalent (quasi-)norms in the space $S^{\bar r}_{p,q}B(\mathbb{T}^d)$:*

$$\|f|S^{\bar r}_{p,q}B(\mathbb{T}^d)\|^{M/M'/\Delta} = \|f|L_p(\mathbb{T}^d)\| + \sum_{\bar\beta\in\{0,1\}^d, |\bar\beta|_1\ge 1} S^{M/M'/\Delta}_{\bar\beta}(f) \quad ,$$

where we define additionally to Theorem 1.14/(i)/(ii) for $|\bar\beta|_1 = n \ge 1$ the quantity

$$S^{M'}_{\bar\beta}(f) = \Bigg(\int\limits_{(0,\infty)^n} \big(\prod_{i=1}^n t_i^{-r_{\delta_i} q}\big) \Big\| \sup_{\substack{|h_i|\le t_i\\ i=1,...,n}} \big|\big(\triangle^{m_{\delta_1}}_{h_1,\delta_1}\circ ...\circ \triangle^{m_{\delta_n}}_{h_n,\delta_n} f\big)(x)\big| \Big|L_p(\mathbb{T}^d)\Big\|^q \frac{d\bar t}{\bar t}\Bigg)^{1/q}$$

with $\bar\delta$ as usual (modification if $q=\infty$).

Remark 1.20 *In the proof of [52, Thm. 3.8.2] we showed even more. Similar to $S^{\bar r}_{p,q}F(\mathbb{T}^d)$ the space $S^{\bar r}_{p,q}B(\mathbb{T}^d)$ can be characterized by $\|\cdot|S^{\bar r}_{p,q}B(\mathbb{T}^d)\|^{M'}$, see Remark 1.16.*

Remark 1.21 *The proof for the isotropic case in [48, 2.5.12] is based on Jackson type inequalities and characterization by approximation. It seems to be not possible to carry over this idea to the dominating mixed scale, since the spaces $S^{\bar r}_{p,q}B(\mathbb{T}^d)$ can not be characterized by quantities of best approximation. Corresponding approximation spaces with respect to trigonometric polynomials with harmonics in hyperbolic crosses and their relation to $S^{\bar r}_{p,q}B$ and $S^{\bar r}_{p,q}F$ are studied in [31].*

Remark 1.22 *The localized versions* $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{M,L}$, $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{\Delta,L}$ *of* $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{M}$ *and* $\|\cdot|S^{\bar{r}}_{p,q}B(\mathbb{T}^d)\|^{\Delta}$ *are equivalent (quasi-)norms in any case. One estimates the* L_p*-(quasi-)norm of the differences of* f *according to the* $(1,\infty)$*-integrals simply by the norm of* f *using the translation invariance. Consequently, these integrals vanish. Recall also Proposition 1.10.*

Chapter 2

Approximation on Sparse Grids

2.1 Introduction

This chapter represents the main part of the thesis. We continue investigations of the approximation properties of trigonometric interpolation with respect to uniform grids, see [17, 18, 43, 45, 33]. The d-variate situation with respect to a sparse grid is studied in detail. Precisely, we investigate the rate of convergence of the Smolyak algorithm (applied to a sampling operator) for functions belonging to a Besov or Triebel-Lizorkin space of dominating mixed smoothness. This also continues former work of Smolyak [40], Temlyakov [43], Wasilkowski, Woźniakowski [57] and Sickel [34, 35].
Section 2.2 deals with interpolation on the torus including the discussion of some examples (de la Vallée Poussin and Dirichlet kernels). Afterwards we switch to the d-dimensional case in Section 2.3. To begin with, we recall the construction of the abstract Smolyak algorithm, discuss a few more or less elementary properties and specify the class of univariate sampling operators, on which it will be applied. We act in a very general setting, so let us refer to the examples given in Section 2.5. Let $A(m, d)$ denote a related Smolyak operator. Since we are interested in the L_p-approximation power of $A(m, d)$, we investigate the norm of the error operator $I - A(m, d)$ acting on both of the scales $S^r_{p,q}B(\mathbb{T}^d)$ and $S^r_{p,q}F(\mathbb{T}^d)$. Unusual is the necessity of different strategies for the B-case and the F-case, respectively. The F-case requires some more restrictive conditions. We just treat the case of Smolyak applied to operators with respect to equidistant sampling knots. We mainly employ Lizorkin's multiplier theorem, see Paragraph 1.3.5, and certain aspects at the field of complex interpolation, see Section 1.5. Nevertheless the result is the same. Section 2.4 contains these general results depending on what has been assumed in Paragraph 2.3.2. Based on that we derive the approximation power of special classical Smolyak constructions among the examples in Section 2.5. This will be compared with the convolution operator resulting after Smolyak applied to classical Fourier partial sums. As one expects, this construction provides better results and is the starting point of a theory named "approximation from hyperbolic crosses", which is widely treated in the literature, see e.g. [3, 4], [2], [6], [9], [10], [15], [19], [22], [25], [30], [31], [36], [39], [40], [46], [45] and [57].

Section 2.6 presents new results for the problem of optimal approximate recovery of functions, which improve existing upper bounds for the quantity ρ_M in the case of Sobolev type spaces, see for instance [45, Chapt. 4]. Finally, Section 2.7 is devoted to the detailed comparison of our results with those existing in the literature.

2.2 Interpolation on the Torus

In this first section we give a short survey about certain aspects of trigonometric interpolation.

2.2.1 Periodic Fundamental Interpolants

Let

$$\mathcal{D}_m(t) := \sum_{|k|\le m} e^{ikt}\,, \qquad t \in \mathbb{T}, \quad m \in \mathbb{N}_0\,,$$

be the Dirichlet kernel of order m and let

$$I_m f(t) := \frac{1}{2m+1} \sum_{\ell=0}^{2m} f(t_\ell)\,\mathcal{D}_m(t-t_\ell)\,, \qquad t_\ell = \frac{2\pi\ell}{2m+1}\,. \tag{2.1}$$

Then $I_m f$ is the unique trigonometric polynomial of degree less than or equal to m which interpolates f at the nodes t_ℓ. This is the prototype for the class of sampling operators on $\mathbb{T}$ we have in mind. To generalize this concept we proceed as follows.
Let $n \in \mathbb{N}$. We put

$$K_n := \left\{\ell \in \mathbb{Z} :\, -\frac{n}{2} \le \ell < \frac{n}{2}\right\} \qquad \text{and} \qquad J_n := \left\{t_\ell = \frac{2\pi\ell}{n} :\, \ell \in K_n\right\}. \tag{2.2}$$

Obviously, the cardinality $|J_n|$ of J_n is equal to n. Here we are interested in periodic fundamental interpolants with respect to this grid J_n, i.e. we consider continuous 2π-periodic functions Λ_n such that

$$\Lambda_n(t_\ell) = \delta_{0,\ell}\,, \qquad \ell \in K_n\,.$$

Here $\delta_{0,\ell}$ is the Kronecker symbol. As in case of the trigonometric interpolation we associate to such a fundamental interpolant a linear operator given by

$$I(\Lambda_n, f)(t) := \sum_{\ell\in K_n} f(t_\ell)\,\Lambda_n(t-t_\ell)\,.$$

In this section our aim consists in deriving some sufficient conditions on Λ_n such that we can estimate the error $f - I(\Lambda_n, f)$ in the L_p-norm for functions from Nikol'skij-Besov spaces. For us it will be convenient to construct a sequence $(\Lambda_n)_n$ from one given function $\Lambda : \mathbb{R} \to \mathbb{R}$. Then the following lemma is known, cf. e.g. [36].

Lemma 2.1 *Let $\Lambda : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$
\begin{array}{lll}
(E1) & \Lambda(2\pi\ell) = \delta_{0\,\ell}\,, & \ell \in \mathbb{Z}\,, \\
(E2) & \sum_{k\in\mathbb{Z}} |\Lambda(x+2\pi k)| \quad \textit{is uniformly convergent on} & [0,2\pi]\,.
\end{array}
$$

(i) *Then, for $n \in \mathbb{N}$,*

$$\Lambda_n^\pi(t) := \sum_{\ell\in\mathbb{Z}} \Lambda(nt + 2\pi\ell n)\,, \qquad t \in \mathbb{R}\,, \tag{2.3}$$

is a continuous 2π-periodic fundamental interpolant with respect to the grid J_n.
(ii) *The Fourier coefficients of these functions are given by*

$$c_\ell(\Lambda_n^\pi) = \frac{1}{n\sqrt{2\pi}}\,\mathcal{F}\Lambda(\ell/n)\,, \qquad \ell \in \mathbb{Z}\,, \quad n \in \mathbb{N}\,. \tag{2.4}$$

2.2.2 The Rate of Convergence

Given an appropriate function Λ we shall investigate the error $f - I(\Lambda_n, f)$ in the L_p-norm. For us it will be sufficient to do this for functions f belonging to some Nikol'skij-Besov space $B^r_{p,\infty}(\mathbb{T})$, see Remark 1.6/(ii) or [32, Chapt. 3].
By $\psi : \mathbb{R} \to \mathbb{R}$ we denote an often used smooth cut-off function, i.e. a compactly supported $\psi \in S(\mathbb{R})$ satisfying $\psi(t) = 1$ if $|t| \le 1$ and $\psi(t) = 0$ if $|t| \ge 2$.

Proposition 2.1 *Let Λ be a continuous function satisfying the hypothesis (E1) and (E2). Let Λ_n^π be defined as in (2.3). Further we assume that for some numbers $0 \le \beta < \alpha$ the function $\mathcal{F}\Lambda$ satisfies:*

(E3) $\mathcal{F}\Lambda(\ell) = \sqrt{2\pi}\,\delta_{0\,,\ell} \quad , \quad \ell \in \mathbb{Z}$*;*

(E4) the functions

$$
\begin{aligned}
A(\xi) &:= \psi\left(\frac{\xi}{2}\right)|\xi|^{-\alpha}\left(1 - \frac{\mathcal{F}\Lambda(\xi)}{\sqrt{2\pi}}\right)\,, \\
B_\ell(\xi) &:= \psi\left(\frac{\xi}{2}\right)|\xi|^{-\alpha}\mathcal{F}\Lambda(\xi + l)\,, \qquad \ell \in \mathbb{Z}\setminus\{0\} \\
C_\ell(\xi) &:= \Big(1 - \psi(2\xi)\Big)|\xi|^{-\beta}\mathcal{F}\Lambda(\xi - \ell)\,, \qquad \ell \in \mathbb{Z}\,,
\end{aligned}
$$

belong to $L_1(\mathbb{R})$,

(E5) the integrals

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\mathcal{F}^{-1}A(w)|\,dw &< \infty\,, \\
\sum_{\ell\neq 0}\int_{-\infty}^{\infty} |\mathcal{F}^{-1}B_\ell(w)|\,dw &< \infty\,, \\
\textit{and} \qquad \sum_{\ell\in\mathbb{Z}}\int_{-\infty}^{\infty} |\mathcal{F}^{-1}C_\ell(w)|\,dw &< \infty
\end{aligned}
$$

are finite.

If $1 \le p \le \infty$, $r > 1/p$ *and* $\beta < r < \alpha$ *then we have*

$$\| I - I(\Lambda_n^\pi, \cdot) \,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r} \,. \tag{2.5}$$

Remark 2.1 *A proof of the estimate from above in (2.5) can be found in [36], at least if* $1 < p < \infty$*. The necessary modifications including the limiting cases are straightforward. For full details we refer to [51]. For* $p = 2$ *the conditions can be simplified, see [37]. The estimate from below can be deduced from the behaviour of the linear widths (approximation numbers) of the embeddings* $B^r_{p,\infty}(\mathbb{T}) \hookrightarrow L_p(\mathbb{T})$*, see Remark 2.2 below.*

Remark 2.2 *Linear widths. For two Banach spaces* X, Y *such that* $X \hookrightarrow Y$ *we define*

$$\lambda_n(I, X, Y) := \inf \Big\{ \| I - L \,|\mathcal{L}(X,Y)\| \; : \quad L \in \mathcal{L}(X,Y), \; \operatorname{rank} L \le n \Big\} \,.$$

Since our operator $I(\Lambda_n^\pi, \cdot)$ *has rank* $\le n$ *we obtain*

$$\lambda_n(I, B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \le \| I - I(\Lambda_n^\pi, \cdot) \,|\mathcal{L}(B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \,\| \,.$$

Since $\lambda_n(I, B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \asymp n^{-r}$*, cf. e.g. [45, 1.4], it is clear that our interpolation operators yield optimal in order approximation.*

2.2.3 Interpolation with de la Vallée Poussin Means

For $0 < \mu < 1/2$ we consider the functions

$$\Lambda_\mu(t) := 2\,\frac{\sin(t/2)\,\sin(\mu t)}{\mu\, t^2}\,, \qquad t \in \mathbb{R} \,. \tag{2.6}$$

Then the Fourier transform is given by

$$\mathcal{F}\Lambda_\mu(\xi) = \sqrt{2\pi} \begin{cases} 1 & \text{if} \;\; |\xi| \le \frac12 - \mu \,, \\ \frac{1}{2\mu}\left(\frac12 + \mu - |\xi|\right) & \text{if} \;\; \frac12 - \mu < |\xi| < \frac12 + \mu \,, \\ 0 & \text{if} \;\; \frac12 + \mu \le |\xi| \,, \end{cases} \tag{2.7}$$

i.e. a piecewise linear function.

Lemma 2.2 *Let* $0 < \mu < 1/2$*. Then the function* Λ_μ *satisfies the restrictions in Proposition 2.1 with* $\beta = 1$ *and* $\alpha > 0$ *arbitrary.*

Proof A proof has been given in [36]. □

Corollary 2.1 *Let* $0 < \mu < 1/2$ *and* Λ_μ *be defined as in (2.6). Let further* $1 \le p \le \infty$ *and* $r > 1/p$*. Then we have*

$$\| I - I(\Lambda_{\mu,n}^\pi, \cdot) \,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r} \,.$$

Proof The corollary becomes a consequence of Proposition 2.1, Lemma 2.2 and complex interpolation for the estimate from above, cf. [36] for details, and Remark 2.2 for the estimate from below. □

Remark 2.3 *Let*

$$v_{2n-1}(t) := \frac{1}{n}\sum_{j=n}^{2n-1} D_j(t)\,, \qquad t\in\mathbb{R}\,, \quad n\in\mathbb{N}\,,$$

denote the de la Vallée Poussin kernels of odd order. Then

$$c_k(v_{2n-1}) = \begin{cases} 1 & \text{if} \quad |k|\le n\,,\\ 2\,(1-|k|/(2n)) & \text{if} \quad n<|k|<2n\,,\\ 0 & \text{if} \quad |k|\ge 2n\,.\end{cases}$$

From Lemma 2.1(ii) we conclude the identity

$$\Lambda^{\pi}_{\mu,3n} = \frac{v_{2n-1}}{3n}\,, \qquad \mu=\frac{1}{6}\,, \quad n\in\mathbb{N}\,.$$

Hence

$$I(\Lambda^{\pi}_{\mu,3n},f)(t) = \frac{1}{3n}\sum_{\ell\in K_{3n}} f(t_\ell)\,v_{2n-1}(t-t_\ell)\,, \qquad \mu=\frac{1}{6}\,.$$

This operator even interpolates on J_{3n}. In contrast to our treatment Temlyakov [45, 1.6] considered the sequence of sampling operators

$$\mathcal{R}_n f(t) := \frac{1}{4n}\sum_{\ell\in K_{4n}} f(t_\ell)\,v_{2n-1}(t-t_\ell)\,, \qquad t_\ell\in J_{4n}\,, \tag{2.8}$$

and proved that these operators also satisfy

$$\|\, I-\mathcal{R}_n\,|B^r_{p,\infty}(\mathbb{T})\to L_p(\mathbb{T})\| \asymp n^{-r}\,,$$

if $1\le p\le\infty$ and $r>1/p$.

2.2.4 Interpolation with the Dirichlet Kernel

The classical case of trigonometric interpolation requires some modifications. It is not covered by Proposition 2.1, however well-known in the literature. Recall, I_n has been defined in (2.1). Then the following is known, see [17, 18, 43, 45, 33].

Proposition 2.2 *Let $1<p<\infty$ and let $r>1/p$. Then we have*

$$\|\, I-I_n\,|B^r_{p,\infty}(\mathbb{T})\to L_p(\mathbb{T})\| \asymp n^{-r}\,.$$

2.2.5 Some other Means

For better comparison we recall some well-known classical approximation properties of the Fourier partial sums and de la Vallée Poussin means, cf. e.g. [32, Chapt. 3] or [45, Chapt. 1]. Let

$$\sigma(t) := \begin{cases} 1 & \text{if} \quad |t| \le 1\,, \\ 2-|t| & \text{if} \quad 1 \le |t| \le 2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For $f \in L_1(\mathbb{T})$ we put

$$\begin{aligned} S_n f(t) &:= \sum_{k=-n}^{n} c_k(f)\, e^{ikt}\,, \\ V_{2n-1} f(t) &:= \sum_{k=-\infty}^{\infty} \sigma(k/n)\, c_k(f)\, e^{ikt}\,, \qquad n \in \mathbb{N}_0\,. \end{aligned}$$

Proposition 2.3 *Let $r > 0$.*
(i) *Let $1 < p < \infty$. Then we have*

$$\| I - S_n \,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r}\,.$$

(ii) *Let $1 \le p \le \infty$. Then we have*

$$\| I - V_{2n-1} \,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r}\,.$$

2.3 The Smolyak Algorithm

2.3.1 Definition and General Properties

Let $d \ge 2$. Let X and Y be Banach spaces such that $X, Y \hookrightarrow L_1(\mathbb{T})$. Further we assume that $P_1, \ldots, P_d : X \to Y$ are continuous linear operators. Then we define its tensor product $P_1 \otimes \ldots \otimes P_d$ to be the linear operator such that:

$$(P_1 \otimes \ldots \otimes P_d)(e^{ik_1\cdot} \cdot \ldots \cdot e^{ik_d\cdot})(x_1, ..., x_d) := \prod_{\ell=1}^{d} P_\ell(e^{ik_\ell\cdot})(x_\ell)$$

$x_\ell \in \mathbb{T}$, $k_\ell \in \mathbb{Z}$, $\ell = 1, \ldots, d$. Formally this operator is defined on trigonometric polynomials only. If X is either $L_p(\mathbb{T})$, $1 \le p < \infty$, or if $X = C(\mathbb{T})$, then, because of the density of trigonometric polynomials, there exists a unique continuous extension of $P_1 \otimes \ldots \otimes P_d$ to either $L_p(\mathbb{T}^d)$ or $C(\mathbb{T}^d)$, respectively. For this extension we shall use the same symbol.
Let either $L_j : L_p(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \le p < \infty$, or $L_j : C(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \le p \le \infty$, $j \in \mathbb{N}_0$, be a sequence of continuous linear operators, denoted by L. Then we put

$$\Delta_j(L) := \begin{cases} L_j - L_{j-1} & \text{if} \quad j \in \mathbb{N}\,, \\ L_0 & \text{if} \quad j = 0\,. \end{cases}$$

Definition 2.1 *Let $m \in \mathbb{N}_0$. The Smolyak-Algorithm $A(m,d,\vec{L})$ relative to the d sequences $L^1 := (L^1_j)_{j=0}^\infty, \dots, L^d := (L^d_j)_{j=0}^\infty$, is the linear operator*

$$A(m,d,\vec{L}) := \sum_{j_1+\ldots+j_d \le m} \Delta_{j_1}(L^1) \otimes \ldots \otimes \Delta_{j_d}(L^d)\,.$$

Remark 2.4 *Originally introduced in [40] there are now hundreds of references dealing with this construction. A few basics and some references can be found in [28] and [57]. In particular the following formula is proved in [57]:*

$$A(m,d,\vec{L}) = \sum_{m-d+1 \le |j|_1 \le m} (-1)^{m-|j|_1} \binom{d-1}{m-|j|_1} L^1_{j_1} \otimes \ldots \otimes L^d_{j_d}\,. \tag{2.9}$$

This will be used later on.

2.3.2 Sampling Operators

Let us now specify sequences $\{L_j\}_j$ for which we want to consider Smolyak's algorithm. Here we shall restrict to a sequence of linear sampling operators of type

$$L_j f(x) = \sum_{\ell=1}^{N_j} f(t^j_\ell)\, \psi^j_\ell(x)\,, \quad f \in C(\mathbb{T})\,, \quad x \in \mathbb{T}\,,$$

where a set

$$\mathcal{T}_j := \{t^j_1, \dots t^j_{N_j}\}$$

of sampling points and continuous periodic functions $\psi^j_1, ..., \psi^j_{N_j}$ are fixed. Now we collect several properties of $L = \{L_j\}_{j=0}^\infty$ in different hypotheses. Referring to these properties gets then very easy.

$\big(H_1(\lambda)\big)$ The operators L_j reproduce trigonometric polynomials with degree at most $\lambda 2^j$, precisely

$$L_j(e^{ik\cdot})(t) = e^{ikt}\,, \quad t \in \mathbb{T}\,, \quad |k| \le \lambda\, 2^j\,, \quad k \in \mathbb{Z}\,, \quad j \in \mathbb{N}_0\,.$$

$\big(H_2(p,r)\big)$ There exists a positive constant $c_{p,r}$ such that

$$\sup_{j=0,1,\ldots} 2^{jr}\, \|\, I - L_j \,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| = c_{p,r} < \infty\,. \tag{2.10}$$

(H_3) We assume the existence of positive constants $C_1 < C_2$ such that

$$C_1\, 2^j \le N_j \le C_2\, 2^j\,, \qquad j \in \mathbb{N}.$$

(H_3') It exists a constant C_3 such that

$$\Big|\mathcal{T}_{j+1} \setminus \bigcup_{k=1}^{j} \mathcal{T}_k\Big| \geq C_3\, 2^{j+1}\,, \qquad j \in \mathbb{N}.$$

(H_4) The sequence of sampling grids is nested , i.e.

$$\mathcal{T}_0 \subset \mathcal{T}_1 \subset ... \subset \mathcal{T}_j \subset \mathcal{T}_{j+1} \subset ...$$

(H_5) The operators L_j are sampling operators with respect to the equidistant grid (see 2.2) of the following type

$$L_j f(x) = \sum_{\ell \in K_{N_j}} f(x_\ell^j)\, \Lambda_j(x - x_\ell^j)\,, \quad f \in C(\mathbb{T})\,, \quad x_\ell^j \in J_{N_j}\,,$$

with a trigonometric polynomial Λ_j, formally given by

$$\Lambda_j(x) = \frac{1}{N_j} \sum_{k \in \mathbb{Z}} \gamma_j(k) e^{ikx}\,.$$

Remark 2.5 *Let us recall the following formula for the Fourier coefficients of such special sampling operators applied to a continuous function f. One has*

$$c_k(L_j f) = \frac{\gamma_j(k)}{N_j} \sum_{\ell \in K_{N_j}} f(x_\ell^j) e^{-ikx_\ell^j} \quad , \quad k \in \mathbb{Z}\,.$$

If f is a trigonometric polynomial, then even the following holds

$$c_k(L_j f) = \gamma_j(k) \sum_{w \in \mathbb{Z}} c_{k+wN_j}(f) \quad , \quad k \in \mathbb{Z}\,. \tag{2.11}$$

Therefore this gives

$$\begin{aligned} L_j\, f(x) =& \sum_{w \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \gamma_j(k) c_{k+wN_j}(f) e^{ikx} \\ =& \sum_{w \in \mathbb{Z}} e^{-iwN_j x} \sum_{k \in \mathbb{Z}} \gamma_j(k - wN_j) c_k(f) e^{ikx} \end{aligned} \tag{2.12}$$

for trigonometric polynomials f.

$\big(H_6(\lambda)\big)$ Let L be a sequence of type (H_5). We assume that for every Λ_j exists a positive number A_j such that the following holds true

$$\gamma_j(k) = \begin{cases} 1 & : \quad |k| \leq \lambda 2^j \\ 0 & : \quad |k| > A_j \end{cases} \quad , \quad \lambda 2^j + A_j < N_j\,.$$

Remark 2.6 *Formula (2.11) implies for the function* $f(x) = e^{imx}$, $m \in \mathbb{N}_0$, *the relation*

$$c_k(L_j(e^{im\cdot})) = \gamma_j(k) \cdot \begin{cases} 1 & : \quad \text{if } N_j \text{ divides } k-m \\ 0 & : \quad \text{otherwise} \end{cases} \quad , \quad k \in \mathbb{Z} \quad , \tag{2.13}$$

which yields immediately the implication

$$H_6(\lambda) \implies H_1(\lambda) \; .$$

(H_7) Let $L = \{L_j\}_{j \in \mathbb{N}_0^d}$ be a sequence of type (H_5). The corresponding sequence of Λ_j is supposed to provide the existence of the following quantity

$$C := \sup_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}} |\gamma_j(k) - \gamma_j(k-1)| \; .$$

We shall say that $\vec{L} = (L^1, ..., L^d)$ satisfies the hypothesis (H_n) if each sequence L^i, $i = 1, \ldots, d$, satisfies (H_n).
The set of sampling points used by L^i_j will be denoted by $\mathcal{T}^i_j$. Then we put

$$\mathcal{G}(m, d, \vec{L}) := \bigcup_{m-d+1 \le |j|_1 \le m} \mathcal{T}^1_{j_1} \times ... \times \mathcal{T}^d_{j_d} \; . \tag{2.14}$$

By (2.9) the operator $A(m, d, \vec{L})$ uses only samples from the grid $\mathcal{G}(m, d, \vec{L})$. To begin with we state a simple property of the standard grid induced by the choice $\mathcal{T}^i_j = J_{2^j}$.

Lemma 2.3 *Let* $\vec{L}$ *be a sequence of operators such that* L^i_j *uses samples from the grid* $\mathcal{T}^i_j = J_{2^j}$, $i = 1, \ldots, d$, $j \in \mathbb{N}_0$ *(see (2.2)). Then the cardinality* $S(m, d)$ *of the grid* $\mathcal{G}(m, d, \vec{L})$ *is given by*

$$S(m, d) = \sum_{j=0}^{d-1} \binom{d-1}{j} 2^{m-j} \binom{m}{j} \; , \tag{2.15}$$

where we put $\binom{m}{j} := 0$ *in case* $m < j$.

Proof *Step 1.* For abbreviation we write $\mathcal{G}(m, d)$ instead of $\mathcal{G}(m, d, \vec{L})$. By using the nestedness of the sequence J_{2^n} we obtain the following recursion formula (see also [24])

$$\begin{aligned} \mathcal{G}(m, d+1) &= \bigcup_{0 \le j_1 + ... + j_{d+1} \le m} J_{2^{j_1}} \times ... \times J_{2^{j_{d+1}}} \\ &= \bigcup_{n=0}^{m} \mathcal{G}(m-n, d) \times J_{2^n} \\ &= (\mathcal{G}(m, d) \times J_1) \cup \Big(\bigcup_{n=1}^{m} \mathcal{G}(m-n, d) \times \big(J_{2^n} \setminus J_{2^{n-1}} \big) \Big) \, , \end{aligned}$$

where $\mathcal{G}(m,1) = J_{2^m}$. Therefore $\mathcal{G}(m,d+1)$ is decomposed into a disjoint union of subsets. This yields

$$S(m,d+1) = S(m,d) + \sum_{n=1}^{m} S(m-n,d)\, 2^{n-1}\,,$$

with $S(m,1) = 2^m$.
Step 2. We proceed by induction with respect to d. From the induction hypothesis (2.15) and our recursion formula we derive

$$\begin{aligned} S(m,d+1) &= \sum_{j=0}^{d-1} \binom{d-1}{j} 2^{m-j} \binom{m}{j} + \sum_{j=0}^{d-1} \binom{d-1}{j} \sum_{n=1}^{m} 2^{m-n-j} \binom{m-n}{j} 2^{n-1} \\ &= \sum_{j=0}^{d-1} \binom{d-1}{j} 2^{m-j} \binom{m}{j} + \sum_{j=0}^{d-1} \binom{d-1}{j} 2^{m-j-1} \sum_{n=1}^{m} \binom{m-n}{j}\,. \end{aligned}$$

Using the identity

$$\sum_{n=j}^{m-1} \binom{n}{j} = \binom{m}{j+1}\,, \qquad j \in \mathbb{N}_0\,, \tag{2.16}$$

we obtain

$$\begin{aligned} S(m,d+1) &= \sum_{j=0}^{d-1} \binom{d-1}{j} 2^{m-j} \binom{m}{j} + \sum_{j=1}^{d} \binom{d-1}{j-1} 2^{m-j} \binom{m}{j} \\ &= \sum_{j=0}^{d} \binom{d}{j} 2^{m-j} \binom{m}{j}\,, \end{aligned}$$

which proves our claim. □
This Lemma can be generalized to the following assertion.

Lemma 2.4 (i) *The hypothesis* (H_3) *should be fulfilled. Then the cardinality* $|\mathcal{G}(m,d,\vec{L})|$ *of the grid* $\mathcal{G}(m,d,\vec{L})$ *satisfies*

$$|\mathcal{G}(m,d,\vec{L})| \le (2C_2)^d\, S(m,d)\ .$$

(ii) *If the hypotheses* (H_3) *and* (H_3') *are fulfilled then*

$$\min(C_1,C_3)^d S(m,d) \le |\mathcal{G}(m,d,\vec{L})| \le (2C_2)^d\, S(m,d)\ , \quad m \in \mathbb{N}_0 \tag{2.17}$$

holds true.

Proof The same arguments as used in Step 1 of the proof of Lemma 2.3 lead to a decomposition of $\mathcal{G}(m,d+1)$ into disjoint subsets

$$\mathcal{G}(m,d+1) = (\mathcal{G}(m,d) \times \mathcal{T}_0^{d+1}) \cup \Big(\bigcup_{n=1}^{m} \mathcal{G}(m-n,d) \times \Big(\mathcal{T}_n^{d+1} \setminus \bigcup_{\ell=0}^{n-1} \mathcal{T}_\ell^{d+1} \Big)\Big)\,.$$

In view of (H_3) and (H_3') this yields

$$\begin{aligned}\min(C_1,C_3)\Big(|\mathcal{G}(m,d)| \;&+\; \sum_{n=1}^{m} |\mathcal{G}(m-n,d)|\, 2^{n-1}\Big) \le |\mathcal{G}(m,d+1)| \\ &\le\; 2C_2 \left(|\mathcal{G}(m,d)| + \sum_{n=1}^{m} |\mathcal{G}(m-n,d)|\, 2^{n-1}\right).\end{aligned}$$

Induction with respect to d by taking (2.17) as induction hypothesis yields the desired result. □

The following property is central for Smolyak's algorithm. First we define a version of the so called " dyadic hyperbolic cross", which was studied very intensively in connection with " dyadic hyperbolic cross approximation". The following lemma points out the connection between Smolyak's algorithm and this field by stating a certain invariance property.

Lemma 2.5 *Let*

$$H(m,d,\lambda) := \Big\{\ell \in \mathbb{Z}^d : \exists u_1,...,u_d \in \mathbb{N}_0 \quad s.t. \quad |\ell_k| \le 2^{u_k}\lambda \text{ and } \sum_{k=1}^{d} u_k = m\Big\} \tag{2.18}$$

be a dyadic hyperbolic cross. Suppose that $\vec{L}$ satisfies $(H_1(\lambda))$ for some $\lambda > 0$. Then

$$A(m,d,\vec{L})\, e^{i\ell\cdot} = e^{i\ell\cdot}\,, \quad \ell \in H(m,d,\lambda)\,.$$

Proof We consider the linear operators

$$T := \sum_{0\le j_1\le m} \cdots \sum_{0\le j_d\le m} \bigotimes_{k=1}^{d} \Delta_{j_k}^k \qquad \text{and} \qquad R := \sum_{|j|_1 \ge m+1} \bigotimes_{k=1}^{d} \Delta_{j_k}^k\,,$$

where we put $\Delta_{j_k}^k := \Delta_{j_k}(L^k)$. Then $A(m,d,\vec{L}) = T - R$. Since $\sum_{j=0}^{m} \Delta_j^k = L_m^k$ we obtain

$$T = \bigotimes_{k=1}^{d} L_m^k\,, \qquad m \in \mathbb{N}_0\,.$$

Obviously, if $\ell \in H(m,d,\lambda)$, i.e. $|\ell_u| \le \lambda\, 2^m$ for all $1 \le u \le d$, then

$$\big(Te^{i\ell\cdot}\big)(x) = \prod_{u=1}^{d} (L_m^u e^{i\ell_u\cdot})(x_u) = e^{i\ell x}\,, \qquad x \in \mathbb{T}^d\,,$$

because of $(H_1(\lambda))$. It remains to prove $Re^{i\ell\cdot} \equiv 0$. Let $j = (j_1,...,j_d)$ be such that $|j|_1 \ge m+1$. Because of $\ell \in H(m,d,\lambda)$ there exist nonnegative integers u_k, $k = 1,...,d$

satisfying $\sum_{k=1}^d u_k = m$ and $|\ell_k| \le 2^{u_k}\lambda$. Thanks to $|j|_1 \ge m+1 > m$ there is at least one component j_k of j with $j_k > u_k$. It follows

$$|\ell_k| \le 2^{u_k}\,\lambda \le 2^{j_k-1}\,\lambda < 2^{j_k}\,\lambda\,.$$

Hence, using again $(H_1(\lambda))$, we find

$$\Delta^k_{j_k} e^{i\ell_k t} = L^k_{j_k} e^{i\ell_k t} - L^k_{j_k-1} e^{i\ell_k t} = e^{i\ell_k t} - e^{i\ell_k t} = 0\,, \qquad t \in \mathbb{T}\,.$$

By definition of R this proves the claim. □

Remark 2.7 *A special case of Lemma 2.5 can be found in [43].*

In addition we need the cardinality of certain subsets of $\mathbb{Z}^d$, especially the size of the hyperbolic cross $H(m,d,1)$, defined in (2.18). For $m \in \mathbb{N}$ we consider also the sets

$$P_0(m,d) = \Big\{(n_1,...,n_d) \in \mathbb{N}_0^d : \sum_{i=1}^d n_i = m\Big\}$$

$$\text{and} \qquad P_1(m,d) = \Big\{(n_1,...,n_d) \in \mathbb{N}^d : \sum_{i=1}^d n_i = m\Big\}\,.$$

Lemma 2.6 *For $m \in \mathbb{N}$ it holds*
(i)

$$|P_0(m,d)| = \binom{m+d-1}{d-1}, \qquad |P_1(m,d)| = \binom{m-1}{d-1},$$

(ii) *and*

$$2^d\,S(m,d) \le |H(m,d,1)| \le 3^d\,S(m,d).$$

Proof Part (i) is an easy consequence of the recursion formulas

$$|P_0(m,d+1)| = \sum_{n=0}^m |P_0(m-n,d)| \quad \text{and} \quad |P_1(m,d+1)| = \sum_{n=1}^{m-d} |P_0(m-n,d)|$$

with $P_0(m,1) = P_1(m,1) = 1$, $m \in \mathbb{N}$ and induction with respect to d using (2.16). The same arguments as used in the proof of the Lemmas 2.3, 2.4 imply

$$\begin{aligned} 2\Big(|H(m,d)| \;&+\; \sum_{n=1}^m |H(m-n,d)|\,2^{n-1}\Big) \le |H(m,d+1)| \\ &\le\; 3\left(|H(m,d)| + \sum_{n=1}^m |H(m-n,d)|\,2^{n-1}\right) \end{aligned}$$

with $2 \cdot 2^m \le H(m,1) \le 3 \cdot 2^m$. Induction with respect to d yields the result. □

Remark 2.8 (i) *Obviously, for fixed d we have*

$$|P_0(m,d)| \asymp m^{d-1} \quad , \quad |P_1(m,d)| \asymp m^{d-1} \quad \text{and} \quad S(m,d) \asymp 2^m\, m^{d-1}\,, \qquad m \in \mathbb{N}\,.$$

We call the grids $\mathcal{G}(m,d,\vec{L})$ sparse because their cardinality is growing only with a logarithmic order with respect to d.
(ii) *Estimates of the cardinality of grids related to Smolyak algorithms are given at various places. What concerns the dependence on d we refer e.g. to [24].*

Lemma 2.7 *Let us assume that*

$$L^i_j f(t) = f(t)\,, \qquad t \in \mathcal{T}^i_j\,, \qquad i = 1,\dots,d\,, \tag{2.19}$$

for all $j \le m$ and all $f \in C(\mathbb{T})$. If now (H_4) (nestedness) is fulfilled then $A(m,d,\vec{L})$ interpolates on $\mathcal{G}(m,d,\vec{L})$, more precisely

$$A(m,d,\vec{L})f(x) = f(x)\,, \qquad x \in \mathcal{G}(m,d,\vec{L})\,, \quad f \in C(\mathbb{T}^d)\,.$$

Proof The proof is similar to the proof of Lemma 2.5. Observe that the nestedness of the grids $\mathcal{T}^i_j$ implies

$$\mathcal{G}(m,d,\vec{L}) = \bigcup_{|j|_1 \le m} \mathcal{T}^1_{j_1} \times \dots \times \mathcal{T}^d_{j_d} = \bigcup_{|j|_1 = m} \mathcal{T}^1_{j_1} \times \dots \times \mathcal{T}^d_{j_d}\,.$$

We employ the same notation and decomposition of $A(m,d,\vec{L}) = T - R$ as in proof of Lemma 2.5. Since L^i_m interpolates on $\mathcal{T}^i_m$ the operator T interpolates on $\mathcal{T}^1_m \times \dots \times \mathcal{T}^d_m$. Hence, it is enough to prove

$$Rf(x) = 0 \qquad \text{for all} \quad x \in \mathcal{T}^1_{k_1} \times \dots \times \mathcal{T}^d_{k_d}\,, \qquad |k|_1 = m\,,$$

and all $f \in C(\mathbb{T}^d)$. We shall prove even more, namely

$$\Big(\Delta^1_{j_1} \otimes \dots \otimes \Delta^d_{j_d}\Big) f(x) = 0\,, \qquad x \in \mathcal{T}^1_{k_1} \times \dots \times \mathcal{T}^d_{k_d}\,, \quad |k|_1 = m\,,$$

$f \in C(\mathbb{T}^d)$ and $|j|_1 > m$.
Let j, $|j|_1 > m$, k, $|k|_1 = m$ and $x \in \mathcal{G}(m,d,\vec{L})$ be given. For $f \in C(\mathbb{T}^d)$ and $1 \le u \le d$ we put $g_u(t) := f(x_1,\dots,x_{u-1},t,x_{u+1},\dots,x_d)$, $t \in \mathbb{T}$. Furthermore, there exists at least one component u such that $k_u < j_u$. This implies $L^u_{j_u} g_u(x_u) = L^u_{j_u-1} g_u(x_u)$ which proves the claim. □

2.4 Main Results

This section contains our main results. As mentioned in the Introduction we will study the approximation power of the Smolyak algorithm for functions taken from Besov, Triebel-Lizorkin and Sobolev type spaces with dominating mixed smoothness $S^r_{p,q}B(\mathbb{T}^d)$, $S^r_{p,q}F(\mathbb{T}^d)$ and $S^r_pW(\mathbb{T}^d)$.

2.4.1 Besov Spaces

Our first result is the following general estimate for sampling operators. Here we shall use that functions from $S^r_{p,q}B(\mathbb{T}^d)$ with $r > 1/p$ have a continuous representative (see Corollary 1.1).

Proposition 2.4 *Let $1 \le p, q \le \infty$ and $r > 1/p$. Let further $\vec{L}$ satisfy the hypotheses $(H_1(\lambda))$ for a certain $\lambda > 0$ and $(H_2(p,s))$ for $1/p < s \le r$. Then there exists a constant $c > 0$ such that*

$$\| I - A(m,d,\vec{L}) \,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(1-1/q)}\, 2^{-mr}$$

holds for all $m \in \mathbb{N}_0$.

Proof *Step 1.* We need first to prove the following lemma, which represents our main tool.

Lemma 2.8 *Let $1 \le p \le \infty$ and $r > 0$. Suppose $P_j \in \mathcal{L}(B^r_{p,p}(\mathbb{T}), L_p(\mathbb{T}))$, $j = 1, \ldots, d$. Then*

$$\| P_1 \otimes \ldots \otimes P_d\, f \,|L_p(\mathbb{T}^d)\| \le \Big(\prod_{j=1}^{d} \| P_j \,|B^r_{p,p}(\mathbb{T}) \to L_p(\mathbb{T})\| \Big) \, \| f \,|S^r_{p,p}B(\mathbb{T}^d)\|$$

holds for all trigonometric polynomials f.

Proof Let $f = \sum_{k \in \mathbb{Z}^d} c_k(f)\, e^{ikx}$ be a trigonometric polynomial. We define $k = (k_1, k')$, $k_1 \in \mathbb{Z}$, $k' \in \mathbb{Z}^{d-1}$, $x = (x_1, x')$, $x_1 \in \mathbb{T}$, $x' \in \mathbb{T}^{d-1}$, and

$$g_{k_1}(x') := \sum_{k' \in \mathbb{Z}^{d-1}} c_k(f) \Big(\prod_{n=2}^{d} P_n(e^{ik_n \cdot})(x_n) \Big), \qquad x' \in \mathbb{R}^{d-1}, \quad k_1 \in \mathbb{Z}\,.$$

Then

$$\begin{aligned} \| (P_1 \otimes \ldots \otimes P_d) f \,|L_p(\mathbb{T}^d)\|^p &= \int_{T^{d-1}} \Big\| P_1\Big(\sum_{k_1 \in \mathbb{Z}} g_{k_1}(x')\, e^{ik_1 \cdot} \Big)(x_1) \,\Big|L_p(\mathbb{T}, x_1)\Big\|^p \, dx' \\ &\le \| P_1 \|^p \int_{T^{d-1}} \Big\| \Big(\sum_{k_1 \in \mathbb{Z}} g_{k_1}(x')\, e^{ik_1 \cdot} \Big)(x_1) \,\Big|B^r_{p,p}(\mathbb{T}, x_1)\Big\|^p \, dx'\,. \end{aligned} \tag{2.20}$$

Now, let $(\varphi_j)_j \in \Phi(\mathbb{R})$ be an appropriate decomposition of unity, see Definition 1.1. Then

$$\begin{aligned} &\Big\| \Big(\sum_{k_1 \in \mathbb{Z}} g_{k_1}(x')\, e^{ik_1 x_1} \Big) \,\Big|B^r_{p,p}(\mathbb{T})\Big\|^p = \sum_{j_1=0}^{\infty} 2^{j_1 rp} \Big\| \sum_{k_1 \in \mathbb{Z}} \varphi_{j_1}(k_1)\, g_{k_1}(x')\, e^{ik_1 x_1} \,\Big|L_p(\mathbb{T}, x_1)\Big\|^p \\ &= \sum_{j_1=0}^{\infty} 2^{j_1 rp} \Big\| P_2\Big(\sum_{k \in \mathbb{Z}^d} c_k(f)\, \varphi_{j_1}(k_1)\, e^{ik_1 x_1} \Big(\prod_{n=3}^{d} P_n(e^{ik_n \cdot})(x_n) \Big)\, e^{ik_2 \cdot} \Big)(x_2) \,\Big|L_p(\mathbb{T}, x_1)\Big\|^p \end{aligned}$$

This identity will be inserted into (2.20). Then we interchange the order of integration and proceed as above:

$$\begin{aligned}
\| (P_1 \otimes \ldots \otimes P_d) f \,|L_p(\mathbb{T}^d)\|^p &\le \| P_1 \|^p \sum_{j_1=0}^{\infty} 2^{j_1 r p} \int_{\mathbb{T}^{d-1}} \int_0^{2\pi} \\
& \Big| P_2\Big(\sum_{k\in\mathbb{Z}^d} c_k(f)\, \varphi_{j_1}(k_1)\, e^{ik_1x_1} \Big(\prod_{n=3}^{d} P_n(e^{ik_n \cdot})(x_n)\Big)\, e^{ik_2\cdot}\Big)(x_2)\Big|^p dx_2\, dx_1\, dx_3 \ldots dx_d \\
\le\ & \| P_1 \|^p \ldots \| P_d \|^p \\
& \times \sum_{j\in\mathbb{N}_0^d} 2^{r|j|_1 p} \int_{\mathbb{T}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k(f)\, \varphi_{j_1}(k_1)\, e^{ik_1x_1} \ldots \varphi_{j_d}(k_d)\, e^{ik_dx_d} \Big|^p dx_1 \ldots dx_d \\
=\ & \| P_1 \|^p \ldots \| P_d \|^p \, \| f \,|S^r_{p,p}B(\mathbb{T}^d)\|^p .
\end{aligned}$$

This proves the claim. □

Let us now proof Proposition 2.4.
Step 2. The aim of this step consists in a description of the decomposition we are going to use. Let us recall the decomposition (1.20) of $f \in S^r_{p,q}B(\mathbb{T}^d)$ into the pieces f_ℓ. Because of $r > 1/p$ we have convergence in $C(\mathbb{T}^d)$, see Corollary 1.1 and Theorem 1.4. Next we need to fix a natural number n_λ such that $2^{-n_\lambda} \le \lambda$. Now we suppose that m is larger than $d(n_\lambda + 1)$. Further we put $s_m := m - d(n_\lambda + 1) \ge 0$ (we drop the parameter λ in all other notations). Let $I_0^m := [0, s_m]$ and $I_1^m := (s_m, \infty)$, respectively. For $b = (b_1, \ldots b_d)$, $b_i \in \{0,1\}$, $i = 1, \ldots, d$, we define

$$Q_b^m := \{\ell \in \mathbb{N}_0^d :\ \ell_n \in I^m_{b_n},\ n = 1, \ldots, d,\ |\ell|_1 > s_m\}\,,$$

This leads to the decomposition

$$f(x) = h(x) + \sum_{b\in\{0,1\}^d} f^b(x)\ ,$$

where

$$f^b(x) := \sum_{\ell\in Q_b^m} f_\ell(x)\,.$$

The function $h(x)$ is a trigonometric polynomial given by

$$h(x) := \sum_{|\ell|_1 \le s_m} f_\ell(x)\,.$$

Observe $c_k(h) \neq 0$ implies $c_k(f_\ell) \neq 0$ for some $\ell \in \mathbb{Z}^d$ satisfying $|\ell|_1 \le s_m$. Hence $|k_n| \le 2^{\ell_n+1} \le 2^{\ell_n+1+n_\lambda}\lambda$ for $n = 1, ..., d$. Therefore $k \in H(m, d, \lambda)$ and consequently $A(m, d, \vec{L})\, h = h$ follows, see Lemma 2.5 and $(H_1(\lambda))$.

Step 3. Estimation (first part). For later use we will take care of the constants and their dependence on the dimension d in all inequalities.
By means of the invariance of h under the application of $A(m,d,\vec{L})$ we find

$$\| f - A(m,d,\vec{L})f \,|L_p(\mathbb{T}^d)\| \le \sum_{b\in\{0,1\}^d} \| f^b - A(m,d,\vec{L})f^b \,|L_p(\mathbb{T}^d)\| \,.$$

Obviously, there exists a number $M_\ell \in \mathbb{N}$, such that the trigonometric polynomial f_ℓ has all its harmonics in the hyperbolic cross $H(M_\ell, d, \lambda)$. For $M_\ell > |\ell|_1 + d(1+n_\lambda)$ Lemma 2.5 implies

$$\begin{aligned} \| f_\ell - A(m,d,\vec{L})f_\ell \,|L_p(\mathbb{T}^d)\| &= \| A(M_\ell,d,\vec{L})f_\ell - A(m,d,\vec{L})f_\ell \,|L_p(\mathbb{T}^d)\| \\ &= \Big\| \sum_{m<|j|_1\le M_\ell} \Big(\bigotimes_{n=1}^d \Delta^n_{j_n}\Big) f_\ell \,\Big|L_p(\mathbb{T}^d)\Big\| \\ &= \Big\| \sum_{j\in\Lambda^m_\ell} \Big(\bigotimes_{n=1}^d \Delta^n_{j_n}\Big) f_\ell \,\Big|L_p(\mathbb{T}^d)\Big\| \,, \end{aligned}$$

where

$$\Lambda^m_\ell := \Big\{ j = (j_1,...,j_d) : \quad |j|_1 > m, \quad j_n \le \ell_n + 1 + n_\lambda,\, n = 1,...,d \Big\} \,.$$

In order to keep simple notation, we used again $\Delta^n_{j_n}$ instead of $\Delta_{j_n}(L^n)$, $n = 1,...,d$, $j_n = 0,1,2,....$ The last step here is a consequence of $(H_1(\lambda))$, the definition of the tensor product and the choice of M_ℓ. We continue by using Lemma 2.8. Let us choose r_0 such that $1/p < r_0 < r$. Then

$$\Big\| \Big(\bigotimes_{n=1}^d \Delta^n_{j_n}\Big) f_\ell \,\Big|L_p(\mathbb{T}^d)\Big\| \le \| f_\ell \,|S^{r_0}_{p,p}B(\mathbb{T}^d)\| \prod_{n=1}^d \| \Delta^n_{j_n} \,|B^{r_0}_{p,p}(\mathbb{T}) \to L_p(\mathbb{T})\| \,.$$

Using hypothesis $(H_2(r_0,p))$, the triangle inequality and Lemma 1.6 this gives

$$\Big\| \Big(\bigotimes_{n=1}^d \Delta^n_{j_n}\Big) f_\ell \,\Big|L_p(\mathbb{T}^d)\Big\| \le C_4^d \, 2^{-mr_0} \, \| f_\ell \,|S^{r_0}_{p,p}B(\mathbb{T}^d)\| \,,$$

where $C_4 := C_0(r_0)\cdot(1+2^{r_0})$, see (2.10). Furthermore

$$\begin{aligned} \| f_\ell \,|S^{r_0}_{p,p}B(\mathbb{T}^d)\| &\le \Big(\sum_{\substack{|j_k-\ell_k|\le 1\\ k=1,\dots,d}} 2^{r_0|j|_1 p} \sup_{j\in\mathbb{N}_0^d} \|(2\pi)^{-d/2}\mathcal{F}^{-1}\varphi_j|L_1(\mathbb{R}^d)\|^p \cdot \|f_\ell|L_p(\mathbb{T}^d)\|^p \Big)^{1/p} \\ &\le C_5 \, 2^{r_0|\ell|_1} \, \| f_\ell \,|L_p(\mathbb{T}^d)\| \,, \end{aligned}$$

where

$$C_5 := 2^{r_0 d} 3^{d/p} (2\pi)^{-d/2} \max_{n=0,1,\dots,d} \| \mathcal{F}^{-1}\varphi_0 \,|L_1(\mathbb{R})\|^n \| \mathcal{F}^{-1}\varphi_1 \,|L_1(\mathbb{R})\|^{d-n} \,.$$

Here we used Lemma 1.1 and the homogeneity of the Fourier transform. Altogether we obtain

$$\| f^b - A(m,d,\vec{L})f^b \,|L_p(\mathbb{T}^d)\| \le C_4^d\, C_5 \sum_{\ell\in Q_b^m} 2^{r_0(|\ell|_1-m)}\, |\Lambda_\ell^m|\, \| f_\ell \,|L_p(\mathbb{T}^d)\|$$

$$= C_4^d\, C_5\, 2^{-r_0 m} \sum_{\ell\in Q_b^m} 2^{(r_0-r)|\ell|_1}\, |\Lambda_\ell^m|\; 2^{r|\ell|_1}\, \| f_\ell \,|L_p(\mathbb{T}^d)\| \,. \tag{2.21}$$

Of course, $|\Lambda_\ell^m|$ denotes the cardinality of the set Λ_ℓ^m. We need to estimate this quantity. Obviously

$$\Lambda_\ell^m \subset \Big[m - \sum_{\substack{n=1\\ n\neq 1}}^{d} (\ell_n+1+n_\lambda), \ell_1+1+n_\lambda\Big] \times \cdots \times \Big[m - \sum_{\substack{n=1\\ n\neq d}}^{d} (\ell_n+1+n_\lambda), \ell_d+1+n_\lambda\Big] \,.$$

This implies

$$|\Lambda_\ell^m| \le \min\Big((|\ell|_1 + d(1+n_\lambda)+1-m)^d \;,\; \prod_{n=1}^{d} (\ell_n+2+n_\lambda)\Big) \,. \tag{2.22}$$

Step 4. Estimation (second part). Depending on the size of $|b|_1$ we continue.
Step 4.1. Let $|b|_1 \le 1$. Without loss of generality we may assume that $b_1 = |b|_1$. For given q let q' be such that $(1/q)+(1/q')=1$. Then we find

$$\begin{aligned}
&2^{-mr_0}\Big(\sum_{\ell\in Q_b^m} 2^{q'(r_0-r)|\ell|_1}\, |\Lambda_\ell^m|^{q'}\Big)^{1/q'}\\
&\le 2^{-mr_0}\Big(\sum_{\ell\in Q_b^m} 2^{q'(r_0-r)|\bar{\ell}|_1}\, (|\ell|_1 + d(n_\lambda+1)+1-m)^{dq'}\Big)^{1/q'}\\
&\le 2^{-mr_0}\Big(\sum_{\ell_2,\ldots,\ell_d=0}^{s_m} \sum_{u=0}^{\infty} 2^{q'(r_0-r)(u+m-d(n_\lambda+1)-1)} u^{dq'}\Big)^{1/q'}\\
&\le 2^{-mr}\Big(m^{d-1} \sum_{u=0}^{\infty} 2^{q'(r_0-r)(u-d(n_\lambda+1)-1)} u^{dq'}\Big)^{1/q'}\\
&\le C_6\, 2^{-mr}\, m^{(d-1)(1-1/q)} \,,
\end{aligned} \tag{2.23}$$

where

$$C_6 := 2^{(r-r_0)(d(n_\lambda+1)+1)} \Big(\sum_{u=0}^{\infty} 2^{q'(r_0-r)u}\, u^{dq'}\Big)^{1/q'} \,.$$

In this case Hölder's inequality and (2.23) lead to

$$\begin{aligned}
&\| f^b - A(m,d,\vec{L})f^b) \,|L_p(\mathbb{T}^d)\|\\
&\le C_4^d\, C_5\, 2^{-mr_0}\Big(\sum_{\ell\in Q_b^m} 2^{q'(r_0-r)|\ell|_1}\, |\Lambda_\ell^m|^{q'}\Big)^{1/q'} \Big(\sum_{\ell\in Q_b^m} 2^{r|\ell|_1 q}\, \| f_\ell \,|L_p(\mathbb{T}^d)\|^q\Big)^{1/q}\\
&\le C_4^d\, C_5\, C_6\, 2^{-mr}\, m^{(d-1)(1-1/q)}\, \| f \,|S^r_{p,q}B(\mathbb{T}^d)\| \,.
\end{aligned} \tag{2.24}$$

Step 4.2. Let $|b|_1 \geq 2$. In this case the estimate becomes easier. We use

$$2^{-mr_0} \sum_{\ell \in Q_b^m} 2^{(r_0-r)|\ell|_1} \, |\Lambda_\ell^m| \leq 2^{-mr_0} \prod_{n=1}^{d} \Big(\sum_{\ell_n \in I_{b_n}^m} 2^{(r_0-r)\ell_n} (\ell_n + 2 + n_\lambda) \Big) ,$$

see (2.22), as well as

$$\begin{aligned} \sum_{\ell_n = s_m+1}^{\infty} & 2^{(r_0-r)\ell_n} (\ell_n + 2 + n_\lambda) \\ &= 2^{m(r_0-r)} 2^{(r-r_0)(d(n_\lambda+1)-1)} \sum_{u=0}^{\infty} 2^{(r_0-r)u} \, (u + m - d(n_\lambda + 1) + 3) \\ &\leq 2^{m(r_0-r)} \, 2^{(r-r_0)(d(n_\lambda+1)-1)} \sum_{u=0}^{\infty} 2^{(r_0-r)u} \, (u + m + 3) \\ &\leq C_7 \, 2^{m(r_0-r)} \, m \end{aligned}$$

with

$$C_7 := 2 \, 2^{(r-r_0)(d(n_\lambda+1)-1)} \Big(\sum_{u=0}^{\infty} 2^{(r_0-r)u} \, (u+3) \Big) ,$$

and

$$\sum_{\ell_n=0}^{s_m} 2^{(r_0-r)\ell_n} (\ell_n + 2 + n_\lambda) \leq m \sum_{u=0}^{\infty} 2^{(r_0-r)u} \leq C_7 \, m \, .$$

Altogether this leads to

$$\begin{aligned} \| \, f_b - A(m,d,\vec{L}) f_b \, | L_p(\mathbb{T}^d) \| &\leq C_4^d \, C_5 \, \| \, f \, | S^r_{p,\infty} B(\mathbb{T}^d) \| \, 2^{-mr_0} \sum_{\ell \in Q_b^m} 2^{(r_0-r)|\ell|_1} |\Lambda_\ell^m| \\ &\leq C_4^d \, C_5 \, C_7^d \, \| \, f \, | S^r_{p,\infty} B(\mathbb{T}^d) \| \, 2^{-mr} m^d 2^{m(r_0-r)(|b|_1-1)} \\ &\leq C_4^d \, C_5 \, C_7^d \, C_8 \, 2^{-mr} \, \| \, f \, | S^r_{p,\infty} B(\mathbb{T}^d) \| \, , \end{aligned} \tag{2.25}$$

where

$$C_8 := \max_{n=1,\ldots,d-1} \, \sup_{m \in \mathbb{N}} m^d \, 2^{m(r_0-r)n} \, ,$$

see (2.21). It remains to sum up over $|b|_1 \leq 1$ in (2.24) and over $2 \leq |b|_1 \leq d$ in (2.25), respectively. This completes the proof of Proposition 2.4. □

Remark 2.9 *Proposition 2.4 generalizes the results obtained in [34] in various directions. In [34] the bivariate case for $1 < p < \infty$ is investigated. In addition the admissible operators $A(m, d, \vec{L})$ are more general now.*

Remark 2.10 *We observe that the dependence on the dimension d is worse than exponential. But this is just because of the constant C_7^d, which has only technical reasons. Consider for instance the operator $A(m,d,D)$, where $\lambda = 1$ and therefore $n_\lambda = 0$. In the case $1 < p < \infty$ we can use the decomposition in Remark 1.10 for the proof (see also Lemma 1.5). The same strategy was used in the proof of [53, Lem. 3]. The quantity s_m then equals m and the factor*

$$2^{(r-r_0)(d(n_\lambda+1)-1)}$$

in the definition of C_7 vanishes. Therefore the result can be rewritten as follows. If $1 < p < \infty$, $1 \le q \le \infty$ and $r > 1/p$ then exists a constant $C > 0$ (independently of d), such that the relation

$$\|I - A(m,d,D)|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le C^d m^{(d-1)(1-1/q)} 2^{-rm} \tag{2.26}$$

holds true for all $m \in \mathbb{N}_0$. Of course, the case $p = q = 2$ is included. Therefore (2.26) holds even for the space $S^r_2W(\mathbb{T}^d)$ (see Remark 1.6/(iii) and Corollary 1.2).

The following assertion gives the estimate from below. Here we restrict to sequences $\vec{L}$ using equidistant sampling knots, see (H_5).

Proposition 2.5 *Let $1 \le p, q \le \infty$ and $r > 1/p$. Let further $\vec{L}$ be given by (H_5), where we assume*

$$N_0 < N_1 < ... < N_j < N_{j+1} < ... \,, \tag{2.27}$$

and satisfy the hypotheses $\big(H_1(\lambda)\big)$ for a certain $\lambda > 0$, (H_3) and (H_4). Then there exists a constant $c > 0$ such that

$$\|\, I - A(m,d,\vec{L})\,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \ge \, c\, m^{(d-1)(1-1/q)}\, 2^{-mr}$$

holds for all $m \in \mathbb{N}_0$.

Step 1. Test functions. Only the estimate from below is of interest. For this reason we construct a sequence of test functions. For $m \ge d^2$ we put

$$f_m(x_1,...,x_d) := \sum_{\substack{u_k \ge d \\ |u|_1 = m}} e^{iN_{u_1}x_1 + ... + iN_{u_d}x_d}\,, \tag{2.28}$$

where $\{N_j\}_j$ is the given sequence of natural numbers according to $\vec{L}$, see hypothesis (H_5) and (2.27). Let us compute $\|f_m|S^r_{p,q}B(\mathbb{T}^d)\|$. Let $\{\varphi_j\}_{j=0}^\infty \in \Phi(\mathbb{R})$. We use the same notation as in Section 1.4 . Now we obtain

$$\begin{aligned} \|f_m|S^r_{p,q}B(\mathbb{T}^d)\| &= \Big(\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{r|\bar{\ell}|_1 q} \Big\| \sum_{k \in \mathbb{Z}^d} \varphi_{\bar{\ell}}(k) c_k(f_m) e^{ik\cdot x} \Big| L_p(\mathbb{T}^d)\Big\|^q\Big)^{1/q} \\ &= \Big(\sum_{\bar{\ell} \in \mathbb{N}_0^d} 2^{r|\bar{\ell}|_1 q} \Big\| \sum_{\substack{u_k \ge d \\ |u|_1 = m}} \varphi_{\bar{\ell}}(N_{u_1},...,N_{u_d}) e^{iN_{u_1}x_1 + ... + iN_{u_d}x_d} \Big| L_p(\mathbb{T}^d)\Big\|^q\Big)^{1/q} \end{aligned}$$

Because of (H_3) we have $N_j \asymp 2^j$. This and Definition 1.1/(i),(ii) imply the existence of $a \in \mathbb{N}$ such that

$$\varphi_n(N_j) \neq 0 \quad \Longrightarrow \quad |j-n| \le a.$$

And therefore

$$\|f_m|S^r_{p,q}B(\mathbb{T}^d)\| \le \Big(\sum_{\bar\ell\in A} 2^{r|\bar\ell|_1 q}\Big\|\sum_{\bar u\in B_{\bar\ell}} \varphi_{\bar\ell}(N_{u_1},...,N_{u_d})e^{iN_{u_1}x_1+...+iN_{u_d}x_d}\Big|L_p(\mathbb{T}^d)\Big\|^q\Big)^{1/q},$$

where

$$\begin{aligned} A &= \{(\ell_1,...,\ell_d)\in\mathbb{N}_0^d : \exists \bar u\in\mathbb{N}_0,\ |\bar u|_1 = m,\ |\ell_k-u_k|\le a\} \quad \text{and} \\ B_{\bar\ell} &= \{(u_1,...,u_d)\in\mathbb{N}_0 : |u_k-\ell_k|\le a,\ u_k\ge d,\ k=1,..,d,\ |\bar u|_1=m\}. \end{aligned}$$

Finally

$$\begin{aligned} \|f_m|S^r_{p,q}B(\mathbb{T}^d)\| &\le c_1 2^{rm}\Big(\sum_{\bar\ell\in A}\sum_{\bar u\in B_{\bar\ell}}\Big\|\varphi_{\bar\ell}(M(u_1),...,M(u_d))e^{iM(u_1)x_1+...+iM(u_d)x_d}\Big|L_p(\mathbb{T}^d)\Big\|^q\Big)^{1/q} \\ &\le c_2 2^{rm}\Big(\sum_{\bar\ell\in A}1\Big)^{1/q} \\ &\le c_3 2^{rm} m^{(d-1)/q} \end{aligned}$$

We refer to Remark 2.8.
With the usual modifications for $q=\infty$ we obtain as well

$$\|f_m|S^r_{p,\infty}B(\mathbb{T}^d)\| \le c\,2^{rm}.$$

Step 2. Calculation of $c_{(0,...,0)}(A(m,d,\vec{L})f_m)$. Let us first study the number $c_0(\Delta^k_{j_k}(e^{iN_{u_k}\cdot}))$. Putting

$$d_M(N) = \begin{cases} 1 & \text{if } \frac{N}{M}\in\mathbb{Z} \\ 0 & \text{otherwise} \end{cases},$$

we derive from (2.13) and $(H_1(\lambda))$

$$c_0(\Delta^k_{j_k}(e^{iN_{u_k}\cdot})) = \begin{cases} d_{N_{j_k}}(N_{u_k}) - d_{N_{j_k-1}}(N_{u_k}) & \text{if} \quad j_k\ge 1 \\ d_{N_0}(N_{u_k}) & \text{if} \quad j_k = 0 \end{cases}.$$

(H_4) and (H_5) together yield $N_{j+1}/N_j \in \mathbb{N}$, $j=0,1,\ldots$
Consequently,

$$c_0(\Delta^k_{j_k}(e^{iN_{u_k}\cdot})) = \begin{cases} -1 & \text{if} \quad j_k = u_k+1 \\ 1 & \text{if} \quad j_k = 0 \\ 0 & \text{otherwise}. \end{cases} \tag{2.29}$$

This yields

$$
\begin{aligned}
c_0\big(A(m,d,\vec{L})\, f_m\big) &= \sum_{\substack{u_k\ge d\\ |u|_1=m}} c_0\big[A(m,d,\vec{L})\big(e^{iN_{u_1}\cdot+\ldots+iN_{u_d}\cdot}\big)\big]\\
&= \sum_{\substack{u_k\ge d\\ |u|_1=m}} \sum_{j\in T_u} c_0\Big[\Big(\bigotimes_{k=1}^{d}\Delta^k_{j_k}\Big)\big(e^{iN_{u_k}\cdot+\ldots+iN_{u_d}}\big)\Big] \qquad (2.30)\\
&= \sum_{\substack{u_k\ge d\\ |u|_1=m}} \sum_{j\in T_u} c_0[\Delta^1_{j_1}(e^{iN_{u_1}\cdot})]\cdot\ldots\cdot c_0[\Delta^d_{j_d}(e^{iN_{u_d}\cdot})]\ ,
\end{aligned}
$$

where

$$
T_u \;=\; \Big\{(j_1,...,j_d)\in\mathbb{N}_0^d:\quad |j|_1\le m \quad\text{and either}\quad j_k=u_k+1 \quad\text{or}\quad j_k=0,\; k=1,...,d\Big\}\,.
$$

Clearly, T_u does not contain $(u_1+1,\ldots,u_d+1)$ because of $|u|_1=m$. Let us decompose the index set T_u into the disjoint subsets $T_u=\bigcup\limits_{\ell=1}^{d} T_u^\ell$, where

$$
T_u^\ell = \{(j_1,...,j_d)\in T_u:\quad \text{exactly } \ell \text{ components of } j \text{ vanish}\}\,,\qquad \ell=1,...,d.
$$

The set T_u^ℓ contains exactly $\binom{d}{\ell}$ elements for every u and because of (2.29) we have

$$
c_0[\Delta^1_{j_1}(e^{iN_{u_1}\cdot})]\cdot\ldots\cdot c_0[\Delta^d_{j_d}(e^{iN_{u_d}\cdot})] = (-1)^{d-\ell}\ ,\qquad j\in T_u^\ell\,.
$$

This together with (2.30) yields

$$
\begin{aligned}
c_0\big(A(m,d,\vec{L})\, f_m\big) &= \sum_{\substack{u_k\ge d\\ |u|_1=m}} \sum_{\ell=1}^{d}\sum_{j\in T_u^\ell}(-1)^{d-\ell}\\
&= \sum_{\substack{u_k\ge d\\ |u|_1=m}} \sum_{\ell=1}^{d}(-1)^{d-\ell}\binom{d}{\ell}\\
&= \sum_{\substack{u_k\ge d\\ |u|_1=m}} \sum_{\ell=0}^{d-1}(-1)^{\ell}\binom{d}{\ell}\,.
\end{aligned}
$$

Because of

$$
\sum_{\ell=0}^{d-1}(-1)^\ell\binom{d}{\ell} = \binom{d-1}{0}+\sum_{\ell=1}^{d-1}(-1)^\ell\Big(\binom{d-1}{\ell-1}+\binom{d-1}{\ell}\Big) = (-1)^{d-1}\binom{d-1}{d-1} = (-1)^{d-1}
$$

we conclude in view of Remark 2.8

$$\Big| \, c_0\big(f_m - A(m,d,\vec{L})f_m\big)\Big| = \Big| \, c_0\big(A(m,d,\vec{L})\, f_m\big)\Big| = \Big| \sum_{\substack{u_k \geq d \\ |u|_1 = m}} (-1)^{d-1} \Big| \asymp m^{d-1}\,.$$

Since we know the behaviour of $\| \, f_m \, | S^r_{p,q}B(\mathbb{T}^d)\|$, we finally get

$$\begin{aligned} \| \, I - A(m,d,\vec{L})\, |S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \;\; &\geq \;\; \frac{\| \, f_m - A(m,d)f_m \, |L_p(\mathbb{T}^d)\|}{\| \, f_m \, |S^r_{p,q}B(\mathbb{T}^d)\|} \\ &\geq \;\; c\, 2^{-rm}\, m^{(d-1)(1-1/q)} \end{aligned}$$

with some positive constant c independent of $m \in \mathbb{N}$. □

Let us collect both propositions to our main theorem.

Theorem 2.1 *Let* $1 \leq p, q \leq \infty$ *and* $r > 1/p$. *Let further* $\vec{L}$ *be given by* (H_5), *where we assume*

$$N_0 < N_1 < ... < N_j < N_{j+1} < ... \quad ,$$

and satisfy additionally the hypotheses $(H_1(\lambda))$ *for a certain* $\lambda > 0$, $\big(H_2(p,s)\big)$ *for* $1/p < s \leq r$, (H_3) *and* (H_4). *Then the relation*

$$\| \, I - A(m,d,\vec{L})\, |S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp \, m^{(d-1)(1-1/q)}\, 2^{-mr}$$

holds true.

2.4.2 Triebel-Lizorkin Spaces

Now we investigate Smolyak's algorithm for approximating functions from $S^r_{p,q}F(\mathbb{T}^d)$. It is worth underlining the result for the scale of Sobolev spaces with dominating mixed smoothness $S^r_pW(\mathbb{T}^d)$. Based on that constructive algorithm we are able to give a new upper bound for the problem of optimal recovery. See the corresponding paragraph below. We start by proving two lemmas, which state the same estimate from above under different conditions. These assertions work as corner results, which will be connected using the method of complex interpolation (see Section 1.5). We only consider the special situation $\vec{L} = (L, ..., L)$ to avoid technical difficulties. To indicate this, we may write $A(m,d,L)$ instead of $A(m,d,\vec{L})$. But nevertheless, the general case can be proved analogously.

Lemma 2.9 *Let* $1 \leq p < \infty$, $p \leq q \leq \infty$, $r > 1/p$ *and* $L = \{L_j\}_{j \in \mathbb{N}_0}$ *satisfying* $(H_1(\lambda))$ *and* $(H_2(p,s))$ *for* $1/p < s \leq r$. *Then there exists a constant* $c > 0$ *such that*

$$\|I - A(m,d,L)|S^r_{p,q}F(\mathbb{T}^d)) \to L_p(\mathbb{T}^d)\| \leq c\, m^{(d-1)(1-1/q)} 2^{-mr}$$

holds for all $m \in \mathbb{N}_0$.

Proof This assertion is a direct consequence of Lemma 1.6 and Proposition 2.4. □

Lemma 2.10 *Let $1 < p,q < \infty$, $r > 1$ and $L = \{L_j\}_{j\in\mathbb{N}_0}$ be given by (H_5) satisfying $(H_2(p,s))$ for $1 < s \le r$, $(H_6(\lambda))$ and (H_7). Then there exists a constant $c > 0$ such that*

$$\|I - A(m,d,L)|S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(1-1/q)}2^{-mr}$$

holds for all $m \in \mathbb{N}_0$.

Proof We start with the same decomposition and estimate as we did in the proof of Proposition 2.4. Here we do not pay any attention to the exact behaviour of the constants with resect to d. Again we obtain

$$\|\, f - A(m,d,L)f\,|L_p(\mathbb{T}^d)\| \le \sum_{b\in\{0,1\}^d} \|\, f^b - A(m,d,L)f^b\,|L_p(\mathbb{T}^d)\|\,.$$

and

$$f^b - A(m,d,L)f^b = \sum_{\ell\in Q^m_b}\sum_{j\in\Lambda^m_\ell}\Big(\bigotimes_{n=1}^d \Delta_{j_n}\Big)f_\ell\,, \tag{2.31}$$

which implies in the case $|b|_1 \ge 2$ the relation

$$\begin{aligned}\|\, f_b - A(m,d,\vec{L})f_b\,|L_p(\mathbb{T}^d)\| &\le 2^{-mr_0}\sum_{\ell\in Q^m_b} 2^{(r_0-r)|\ell|_1}|\Lambda^m_\ell|\,\|\, f\,|S^r_{p,\infty}B(\mathbb{T}^d)\|\\ &\le c_1\,2^{-mr}m^d 2^{m(r_0-r)(|b|_1-1)}\|\, f\,|S^r_{p,\infty}B(\mathbb{T}^d)\| \qquad (2.32)\\ &\le c_2\,2^{-mr}\|\, f\,|S^r_{p,\infty}B(\mathbb{T}^d)\|\\ &\le c_3 2^{-mr}\|f\,|S^r_{p,q}F(\mathbb{T}^d)\,.\end{aligned}$$

The last estimate follows by Lemma 1.6/(1.28). For the case $|b|_1 \le 1$ let us go back to (2.31) and estimate

$$|f^b - A(m,d,L)f^b| \le \sum_{\ell\in Q^m_b}\sum_{j\in\Lambda^m_\ell}\sum_{u\in U_j}\Big|\Big(\bigotimes_{n=1}^d L_{u_n}\Big)f_\ell\Big|\,, \tag{2.33}$$

where

$$U_j = \big\{(u_1,...,u_d)\in\mathbb{N}^d_0 : u_k = j_k \text{ or } u_k = j_k - 1\big\}\,.$$

Of course, the cardinality $|U_j|$, $j\in\mathbb{N}^d_0$, is less or equal to 2^d. Next we use (2.12) to obtain

$$\Big|\Big(\bigotimes_{k=1}^d L_{u_k}\Big)f_\ell(x)\Big| \le \sum_{w\in\mathbb{Z}^d}\Big|\underbrace{\sum_{k\in\mathbb{Z}^d}\gamma_{u_1}(k_1 - w_1N_{u_1})\cdot\ldots\cdot\gamma_{u_d}(k_d - w_dN_{u_d})c_k(f_\ell)e^{ikx}}_{=:f_{\ell,u,w}(x)}\Big|\,.$$

Inserted into (2.33) and taking the L_p-norm afterwards this leads to

$$\|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\| \le \Big\| \sum_{\ell\in Q_b^m} \sum_{j\in\Lambda_\ell^m} \sum_{u\in U_j} \sum_{w\in\mathbb{Z}^d} |f_{\ell,u,w}(x)| \Big| L_p(\mathbb{T}^d)\Big\| .$$

Let us restrict the range for w in the corresponding sum. We do not need the entire $\mathbb{Z}^d$ here. Suppose that the trigonometric polynomial $f_{\ell,u,w}(x)$ does not vanish. Taking $(H_6(\lambda))$ into account, this implies the existence of a vector $k\in\mathbb{Z}^d$ such that

$$|k_i - w_i N_{u_i}| \le A_{u_i} \quad \text{and} \quad |k_i| \le 2^{\ell_i} \quad , \quad i=1,...,d\,. \tag{2.34}$$

Now again hypothesis $(H_6(\lambda))$ comes into play. The first condition in (2.34) implies for all $i=1,...,d$

$$\begin{aligned} |k_i| &\ge |w_i|N_{u_i} - A_{u_i} \\ &\ge |w_i|N_{u_i} - (N_{u_i} - \lambda 2^{u_i}) \\ &\ge C_1(|w_i|-1)2^{u_i}\,. \end{aligned}$$

Because of $u\in U_j$ and $j\in\Lambda_\ell^m$ there exists a number $\eta>0$ such that for $|w_i|\ge 2^{\ell_i-u_i+\eta}>1$ holds

$$C_1(|w_i|-1)2^{u_i} > 2^{\ell_i}\,.$$

This means, the condition $|w_i|\ge 2^{\ell_i-u_i+\eta}$ implies $|k_i|>2^{\ell_i}$. Altogether (2.34) leads to the following index set for w:

$$W_{\ell,u} := \big\{(w_1,...,w_d)\in\mathbb{Z}^d : |w_i|\le 2^{\ell_i-u_i+\eta}, i=1,...,d\big\} \quad , \quad \text{for some } \eta>0\,. \tag{2.35}$$

We go further in estimating $\|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\|$ by Hölder's inequality using $1/q + 1/q' = 1$ and obtain

$$\begin{aligned} \|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\| \le &\Big(\sum_{\ell\in Q_b^m} \sum_{j\in\Lambda_\ell^m} \sum_{u\in U_j} \sum_{w\in W_{\ell,u}} 2^{-r|\ell|_1 q'} |\Lambda_\ell^m|^{q'/q} |W_{u,\ell}|^{q'/q}\Big)^{1/q'} \times \\ &\times \Big\| \Big(\sum_{\ell\in Q_b^m} \sum_{j\in\Lambda_\ell^m} \sum_{u\in U_j} \sum_{w\in W_{\ell,u}} 2^{r|\ell|_1 q} |\Lambda_\ell^m|^{-1} |W_{\ell,u}|^{-1} |f_{\ell,u,w}(x)|^q\Big)^{1/q} \Big| L_p(\mathbb{T}^d)\Big\| . \end{aligned}$$

Because of (H_7), Remark 1.3 can be considered in this situation. Lizorkin's multiplier theorem, see Proposition 1.8, is applicable and we obtain from the previous estimate

$$\begin{aligned} \|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\| \le c_9 &\Big(\sum_{\ell\in Q_b^m} \sum_{j\in\Lambda_\ell^m} \sum_{u\in U_j} 2^{-r|\ell|_1 q'} |\Lambda_\ell^m|^{q'/q} |W_{u,\ell}|^{q'/q+1}\Big)^{1/q'} \times \\ &\times \Big\| \Big(\sum_{\ell\in Q_b^m} \sum_{j\in\Lambda_\ell^m} \sum_{u\in U_j} \sum_{w\in W_{\ell,u}} 2^{r|\ell|_1 q} |\Lambda_\ell^m|^{-1} |W_{\ell,u}|^{-1} |f_\ell(x)|^q\Big)^{1/q} \Big| L_p(\mathbb{T}^d)\Big\| . \end{aligned}$$

This is a very comfortable situation since we can compute each sum one after another inside the second factor. Taking into account $q'/q + 1 = q'$ we obtain

$$\begin{aligned}\|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\| \le & c_{10}\Big\|\Big(\sum_{\ell\in Q_b^m} 2^{r|\ell|_1 q}|f_\ell(x)|^q\Big)^{1/q}\Big|L_p(\mathbb{T}^d)\Big\|\times \\ & \times\Big(\sum_{\ell\in Q_b^m}\sum_{j\in\Lambda_\ell^m}\sum_{u\in U_j} 2^{-r|\ell|_1 q'}|\Lambda_\ell^m|^{q'/q}|W_{u,\ell}|^{q'}\Big)^{1/q'}.\end{aligned}$$

It remains to estimate the second factor in the previous inequality. We start by estimating the size of the index sets Λ_ℓ^m and $W_{\ell,u}$. Because of $u_i \ge j_i - 1$ and $|j|_1 > m$ the definition of $W_{\ell,u}$, see (2.35), implies

$$|W_{\ell,u}| \le 2^{|\ell|_1 - |j|_1 + d(\eta+3)} \le 2^{|\ell|_1 - m + d(\eta+3)}.$$

On the basis of

$$\Lambda_\ell^m \subset \Big[m - \sum_{\substack{n=1\\ n\neq 1}}^{d}(\ell_n + n_\lambda + 1), \ell_1 + n_\lambda + 1\Big] \times\cdots\times \Big[m - \sum_{\substack{n=1\\ n\neq d}}^{d}(\ell_n + n_\lambda + 1), \ell_d + n_\lambda + 1\Big]$$

we derive

$$|\Lambda_\ell^m| \le \big(|\ell|_1 + d(n_\lambda+1) + 1 - m\big)^d.$$

Thus it holds

$$\|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\| \le c_{11}\,\|f|S^r_{p,q}F(\mathbb{T}^d)\|\Big(\sum_{\ell\in Q_b^m} 2^{-r|\ell|_1 q'}|\Lambda_\ell^m|^{q'}2^{(|\ell|_1-m)q'}\Big)^{1/q'}.$$

Without loss of generality we may assume that $b_1 = |b|_1$. The index transform $u := |\ell|_1 + d(n_\lambda+1) + 1 - m$ yields

$$\begin{aligned}&\Big(\sum_{\ell\in Q_b^m} 2^{-r|\ell|_1 q'}|\Lambda_\ell^m|^{q'}2^{(|\ell|_1-m)q'}\Big)^{1/q'}\\ &\le \Big(\sum_{\ell\in Q_b^m} 2^{-r|\ell|_1 q'}\big(|\ell|_1 + d(n_\lambda+1)+1-m\big)^{dq'}2^{(|\ell|_1-m)q'}\Big)^{1/q'}\\ &\le c_{12}\Big(\sum_{\ell_2,\dots,\ell_d=0}^{s_m}\sum_{u=0}^{\infty} 2^{-r(u+m)q'}u^{dq'}2^{uq'}\Big)^{1/q'}\\ &= c_{12}\,2^{-mr}\Big(\sum_{\ell_2,\dots,\ell_d=0}^{s_m}\sum_{u=0}^{\infty} 2^{u(1-r)q'}u^{dq'}\Big)^{1/q'}.\end{aligned}$$

Finally the condition $r > 1$ enters the stage. It is needed to ensure the finiteness of the sum with respect to u. Altogether everything leads to

$$\begin{aligned}\|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\| &\le c_{13}\,2^{-mr}m^{(d-1)/q'}\|f|S^r_{p,q}F(\mathbb{T}^d)\|\\ &= c_{13}\,2^{-mr}m^{(d-1)(1-1/q)}\|f|S^r_{p,q}F(\mathbb{T}^d)\|.\end{aligned}$$

Summing up over b by taking (2.32) into account this completes the proof. □

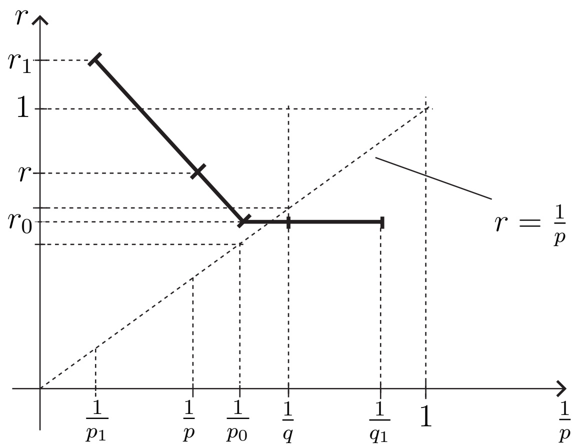
Next we connect Lemma 2.9 and 2.10 by complex interpolation to obtain the result below.

Proposition 2.6 *Assume that* (p,q) *belongs to the set* $[1,\infty)\times(1,\infty]\cup\{(1,1)\}$. *Let further* $r > \max(1/p, 1/q)$. *The sequence* $L = \{L_j\}_{j=0}^{\infty}$ *is given by* (H_5) *and supposed to satisfy* $\big(H_2(u,s)\big)$ *for all* $1 < u < \infty$, $s > 1/u$ *as well as* $\big(H_2(p,s)\big)$ *for all* $s > 1/p$. *Further* $\big(H_6(\lambda)\big)$ *for a certain* $\lambda > 0$ *and* (H_7) *is assumed. Then it exists a constant* $C > 0$ *such that*

$$\|I - A(m,d,L) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le C 2^{-rm} m^{(d-1)(1-1/q)} \tag{2.36}$$

holds for all $m \in \mathbb{N}_0$.

Proof Having Lemma 2.9 and Lemma 2.10 it remains to prove (2.36) for $1 < q < p < \infty$ and $1/q < r \le 1$. We use complex interpolation and the result stated in Theorem 1.7. Furthermore we shall use the relations stated in Lemma 2.9 and 2.10 as corner information to interpolate the norm of the operator $I - A(m,d,L)$. To do this we have to construct appropriate parameter triples (p_0, q_0, r_0) and (p_1, q_1, r_1). Firstly, they have to satisfy the conditions of Lemma 2.10 and Lemma 2.9, respectively. And secondly, the triple (p,q,r) has to be an intermediate tuple. The figure below gives an idea what we have to realize.



Step 1. We start by choosing p_0 such that

$$\frac{1}{p} < \frac{1}{p_0} < \frac{q}{p(q-1)+q} < \frac{1}{q}\,. \tag{2.37}$$

This is possible because

$$\frac{1}{p} = \frac{q}{pq} = \frac{q}{p(q-1)+p} \underset{(p>q)}{<} \frac{q}{p(q-1)+q} \underset{(p>q)}{<} \frac{q}{q^2-q+q} = \frac{1}{q}\,.$$

Furthermore we choose r_0 such that

$$\frac{1}{p_0} < r_0 < r - \frac{1/q - 1/p_0}{1 - 1/q}(1-r) \le r\,.$$

This is possible because of

$$0 \le \frac{1/q - 1/p_0}{1-1/q}(1-r) < \frac{1/q - 1/p_0}{1-r}(1-r) < r - 1/p_0\,.$$

This choice has the following consequences

(a)

$$\frac{1-r}{r-r_0} < \frac{1-1/q}{1/q - 1/p_0}$$

(b) and

$$\frac{1/p_0 - 1/p}{1/p} < \frac{1/q - 1/p_0}{1-1/q} \quad .$$

Let us give a short comment on (b). Having (2.37) we conclude

$$\frac{1}{p_0} < \frac{q}{p(q-1)+q} = \frac{q-1+1}{p(q-1)+q} = \frac{1+\frac{1}{q-1}}{p+\frac{q}{q-1}}$$

which implies (b). The inequalities in (a) and (b) allow us to choose a number $1 < q_1 < q$ such that it holds simultaneously

$$\frac{1-r}{r-r_0} < \frac{1/q_1 - 1/q}{1/q - 1/p_0} \quad \text{and} \quad \frac{1/p_0 - 1/p}{1/p} < \frac{1/q - 1/p_0}{1/q_1 - 1/q}\,.$$

And finally this implies the existence of $r_1 > 1$, $1/p_1 < 1/p$ and $0 < \vartheta < 1$ such that

$$\frac{1-\vartheta}{\vartheta} = \frac{r_1 - r}{r - r_0} = \frac{1/q_1 - 1/q}{1/q - 1/p_0} \quad \text{and} \quad \frac{\vartheta}{1-\vartheta} = \frac{1/p_0 - 1/p}{1/p - 1/p_1} = \frac{1/q - 1/p_0}{1/q_1 - 1/q}\,.$$

And therefore with $q_0 := p_0$ it follows

$$(r, 1/p, 1/q) = (1-\vartheta)\big(r_0, 1/p_0, 1/q_0\big) + \vartheta\big(r_1, 1/p_1, 1/q_1\big)\,, \tag{2.38}$$

where $r_0 > 1/p_0, 1 < p_0 = q_0 < \infty$ and $r_1 > 1$, $1 < p_1, q_1 < \infty$.

Step 2. As a consequence of Lemma 2.10 we obtain

$$\|I - A(m,d,L) : S^{r_0}_{p_0,q_0}F(\mathbb{T}^d) \to L_{p_0}(\mathbb{T}^d)\| \le c_1 2^{-mr_0} m^{(d-1)(1-1/q_0)}\,.$$

Additionally Lemma 2.9 implies

$$\|I - A(m,d,L) : S^{r_1}_{p_1,q_1}F(\mathbb{T}^d) \to L_{p_1}(\mathbb{T}^d)\| \le c_2 2^{-mr_1} m^{(d-1)(1-1/q_1)}\,.$$

We finish the proof by applying Theorem 1.7, Lemma 1.8, (1.42) and (2.38) to obtain

$$\begin{aligned}&\|I - A(m,d,L) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \\ &\leq \|I - A(m,d,L) : S^{r_0}_{p_0,q_0}F \to L_{p_0}\|^{1-\vartheta} \cdot \|I - A(m,d,L) : S^{r_1}_{p_1,q_1}F \to L_{p_1}\|^{\vartheta} \\ &\leq c_3\, 2^{-mr_0(1-\vartheta)} m^{(d-1)(1-1/q_0)(1-\vartheta)} \cdot 2^{-mr_1\vartheta} m^{(d-1)(1-1/q_0)\vartheta} \\ &= c_3\, 2^{-mr} m^{(d-1)(1-1/q)}\,.\end{aligned}$$

□

The estimate from below is covered by the following assertion.

Proposition 2.7 *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $r > 1/p$ and $L = \{L_j\}_{j=0}^{\infty}$ be given by (H_5), where we assume*

$$N_0 < N_1 < ... < N_j < N_{j+1} < ... \quad .$$

Let further $(H_1(\lambda))$ hold true for a certain $\lambda > 0$ and (H_3) as well as (H_4) are satisfied. Then there exists a constant $C > 0$ such that

$$\|I - A(m,d,L) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \geq C\, 2^{-rm} m^{(d-1)(1-1/q)}\,, \quad m \in \mathbb{N}_0\,.$$

Proof The proof can be transferred almost word by word from Proposition 2.5. By the same arguments we obtain

$$\|f_m | S^r_{p,q}F(\mathbb{T}^d)\| \leq c\, 2^{rm} m^{(d-1)/q}\,.$$

What remains is completely the same as there. □

We combine Proposition 2.6 and 2.7 to the main theorem for the F-scale. It reads as follows.

Theorem 2.2 *Assume that (p,q) belongs to the set $[1,\infty) \times (1,\infty] \cup \{(1,1)\}$. Let further $r > \max(1/p, 1/q)$. The sequence $L = \{L_j\}_{j=0}^{\infty}$ is given by (H_5), where we assume*

$$N_0 < N_1 < ... < N_j < N_{j+1} < ... \quad ,$$

and supposed to satisfy $\big(H_2(u,s)\big)$ for all $1 < u < \infty$, $s > 1/u$ as well as $\big(H_2(p,s)\big)$ for all $s > 1/p$. Further (H_3), (H_4), $\big(H_6(\lambda)\big)$ for a certain $\lambda > 0$ and (H_7) are satisfied. Then it holds

$$\|I - A(m,d,L) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-rm} m^{(d-1)(1-1/q)} \quad , \quad m \in \mathbb{N}_0\,.$$

2.4.3 Sobolev Spaces

Let us formulate an important result for Sobolev spaces with dominating mixed smoothness, defined in Paragraph 1.4.4. The following corollary is a special case of Theorem 2.2 applied with $q = 2$ (see also Corollary 1.2).

Corollary 2.2 *Let $1 < p < \infty$. Let further $r > \max(1/p, 1/2)$. The sequence $L = \{L_j\}_{j=0}^{\infty}$ is given by (H_5), where we assume*

$$N_0 < N_1 < ... < N_j < N_{j+1} < ... ,$$

and supposed to satisfy $\big(H_2(u,s)\big)$ for all $1 < u < \infty$ and $s > 1/u$. Moreover, we assume (H_3), (H_4), $\big(H_6(\lambda)\big)$ for a certain $\lambda > 0$ and (H_7). Then it holds

$$\|I - A(m,d,L) : S_p^r W(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-rm} m^{(d-1)/2} \quad , \quad m \in \mathbb{N}_0 \, .$$

2.4.4 Non-Equidistant Knots

Using a simple bump function argument it turns out that we cannot improve the approximation properties of Smolyak's algorithm by admitting general sequences L, at least in the case $q = 1$.

Theorem 2.3 *Let $\vec{L}$ satisfy the hypotheses (H_1), (H_2), (H_3) and (H_3'). Then the following two relations hold true.*

(i) If $1 \leq p \leq \infty$ and $r > 1/p$ then

$$\|I - A(m,d,\vec{L}) | S_{p,1}^r B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-rm} \, .$$

(ii) If $1 \leq p < \infty$ and $r > 1/p$ then there exist two constants c_1 and c_2 such that

$$c_1 \, 2^{-rm} \leq \|I - A(m,d,\vec{L}) | S_{p,1}^r F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \leq c_2 \, 2^{-rm} m^{(d-1)(1-1/p)} \, .$$

Proof *Step 1.* We shall use the concept of periodic bump functions. Let $\widetilde{B}$ be a compactly supported smooth function such that $\operatorname{supp} \widetilde{B} \subset \{x \in \mathbb{R}^d : |x| \leq 1\}$. Its 2π-periodic extension is denoted by B. Obviously, $B \in S_{p,q}^r A(\mathbb{T}^d)$, $1 \leq p,q \leq \infty$, $r \geq 0$. Furthermore, if $\lambda = (\lambda_1, \ldots, \lambda_d) \geq 1$ is given then $B(\lambda \cdot)$ denotes the 2π-periodic extension of $\widetilde{B}(\lambda \cdot)$. Let us collect some properties in the following lemma.

Lemma 2.11 *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\lambda = (\lambda_1, ..., \lambda_d)$. Let $r > 1/p$. Then there exists a positive constant c such that*

$$\| \, B(\lambda \cdot) \, | S_{p,q}^r B(\mathbb{T}^d)\| \leq c \, \lambda_1^{r-1/p} \cdot \, \ldots \, \cdot \lambda_d^{r-1/p} \, \| \, B(\cdot) \, | S_{p,q}^r B(\mathbb{T}^d)\| \qquad (2.39)$$

and

$$\| \, B(\lambda \cdot) \, | L_p(\mathbb{T}^d)\| = \lambda_1^{-1/p} \cdot ... \cdot \lambda_d^{-1/p} \, \| \, B(\cdot) \, | L_p(\mathbb{T}^d)\|$$

holds for all $\lambda \geq 1$. If $p < \infty$ and additionally $r > 1/q$, then also

$$\| \, B(\lambda \cdot) \, | S_{p,q}^r F(\mathbb{T}^d)\| \leq c \, \lambda_1^{r-1/p} \cdot ... \cdot \lambda_d^{r-1/p} \, \| \, B(\cdot) \, | S_{p,q}^r F(\mathbb{T}^d)\| \, .$$

Proof These are simple consequences of characterizations of $S^r_{p,q}B(\mathbb{T}^d)$ and $S^r_{p,q}F(\mathbb{T}^d)$ by differences. We refer to Theorem 1.12 and 1.14 and obtain in the B-case

$$\| f \,|S^r_{p,q}B(\mathbb{T}^d)\|^{\Delta} := \| f \,|L_p(\mathbb{T}^d)\| + \sum_{n=1}^{d} \sum_{\bar\beta\in\{0,1\}^d, |\bar\beta|_1=n} S^{\Delta}_{\bar\beta}(f)\,,$$

where for $|\bar\beta|_1 = n \geq 1$

$$S^{\Delta}_{\bar\beta}(f) := \Bigg[\int\limits_{\mathbb{R}^n} \big(\prod_{i=1}^{n} |h_i|^{-rq}\big) \big\| \big(\triangle^m_{h_1,\delta_1} \circ \cdots \circ \triangle^m_{h_n,\delta_n} f\big)(x) \big| L_p(\mathbb{T}^d)\big\|^q \, \frac{dh_1}{|h_1|} \cdots \frac{dh_n}{|h_n|}\Bigg]^{1/q} \quad ,$$

and $\delta = (\delta_1, \ldots, \delta_n)$ is defined by means of $\bar\beta_{\delta_i} = 1,\ i = 1, ..., n$.
For abbreviation we put $B_\lambda(x) = B(\lambda x)$. It is not difficult to recognize

$$\big(\triangle^m_{h_1,\delta_1} \circ \cdots \circ \triangle^m_{h_d,\delta_d} B_\lambda\big)(x) = \big(\triangle^m_{\lambda_{\delta_1}\cdot h_1,\delta_1} \circ \cdots \circ \triangle^m_{\lambda_{\delta_d}\cdot h_d,\delta_d} B\big)(\lambda_1 x_1, ..., \lambda_d x_d)\ ,$$

which corresponds to the well-known formula $(\triangle^M_h f(\lambda\cdot)(t) = \triangle^M_{\lambda h} f(\lambda t)$, $t \in \mathbb{R}$. Finally, a change of variable yields

$$S^{\Delta}_{\bar\beta}(B_\lambda) = \lambda^{r-1/p}_{\delta_1} \cdot ... \cdot \lambda^{r-1/p}_{\delta_d} S^{\Delta}_{\bar\beta}(B)\ .$$

Now (2.39) is a consequence of $\lambda_i \geq 1$ and $r > 1/p$. □

Step 2. Proof of Theorem 2.3. Only the estimate from below in the B-case is of interest. The F-case follows from elementary embeddings, see Lemma 1.6/(iii). Associated to $L = (L^1, ..., L^d)$ is the sequence of grids $\mathcal{G}(m, d, \vec{L})$, $m \in \mathbb{N}_0$, see (2.14). For simplicity we concentrate for the moment on the first component. Because of (H_3) we find

$$\Big|\bigcup_{j=0}^{m} \mathcal{T}^1_j\Big| \leq C_2\, 2^{m+1}\,.$$

Consequently, for every $m \in \mathbb{N}_0$ there exists an open interval $\mathcal{I}_m \subset [-\pi, \pi]$, $|\mathcal{I}_m| = \frac{1}{C_2} 2^{-(m+1)}$, such that

$$\mathcal{I}_m \cap \Big(\bigcup_{j=0}^{m} \mathcal{T}^1_j\Big) = \emptyset\,.$$

Therefore we can find a rectangle $R_m := \mathcal{I}_m \times [-\pi, \pi] \times ... \times [-\pi, \pi]$ providing

$$R_m \cap \mathcal{G}(m, d, \vec{L}) = \emptyset\,.$$

Let B denote the function investigated in Lemma 2.11. We choose $\lambda_1 = C_2\, 2^{m+1}$ and $\lambda_2 = ... = \lambda_d = 1$. If x^m denotes the centre of R_m the function $B(\lambda(\cdot\, - x^m))$ vanishes in $\mathcal{G}(m, d, \vec{L})$. In view of Lemma 2.11 this implies

$$\frac{\| B(\lambda(\cdot\, - x^m)) - A(m, d, \vec{L})(B(\lambda(\cdot\, - x^m)))\, |L_p(\mathbb{T}^d)\|}{\| B(\lambda(\cdot\, - x^m))\, |S^r_{p,1}B(\mathbb{T}^d)\|} \geq c\lambda_1^{-r}$$

where the corresponding constants do not depend on m. □

2.5 Examples

2.5.1 Sampling Operators

First we consider interpolation with de la Vallée Poussin means, cf. Subsection 2.2.3. We put

$$A(m,d,\mu) := A(m,d,L)\,, \qquad L_j := I(\Lambda^{\pi}_{\mu,2^j},\,\cdot\,)\,,\, j\in\mathbb{N}_0\,. \tag{2.40}$$

L is of type (H_5). As an immediate consequence of Corollary 2.1 we obtain that L satisfies $(H_2(u,s))$ for every admissible pair (u,s). A simple calculation shows that $(H_6(\lambda))$ is satisfied with $\lambda = 1/2-\mu$ and therefore also $(H_1(\lambda))$. Since $I(\Lambda^{\pi}_{\mu,2^j},\,\cdot\,)$ uses function values from the standard grid J_{2^j} also (H_3), (H_3') and (H_4) are fulfilled. Obviously also (H_7) holds true, which follows from (2.7) together with (2.4). Altogether Theorem 2.1, 2.2 and Corollary 2.2 yield the following.

Corollary 2.3 *Let $0<\mu<1/2$.*

(i) If $1\le p,q\le\infty$ and $r>1/p$ then

$$\|\,I - A(m,d,\mu)\,|S^r_{p,q}B(\mathbb{T}^d)\to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)(1-1/q)}\,2^{-mr}\,.$$

(ii) If $(p,q)\in[1,\infty)\times(1,\infty]\cup\{(1,1)\}$ and $r>\max(1/p,1/q)$ then

$$\|\,I - A(m,d,\mu)\,|S^r_{p,q}F(\mathbb{T}^d)\to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)(1-1/q)}\,2^{-mr}\,.$$

(iii) If $1<p<\infty$ and $r>\max(1/p,1/2)$ then

$$\|\,I - A(m,d,\mu)\,|S^r_{p}W(\mathbb{T}^d)\to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)/2}2^{-mr}\,.$$

As a second example we consider Smolyak's algorithm associated to the interpolation operators $\mathcal{R}_n$, cf. (2.8) or [45, 1.6]. Putting

$$A(m,d,\mathcal{R}) := A(m,d,L) \quad , \quad \text{where} \quad L_j := \mathcal{R}_{2^j}\,, j\in N_0\,,$$

we obtain with exactly the same arguments as above.

Corollary 2.4 *(i) If $1\le p,q\le\infty$ and $r>1/p$ then*

$$\|\,I - A(m,d,\mathcal{R})\,|S^r_{p,q}B(\mathbb{T}^d)\to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)(1-1/q)}\,2^{-mr}\,.$$

(ii) If $(p,q)\in[1,\infty)\times(1,\infty]\cup\{(1,1)\}$ and $r>\max(1/p,1/q)$ then

$$\|\,I - A(m,d,\mathcal{R})\,|S^r_{p,q}F(\mathbb{T}^d)\to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)(1-1/q)}\,2^{-mr}\,.$$

(iii) If $1<p<\infty$ and $r>\max(1/p,1/2)$ then

$$\|\,I - A(m,d,\mathcal{R})\,|S^r_{p}W(\mathbb{T}^d)\to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)/2}2^{-mr}\,.$$

Remark 2.11 *Lemma 2.7 and Remark 2.3 imply that $A(m,d,1/6)\,f$ interpolates every continuous function $f \in C(\mathbb{T}^d)$ on the used sampling grid $\mathcal{G}(m,d,1/6)$ given by*

$$\mathcal{G}(m,d,1/6) := \Big\{ \Big(\frac{2\pi\ell_1}{3\cdot 2^{j_1}}, \ldots , \frac{2\pi\ell_d}{3\cdot 2^{j_d}} \Big) : \\ -3\cdot 2^{j_i-1} \le \ell_i < 3\cdot 2^{j_i-1}, i=1,\ldots,d, \quad m-d+1 \le |j|_1 \le m \Big\} .$$

The last example of a sparse grid sampling operator is the following classical one. We consider interpolation by means of the Dirichlet kernel, i.e. we put

$$A(m,d,D) := A(m,d,L)\,, \qquad L_j := I_{2^j}\,, \quad j \in \mathbb{N}_0\,.$$

Here $H_2(u,s)$ is satisfied for $1 < u < \infty$ and $s > 1/u$. Contrarily (H_4) is not satisfied and hence the Propositions 2.5, 2.7 and the Theorems 2.1, 2.2 are not applicable in the stated form. With specific modifications of the family of testfunctions used in the proof, see (2.28), it is also possible to obtain a sharp estimate from below (see [35] for the bivariate case). Nevertheless, Proposition 2.4 and 2.6 give us the following.

Corollary 2.5 *(i) If $1 < p < \infty$, $1 \le q \le \infty$ and $r > 1/p$ then there a constant $c > 0$ such that*

$$\| I - A(m,d,D)\,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(1-1/q)}\, 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

(ii) If $1 < p,q < \infty$ and $r > \max(1/p,1/q)$ then there exists a $c > 0$ such that

$$\| I - A(m,d,D)\,|S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(1-1/q)}\, 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

(iii) If $1 < p < \infty$ and $r > \max(1/p,1/2)$ then there exists a $c > 0$ such that

$$\| I - A(m,d,D)\,|S^r_{p}W(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)/2} 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

Remark 2.12 (i) *The operator $A(m,d,D)$ uses function values from the grid*

$$\mathcal{G}^*(m,d) := \Big\{ \Big(\frac{2\pi\ell_1}{2^{j_1+1}+1}, \ldots , \frac{2\pi\ell_d}{2^{j_d+1}+1} \Big) : \\ 0 \le \ell_i \le 2^{j_i+1}, i=1,\ldots,d, \quad m-d+1 \le |j|_1 \le m \Big\} .$$

But in general $A(m,d,D)$ is not interpolating. To see this, it is sufficient to consider the operator $A(1,2,D)$ applied to the function $f(x_1,x_2) = e^{i3x_1}$ at the point $(2\pi/3,0) \in \mathcal{G}^*(1,2)$. Of course, the reason for this consists in

$$\Big\{ \frac{2\pi\ell}{2^{j+1}+1} : \quad 0 \le \ell \le 2^{j+1} \Big\} \not\subset \Big\{ \frac{2\pi\ell}{2^{j}+1} : \quad 0 \le \ell \le 2^{j} \Big\} .$$

(ii) *Let us return to the operator $A(m,d,\mathcal{R})$. Here Lemma 2.7 is not applicable in the stated form. The condition* (2.19) *is not satisfied, however* (H_4) *is. In the case* $d=2$ *we have the following*

$$\begin{aligned} A(2,2,\mathcal{R})\, f_2(x_1,x_2) &= e^{i2x_1}+e^{i2x_2}+e^{i4x_1}+e^{i4x_2} \quad , \\ A(3,2,\mathcal{R})\, f_3(x_1,x_2) &= e^{i2x_1}+e^{i2x_2}+e^{i4x_1}+e^{i4x_2}+e^{i8x_1}+e^{i8x_2} \quad , \end{aligned} \tag{2.41}$$

where the test functions f_m are defined in Section 2.7, cf. Remark 2.17. This shows immediately that $A(m,d,\mathcal{R})$ does not interpolate.

2.5.2 Convolution Operators

It turned out that the Smolyak algorithm applied to sampling operators yields a worst case within a wider class of Smolyak algorithms. In particular, applied to the partial sum of the Fourier series, it behaves better in approximation order than the Smolyak algorithm with respect to a sampling operator. Precisely, we do the following. Let us denote by $A(m,d,S)$ the operator $A(m,d,L)$ associated to the sequence

$$L_j := S_{2^j} \quad , \quad j \in \mathbb{N}_0 \, .$$

Of course Definition 2.1 makes also sense in this situation. This operator and its approximation properties have been studied very intensively in the former Soviet Union, for instance by Bugrov, Dinh Dung, Galeev, Nikol'skaya, Nikol'skij, Romanyuk and Temlyakov. However, in contrast to the previous considerations, the operator $A(m,d,S)$ leads to better results, namely:

Theorem 2.4 *Let $1<p<\infty$, $1\le q\le\infty$ and $r>0$. Then*

$$\begin{aligned} &\| \, I - A(m,d,S)\, | S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \\ &\quad \asymp \begin{cases} m^{(d-1)(\frac{1}{p}-\frac{1}{q})}\, 2^{-mr} & \text{if } \; 1<p\le 2 \text{ and } \; p\le q\le\infty\, , \\ m^{(d-1)(\frac{1}{2}-\frac{1}{q})}\, 2^{-mr} & \text{if } \; 2<p<\infty \text{ and } \; 2<q\le\infty\, , \\ 2^{-mr} & \text{otherwise}\, , \end{cases} \end{aligned}$$

$m \in \mathbb{N}_0$.

Proof A proof can be found in [39, Thm. 5] and further references. □

Furthermore in the F-case we have the following result.

Theorem 2.5 *Let $1<p<\infty$, $1<q<\infty$ and $r>0$. Then we have*

$$\|I - A(m,d,S) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-mr} m^{(d-1)(1/2-1/q)_+} \quad , \quad m \in \mathbb{N}\, .$$

Proof A proof can be found in [53, Prop. 5]. □

Of significant interest is the behaviour for Sobolev type spaces. Here we obtain as a special case ($q = 2$, see Corollary 1.2) of the previous theorem.

Theorem 2.6 *Let $1 < p < \infty$ and $r > 0$. Then we have*

$$\| I - A(m,d,S) : S^r_p W(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \asymp 2^{-mr} \quad , \quad m \in \mathbb{N} .$$

2.6 Optimal Recovery of Functions

In this section we study the question of optimal recovery of a function from a finite number of function values. In order to define the quantity ρ_M, already mentioned in the introduction, we introduce the following framework.
Let

$$\Psi_M(f,\xi)(x) := \sum_{j=1}^{M} f(\xi^j)\, \psi_j(x)$$

denote a general sampling operator for a class F of continuous, periodic functions defined on $\mathbb{T}^d$, where

$$\xi := \left\{ \xi^1, ..., \xi^M \right\}, \qquad \xi^i \in \mathbb{T}^d , \quad i = 1, 2, \ldots , M ,$$

is a fixed set of sampling points and $\psi_j : \mathbb{T}^d \to \mathbb{C}$, $j = 1, \ldots , M$, are fixed, continuous, periodic functions. Then the quantity

$$\rho_M(F, L_p(\mathbb{T}^d)) := \inf_{\xi} \inf_{\psi_1,\ldots,\psi_M} \sup_{\|f|F\| \le 1} \| f - \Psi_M(f,\xi)\, |L_p(\mathbb{T}^d)\|$$

measures the optimal rate of approximate recovery of the functions taken from F. We are interested in the case, when $F = S^r_p W(\mathbb{T}^d)$, $1 < p < \infty$, $r > 1/p$, $F = S^r_{p,q} F(\mathbb{T}^d)$, $1 \le p < \infty$, $1 \le q \le \infty$, $r > 1/p$ and $F = S^r_{p,q} B(\mathbb{T}^d)$, where $1 \le p, q \le \infty$ and $r > 1/p$. Observe, that the operator $A(m,d,\mu)$ uses $M = M(m,d) \asymp 2^m\, m^{d-1}$ function values from its argument (see (2.40) and Remark 2.8). Therefore $m \le c \log M$ with some c independent of m and hence

$$2^{-rm} m^{(d-1)(1-1/q)} \le M^{-r} (c \log M)^{(d-1)(r+1-1/q)} .$$

In view of Corollary 2.3 this implies the upper bound given below.

Corollary 2.6 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r > 1/p$. Then there exist positive constants c_1 and c_2 such that for all $M \in \mathbb{N}$*

$$\begin{aligned} c_1\, M^{-r} (\log M)^{(d-1)r} \eta(M,d,p,q) &\le \rho_M(S^r_{p,q} B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \\ &\le c_2\, M^{-r} (\log M)^{(d-1)(r+1-1/q)} , \end{aligned}$$

where

$$\eta(M,d,p,q) := \begin{cases} (\log M)^{(d-1)(\frac{1}{2}-\frac{1}{q})} & \textit{if} \quad 2 \le p,q\,, \\ (\log M)^{(d-1)(\frac{1}{p}-\frac{1}{q})} & \textit{if} \quad 1<p<2 \quad \textit{and} \quad p \le q\,, \\ 1 & \textit{otherwise}\,. \end{cases} \tag{2.42}$$

Proof The estimate from above follows from Corollary 2.3. It holds even in the case $1 \le p,q \le \infty$. For the estimate from below we shall use some well-known results about Kolmogorov numbers of those embedding operators. Recall, for a Banach space $F \hookrightarrow L_p(\mathbb{T}^d)$ we put

$$d_M(F, L_p(\mathbb{T}^d)) := \inf_{\{u_i\}_{i=1}^M \subset L_p(\mathbb{T}^d)} \sup_{\|f|F\| \le 1} \inf_{c_1,\cdots,c_M} \Big\| f - \sum_{i=1}^M c_i\, u_i \Big| L_p(\mathbb{T}^d)\Big\|\,.$$

Hence $d_M \le \rho_M$. In case of $F = S^r_{p,\infty}B(\mathbb{T}^d)$ one has the convenient references [50, 11.4.11] and [45, Thm. 3.4.5], but with some additional restrictions what concerns r and p. For the general case we refer to [16]. Galeev considered a bit different spaces. However, by some standard arguments his estimates carry over to our situation, see e.g. [45, Introduction to Chapt. 3] and Section 2.7/(2.52). For $1 < p < \infty$, $1 \le q \le \infty$ and $r > 0$ this leads to

$$d_M(I, S^r_{p,q}B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \asymp M^{-r}(\log M)^{(d-1)r}\eta(M,d,p,q)\,,$$

where $\eta(M,d,p,q)$ is defined in (2.42). □

Remark 2.13 *For the estimate from below one can also use entropy and approximation numbers. For a definition of these quantities we refer, e.g., to [13], [50] or [56]. Let* $e_n(I,X,Y)$ *denote the* n*-th dyadic entropy number of the embedding operator* I *which maps the Banach space* X *into the Banach space* Y *and let* $\lambda_n(I,X,Y)$ *denote the* n*-th approximation number (linear width, see Remark 2.2) of this embedding. Then trivially* $\lambda_M \le \rho_M$ *and furthermore* $e_n \le c\,\lambda_n$ *under certain weak conditions on* X *and* Y *which are satisfied in our context, see Theorem 1.3.3 in [13]. So, entropy numbers can be used as well for deriving lower bounds of* ρ_M*. The estimates*

$$e_M(I, S^r_{p,q}B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \ge c \begin{cases} M^{-r}(\log M)^{(d-1)(r+\frac{1}{2}-\frac{1}{q})_+} & \textit{if} \quad 1<p<\infty\,, \\ M^{-r}(\log M)^{(d-1)(r-1/q)_+} & \textit{if} \quad p = 1,\infty\,, \end{cases}$$

with some positive constant c *(independent of* M*) are known, at least in a situation very close to ours. For (non-periodic) function spaces on domains it has been proved in [56, Thm. 4.11]. This can be transferred to the periodic situation. In our case it is enough to construct a bounded linear extension operator from* $S^r_{p,q}B((-1,1)^d)$ *to* $S^r_{p,q}B(\mathbb{T}^d)$ *and to apply the multiplicativity of the entropy numbers, see [13, 1.3.1]. We omit details and refer to [8] where a similar situation is investigated. Under additional restrictions on* p *and* q *entropy numbers of the embeddings* $I : S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d))$ *are studied in [5], [12] and [44].*

Remark 2.14 *In case $q = \infty$ Temlyakov proved the estimate from above in Corollary 2.6, cf. [43] and [45, 4.5].*

For Triebel-Lizorkin spaces $S^r_{p,q}F(\mathbb{T}^d)$ the situation is similar.

Corollary 2.7 *Let $(p,q) \in [1,\infty) \times (1,\infty] \cup \{(1,1)\}$, $r > \max(1/p, 1/q)$. Then there exist two positive constants c_1 and c_2 such that for all $M \in \mathbb{N}$*

$$\begin{aligned} c_1\, M^{-r}(\log\, M)^{(d-1)(r+1/2-1/q)_+} &\leq \rho_M(S^r_{p,q}F(\mathbb{T}^d), L_p(\mathbb{T}^d)) \\ &\leq c_2\, M^{-r}(\log M)^{(d-1)(r+1-1/q)}\,. \end{aligned} \tag{2.43}$$

Proof The upper bound is obtained by Corollary 2.3. For the lower bound we refer to Remark 2.13. The same asymptotic lower bound holds for the quantity $e_M(I, S^r_{p,q}F(\mathbb{T}^d), L_p(\mathbb{T}^d))$. The consequence for the scale of Sobolev spaces can be formulated as follows.

Corollary 2.8 *Let $1 < p < \infty$ and $r > \max(1/p, 1/2)$. Then there exist positive constants c_1, c_2 such that*

$$c_1\, M^{-r}(\log M)^{(d-1)r} \;\leq\; \rho_M(S^r_pW(\mathbb{T}^d), L_p(\mathbb{T}^d)) \leq c_2\, M^{-r}(\log M)^{(d-1)(r+1/2)}\,, \quad M \in \mathbb{N}$$

holds.

Remark 2.15 *The Smolyak algorithm uses samples of a very specific structure. The Corollaries 2.6, 2.7, 2.8 tell us that allowing arbitrary sets of sampling points of the same cardinality we can not do much better. The difference is at most $(\log M)^{(d-1)/2}$ if $1 < p < \infty$.*

2.7 Comparison with Known Results

This section contains a brief summary of several results available in literature. We intend to discuss our results obtained in the Sections 2.4 and 2.6 in that context.
Let us start with Sickel and [34], [35]. The estimates given there for the operator $A(m, 2, D)$ can be extended to the d-dimensional setting. The use of more sophisticated constructions, like interpolation with de la Vallée Poussin means, made it also possible to prove counterparts for the limiting situation $p = 1$ and $p = \infty$ (see Remark 2.9) .
Furthermore, periodic function spaces with dominating mixed smoothness in connection with topics in approximation theory have especially been studied by Temlyakov, cf. [43, 44, 45] . In his monograph [45] he deals with several approximation techniques. In particular, approximation from hyperbolic crosses and sampling (cf. [45, 1.6, 4.5]). For instance, the sampling operators I_n and $\mathcal{R}_n$ are investigated there (see (2.8) in Remark 2.3). Also the problem of approximate optimal recovery of functions is posed in the setting which we used in Section 2.6. In fact, our approach for constructing sampling operators on $\mathbb{T}^d$ is basically the same as used in [45, 4.5] . Smolyak's algorithm with respect to the operators $\mathcal{R}_n$ is

considered there. Following the Russian tradition he chose a slightly different approach for defining classes of periodic functions with dominating mixed derivative. In order to make a serious comparison of his results and the ones obtained in Chapter 2, it is necessary to have a closer look at the classes $MH^r_{p,\ell}(\mathbb{T}^d)$ and $MW^r_p(\mathbb{T}^d)$, cf. [45, 3.3]. Their definition can be summarized as follows. For $r > 0$ we consider the periodic function

$$F_r(t) = 1 + \sum_{k=-\infty}^{\infty} |k|^{-r} e^{ikt} \quad , \quad t \in \mathbb{T}\,.$$

By using Lemma 1.2 it is easy to prove that F_r belongs to $B^r_{1,\infty}(\mathbb{T})$ and therefore to $L_1(\mathbb{T})$. Suppose that $\varphi \in L_p(\mathbb{T})$ for $1 \le p \le \infty$. The function $\varphi * F_r$ defined by the convolution

$$(\varphi * F_r)(t) = \frac{1}{2\pi} \int\limits_{\mathbb{T}} \varphi(y) F_r(t-y)\, dy$$

belongs to $L_p(\mathbb{T})$ and moreover

$$\|\varphi * F_r | L_p(\mathbb{T})\| \le c \|F_r|L_1(\mathbb{T})\| \cdot \|\varphi|L_p(\mathbb{T})\|\,. \tag{2.44}$$

The Fourier coefficients of this function are given by

$$c_k(\varphi * F_r) = c_k(\varphi) \cdot c_k(F_r)\,. \tag{2.45}$$

Let us now consider the tensor product $F_r(x) \in L_1(\mathbb{T}^d)$, denoted by the same symbol and given by

$$F_r(x_1, ..., x_d) = F_r(x_1) \cdot ... \cdot F_r(x_d) \quad , \quad x \in \mathbb{T}^d\,.$$

Similar to the case $d = 1$ we consider $\varphi * F_r$ as an $L_p(\mathbb{T}^d)$-function for $\varphi \in L_p(\mathbb{T}^d)$. The classes $MW^r_p(\mathbb{T}^d)$ are now defined as follows (see [45, 3.3]). Let $1 \le p \le \infty$ and $r > 0$. Then the classes $MW^r_p(\mathbb{T}^d)$ are defined by

$$\begin{aligned} MW^r_p(\mathbb{T}^d) = \Big\{ f : \mathbb{T}^d \to \mathbb{C} : f(x) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{T}^d} \varphi(y) F_r(x-y)\, dy \\ \text{for some } \varphi \in L_p(\mathbb{T}^d) \text{ such that } \|\varphi|L_p(\mathbb{T}^d)\| \le 1 \Big\}\,. \end{aligned} \tag{2.46}$$

Recall the definition of Sobolev spaces with dominating mixed smoothness $S^r_pW(\mathbb{T}^d)$ in Paragraph 1.4.4/(1.40). The following theorem will establish the relation between these two classes in the case $1 < p < \infty$. It turns out that $MW^r_p(\mathbb{T}^d)$ in some sense is the unit ball in $S^r_pW(\mathbb{T}^d)$.

Theorem 2.7 *Let $1 < p < \infty$ and $r > 0$. There are constants $c_1, c_2 > 0$ such that*
(i) *$f \in MW^r_p(\mathbb{T}^d)$ implies $\|f|S^r_pW(\mathbb{T}^d)\| \le c_1$ and*
(ii) *$f \in S^r_pW(\mathbb{T}^d)$ such that $\|f|S^r_pW(\mathbb{T}^d)\| \le c_2$ implies $f \in MW^r_p(\mathbb{T}^d)$.*

Consequently, the classes $MW^r_p(\mathbb{T}^d)$ and $S^r_pW(\mathbb{T}^d)$ are equivalent for our purposes. Let us give a short proof of this assertion.

Proof *Step 1.* Let $f \in MW^r_p(\mathbb{T}^d)$. Then a function $\varphi \in L_p(\mathbb{T}^d)$ exists such that $\|\varphi|L_p(\mathbb{T}^d)\| \le 1$ and $f = \varphi * F_r$. For the Fourier coefficients of f we have again (2.45). It is enough to check whether the function $f^{(r)}(x)$ with the Fourier series

$$\sum_{k\in\mathbb{Z}^d} c_k(f)(1+|k_1|^2)^{r/2}\,\ldots\,(1+|k_d|^2)^{r/2}\,e^{ikx}$$

belongs to $L_p(\mathbb{T}^d)$. We consider the formal decomposition

$$f^{(r)} = \sum_{\bar\beta\in\{0,1\}^d} f^{(r)}_{\bar\beta} \quad ,$$

where $I_{\bar\beta}$ is defined as in Paragraph 1.6.1 and obtain

$$f^{(r)}_{\bar\beta}(x) = \sum_{k\in I_{\bar\beta}} \frac{1+|k_1|^2)^{r/2}}{|k_1|^{\beta_1 r}}\,\ldots\,\frac{(1+|k_d|^2)^{r/2}}{|k_d|^{\beta_d r}}\,c_k(\varphi)\,e^{ikx}\,. \tag{2.47}$$

Analogously we decompose $\varphi \in L_p(\mathbb{T}^d)$ into $\sum_{\bar\beta\in\{0,1\}^d}\varphi_{\bar\beta}$. Now Fourier multiplier assertions come into play. One uses a scalar version of the multiplier theorem by Lizorkin (cf. Proposition 1.8) and a periodic version of a classical theorem by Michlin-Hörmander (cf. [32, 1.7.7]). Both require $1 < p < \infty$. The result is on the one hand $\|f^{(r)}_{\bar\beta}|L_p(\mathbb{T}^d)\| \le c_1\|\varphi_{\bar\beta}|L_p(\mathbb{T}^d)\|$ and on the other hand $\|\varphi_{\bar\beta}|L_p(\mathbb{T}^d)\| \le c_2\|\varphi|L_p(\mathbb{T}^d)\|$. Consequently, $\|f^{(r)}|L_p(\mathbb{T}^d)\| \le \|\varphi|L_p(\mathbb{T}^d)\|$.

Step 2. Let $f \in S^r_pW(\mathbb{T}^d)$ such that

$$\Big\|\sum_{k\in\mathbb{Z}^d} c_k(f)(1+|k_1|^2)^{r/2}\,\ldots\,(1+|k_d|^2)^{r/2}\,e^{ikx}\Big|L_p(\mathbb{T}^d)\Big\| \le 1\,.$$

The function φ given by

$$\sum_{\bar\beta\in\{0,1\}^d}\varphi_{\bar\beta}\,,$$

where

$$\sum_{k\in I_{\bar\beta}} c_k(f)|k_1|^{\beta_1 r}\,\ldots\,|k_d|^{\beta_d r}\,e^{ikx} \quad ,$$

belongs to $L_p(\mathbb{T}^d)$ and satisfies $\|\varphi|L_p(\mathbb{T}^d)\| \le c_4\|f|S^r_pW(\mathbb{T}^d)\|$, where c_4 is independent of $f \in S^r_pW(\mathbb{T}^d)$. This follows by the same arguments as used in Step 1. Moreover, we have $f = \varphi * F_r$ which completes the proof. □

Remark 2.16 *Recall Paragraph 1.4.4. In the same manner one proves the equivalence of the Sobolev norms (1.39) and (1.40).*

The Nikol'skij-classes of type $MH^r_{p,\ell}(\mathbb{T}^d)$, $1 \le p \le \infty$, $0 < r < \ell \in \mathbb{N}$ are defined via: $f \in L_p(\mathbb{T}^d)$ belongs to $MH^r_{p,\ell}(\mathbb{T}^d)$ if and only if for every $\bar{\beta} \in \{0,1\}^d$

$$S^{\Delta}_{\bar{\beta}}(f) = \sup_{\substack{|h_i|>0\\ i=1,..,n}} \Big(\prod_{i=1}^{n} |h_i|^{-r}\Big) \big\|(\triangle^{\ell}_{h_1,\delta_1} \circ \cdots \circ \triangle^{\ell}_{h_n,\delta_n} f)(x)\big|L_p(\mathbb{T}^d)\big\| \le 1$$

holds true. See Paragraph 1.6.1 concerning notation. By Theorem 1.14 and Remark 1.19 in Section 1.6 we obtain a counterpart of Theorem 2.7, namely the following:

Theorem 2.8 *Let $1 \le p \le \infty$ and $0 < r < \ell \in \mathbb{N}$. Then there exist $c_1, c_2 > 0$ such that*
(i) *$f \in MH^r_{p,\ell}(\mathbb{T}^d)$ implies $\|f|S^r_{p,\infty}B(\mathbb{T}^d)\| \le c_1$ and*
(ii) *$f \in S^r_{p,\infty}B(\mathbb{T}^d)$ such that $\|f|S^r_{p,\infty}B(\mathbb{T}^d)\| \le c_2$ implies $f \in MH^r_{p,\ell}(\mathbb{T}^d)$.*

Temlyakov obtained the following results in both cases.

Theorem 2.9 *Let $1 \le p \le \infty$ and $r > 0$. Then there exists a constant $c > 0$ such that*

$$\|f - A(m,d,\mathcal{R})f|L_p(\mathbb{T}^d)\| \le c\, 2^{-rm} m^{d-1} \tag{2.48}$$

holds true for all $m \in \mathbb{N}_0$ and $f \in MH^r_{p,\ell}(\mathbb{T}^d)$.

As a consequence of the inclusion $MW^r_p(\mathbb{T}^d) \subset MH^r_{p,\ell}(\mathbb{T}^d)$ the relation (2.48) is also valid for $f \in MW^r_p(\mathbb{T}^d)$. These results can now be compared with the results stated in Section 2.5. The relation (2.48) turns out to be a special case of Corollary 2.4. However, the result for $MW^r_p(\mathbb{T}^d)$, $1 < p < \infty$, $r > 0$, is improved by $(d-1)/2$ in the power of m. This yields the same improvement in the estimate for $\rho_M(MW^r_p(\mathbb{T}^d), L_p(\mathbb{T}^d))$, see [45, 4.5] and Corollary 2.8. To be precise, in case $2 < p < \infty$ we have to pay the price of a more restrictive assumption, namely $r > 1/2$. We do not know if this represents a necessary condition. Let us turn to estimates from below.

Remark 2.17 *In [45, 4.5] there is also stated the following inequality*

$$\sup_{f \in MH^r_{p,\ell}} \|f - A(m,d,\mathcal{R})f\| \ge c\, 2^{-mr} m^{d-1}\,. \tag{2.49}$$

We add some comments, why the arguments given by Temlyakov do not apply. In [45, 4.5] the following test functions are considered

$$f_m(x) = \sum_{|s|_1 = m} e^{i2^{s_1+1}x_1 + \ldots + i2^{s_d+1}x_d} \quad , \quad m \in \mathbb{N}_0\,. \tag{2.50}$$

Let $f^s(x) = e^{i2^{s_1+1}x_1} \cdot \ldots \cdot e^{i2^{s_d+1}x_d}$ *for* $s \in \mathbb{N}_0^d$, $x \in \mathbb{T}^d$. *Let further* $\mu \in \mathbb{N}_0$. *Now (2.11) implies*

$$\mathcal{R}_{2^\mu} f^s(t) = \begin{cases} f^s(t) & : \mu \geq s+1 \\ 0 & : \mu = s \\ 1 & : \mu < s \end{cases}$$

and therefore

$$\Delta_\mu f^s(x) = \begin{cases} 0 & : \mu \geq s+2 \\ f^s(x) & : \mu = s+1 \\ -1 & : \mu = s \geq 1 \\ 0 & : 1 \leq \mu < s \\ 1 & : 0 = \mu < s \\ 0 & : 0 = \mu = s\,. \end{cases} \tag{2.51}$$

This has the following consequence for the Fourier coefficient $c_{(0,\ldots,0)}(A(m,d,\mathcal{R})f^s)$, $s \in \mathbb{N}_0^d$, $|s|_1 = m$. *Because of (2.51) this coefficient vanishes in case* $\prod_{j=1}^d s_j = 0$. *It remains to consider the case* $\prod_{j=1}^d s_j \neq 0$. *Using (2.51) we obtain*

$$\begin{aligned} c_{(0,\ldots,0)}(A(m,d,\mathcal{R})\, f^s) &= \sum_{u\in\{0,1\}^d} c_{(0,\ldots,0)}\Big[\Big(\bigotimes_{j=1}^d \Delta_{u_j\cdot s_j}\Big)\, f^s\Big] \\ &= \sum_{u\in\{0,1\}^d} (-1)^{|u|_1} \\ &= \sum_{\ell=0}^d (-1)^\ell \binom{d}{\ell} \\ &= (-1)^d + \binom{d-1}{0} + \sum_{\ell=1}^{d-1}(-1)^\ell\Big(\binom{d-1}{\ell-1} + \binom{d-1}{\ell}\Big) \\ &= (-1)^d + (-1)^{d-1} = 0\,. \end{aligned}$$

Hence, the family f_m *does not qualify for test-functions. For the necessary modifications of* f_m *see Proposition 2.5.* □

Temlyakov also considered spaces with zero means. Here $L_p^0(\mathbb{T}^d)$ denotes the usual $L_p(\mathbb{T}^d)$ space with additional conditions given by

$$\int_{\mathbb{T}} f(x)\, dx_j = 0 \quad , \quad j = 1,..,d\,.$$

Then $M^0H^r_{p,\ell}(\mathbb{T}^d)$ can be characterized by

$$\sup_{\bar{h}\in\mathbb{R}^d, h_i\neq 0} \Big(\prod_{i=1}^d |h_i|^{-r}\Big)\|(\triangle^\ell_{h_1,1}\circ\cdots\circ\triangle^\ell_{h_d,d}f)(x)|L_p(\mathbb{T}^d)\| \leq 1\,. \tag{2.52}$$

Classes of this type were also considered by Galeev [16] to obtain estimates for several kinds of widths. His results are also important for us (see Corollary 2.6). In [45, Intr. to Chapt. 3] Temlyakov pointed out that (in a certain sense) $MH^r_{p,\ell}(\mathbb{T}^d)$ is equivalent to a sum of spaces of type $M^0H^r_{p,\ell}(\mathbb{T}^n)$, where $1 \leq n \leq d$.
This is the starting point for the theory which is presented by Dinh Dung in [11]. In particular, he considered periodic Hölder spaces $H^A_p(\mathbb{T}^d)$ with mixed smoothness, where $A \subset \mathbb{R}^d$, $1 \leq p \leq \infty$ and

$$H^A_p(\mathbb{T}^d) := \bigcap_{\bar{\alpha} \in A} H^{\bar{\alpha}}_p(\mathbb{T}^d)\,.$$

In the case $A = \{(r, ..., r)\}$, $r > 0$ these spaces coincide with $M^0H^r_p(\mathbb{T}^d)$. His attempt for estimating ρ_M is characterized by some geometrical aspects. He solved an $\mathbb{R}^d$-optimization problem instead, where the set A is involved. A special case of [11, Thm. 4.2] is the estimate

$$\rho_M(H^{(r,...,r)}_p(\mathbb{T}^d), L_p(\mathbb{T}^d)) \leq cM^{-r}(\log M)^{(d-1)(r+1)}$$

for $r > 0$ and $1 \leq p \leq \infty$. This corresponds to Temlyakov's result (see Theorem 2.9) as well as our results stated (special case $q = \infty$) in Section 2.6.
Let us recall the results from Wasilkowski-Wozniakowski [57]. A general framework of tensor products of Hilbert spaces is presented. This setting is also applicable to Sobolev spaces with dominating mixed smoothness $S^r_2W(\mathbb{T}^d)$. Under very general assumptions, which essentially correspond to our hypothesis $(H_2(p, r))$ in Paragraph 2.3.2, they can prove the following assertion.

Theorem 2.10 *Let $r > 0$ and $L = \{L_j\}_j$ be a sequence of operators satisfying*
(i) $\|I - L_j|W^r_2(\mathbb{T}) \to L_2(\mathbb{T})\| \leq C2^{-rj}$ *and*
(ii) $\|L_j - L_{j-1}|W^r_2(\mathbb{T}) \to L_2(\mathbb{T})\| \leq D2^{-rj}$,
where C, D are fixed constants. Then a constant $C > 0$ exists (independent of d) such that

$$\|I - A(m, d, L) : S^r_2W(\mathbb{T}^d) \to L_2(\mathbb{T}^d)\| \leq CH^{d-1}\binom{m}{d-1}2^{-rm}$$

holds for every $m \in \mathbb{N}_0$. The constant H is given by $H = \max(2^r, D)$.

Again this overlaps with Temlyakov's results. Here they took care about the explicit d-dependence of the constant and obtained exponential dependence (but without an estimate from below). This can be compared with the result stated in Remark 2.10. Let us also refer to [39]. In [39, Thm. 1,2] we also considered a more general setting, similar to [57], and obtained better results. However, the main difference to [57] is the necessity of a projection property of the one-dimensional operators, see hypothesis (H2) in [39], which corresponds to hypothesis $(H_1(\lambda))$ in Paragraph 2.3.2. See also the next chapter.

Chapter 3

Conclusion and Outlook

The final chapter is intended to present several unsolved problems and interesting open questions related to the investigations in this thesis. It is divided into two sections. The first one deals with further questions in the theory of periodic functions spaces. In the second part we will critically discuss the results obtained in Chapter 2 and give some stimulation for further investigation.

3.1 Open Problems for Periodic Function Spaces

Several open problems occurred while studying the classes $S^{\bar{r}}_{p,q}B(\mathbb{T}^d)$ and $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$.
The first one appeared during Section 1.6, especially in Paragraph 1.6.5. We considered the question, whether it is possible to replace the integrals with respect to $(0,\infty)$ by integrals with respect to $(0,1)$. This was titled "Localization". We conjectured that the localized characterizations for $S^{\bar{r}}_{p,q}F(\mathbb{T}^d)$ are extendable to the full range of parameters. In Theorem 1.11 the case of (quasi-)Banach spaces remains open. Remark 1.17 gives only for $0 < q \leq p < \infty$ a positive answer . However, the case $0 < p < q \leq \infty$ also remains open.
Another open question concerns the B-case. It would be of big interest to extend the characterization by moduli of smoothness (see Theorem 1.14) to $0 < p < 1$ and $\bar{r} > \sigma_p$ as it holds true in the isotropic case. At the moment we have a corresponding result only for $\bar{r} > 1/p$ (see Theorem 1.15).

A further interesting problem is the characterization of the given classes via atoms and wavelets. There are several motivations to develop a corresponding theory also for periodic spaces with dominating mixed smoothness. For the non-periodic situation (i.e. the scales $S^{\bar{r}}_{p,q}B(\mathbb{R}^d)$ and $S^{\bar{r}}_{p,q}F(\mathbb{R}^d)$) this has already been done by Vybíral in [56].

3.2 Further Remarks on Chapter 2

In this Section we want to present open problems regarding Chapter 2 as well as ideas for some future work. Let us start with some remarks concerning the class of admissible sampling operators (see Paragraph 2.3.2). In our opinion the class of admissible operators is still too small. This is the consequence after comparing our results with the ones by Wasilkowski-Woźniakowski [57], see Theorem 2.10. Although they use the Hilbert space setting, their assumptions are in some sense more general. The projection conditions, formulated in the hypotheses $H_1(\lambda)$ and $H_6(\lambda)$ in Paragraph 2.3.2, represent strong assumptions. As Theorem 2.10 indicates, it would be of interest to study the problem without these conditions. This opens the door to an investigation of spline approximation on non-periodic spaces with dominating mixed smoothness.

Moreover, the knowledge of the optimal constants in Proposition 2.4 and especially in Remark 2.10 is unsatisfactory. Investigations in this direction are necessary in order to confirm the algorithm as a tool for problems in high dimensions. A first step could be the computation of the approximation error in the case of Hilbert spaces (i.e. $S_2^rW(\mathbb{T}^d)$ and $L_2(\mathbb{T}^d)$).

Let us turn to an interesting open problem from the theoretical point of view. Our results in Section 2.6 give estimates from above for the problem of optimal recovery. It would be interesting to prove (or disprove), that our constructive methods are optimal. If not, one should try to find another constructive (and easy to handle) method which realizes the optimum. Even the knowledge that a better method exists would be a progress. To derive estimates from below for the quantity ρ_M one can try to extend the bump function argument used in Paragraph 2.4.4 to all q. Characterizations of the spaces with periodic wavelets should be very helpful in this context. See for instance [29], where the isotropic case is treated. A first step is therefore to investigate atomic as well as wavelet decompositions of functions from $S^r_{p,q}B(\mathbb{T}^d)$ and $S^r_{p,q}F(\mathbb{T}^d)$ (see the previous section). However, from the applied point of view it would make more sense to investigate algorithms with good behavior in high dimensions.

Finally, we return to the classes $S^{1/p}_{p,1}B(\mathbb{T}^d)$, $1 \le p \le \infty$, and $S^1_{1,q}F(\mathbb{T}^d)$, $1 \le q \le \infty$. Having the embedding in $C(\mathbb{T}^d)$, they represent a large class of continuous functions (see Theorem 1.4 and Theorem 1.5). From this point of view it would make sense to investigate the behavior of Smolyak's algorithm. The proof techniques used in Section 2.4 do not apply in this situation.

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