Smolyak's Algorithm, Sampling on Sparse Grids and Sobolev Spaces of Dominating Mixed Smoothness

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Abstract

We investigate the rate of convergence in $\| \cdot |L_p\|$, $1 \le p \le \infty$, of the d-dimensional Smolyak algorithm, associated to a sequence of sampling operators in the framework of periodic Sobolev spaces with dominating mixed smoothness.

Key Words. Trigonometric interpolation, sampling operators, sampling numbers, blending operators, Smolyak algorithm, rate of convergence, Sobolev spaces of dominating mixed smoothness, approximate recovery.
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1 Introduction

Let $(a_j^1)_{j=0}^\infty, \dots, (a_j^d)_{j=0}^\infty$ be convergent sequences of complex numbers. The respective limits are denoted by $a^1, \dots, a^d$. In addition we put $a_{-1}^\ell = 0$, $\ell = 1, \dots, d$. Then $a^\ell = \sum_{j=0}^\infty (a_j^\ell - a_{j-1}^\ell)$ and hence

$$a^1 \cdot \ldots \cdot a^d = \sum_{j_1,\dots,j_d=0}^\infty \prod_{\ell=1}^d (a_{j_\ell}^\ell - a_{j_\ell-1}^\ell)\,.$$

It has been the idea of Smolyak [17] to use the sequence

$$\sum_{j_1+\ldots+j_d\le m} \prod_{\ell=1}^d (a_{j_\ell}^\ell - a_{j_\ell-1}^\ell)\,, \qquad m = 0, 1, \dots\,,$$

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to approximate the product $a^1 \cdot \ldots \cdot a^d$. Now, if

$$a_j^1 = a_j^2 = \ldots = a_j^d = I_j f(x)\,, \qquad x \in \mathbb{T}\,,$$

where I_j denotes a sampling operator with respect to a certain set $\mathcal{T}_j$ of sample points, then the suggested approximation procedure results in an operator which uses samples from a sparse grid in $\mathbb{T}^d$ only, cf. Section 3 for details. Furthermore, the sequence of sampling operators constructed in such a way, should yield good approximations of tensor products $f_1 \otimes \ldots \otimes f_d$ of functions $f_\ell : \mathbb{T} \to \mathbb{C}$. In this paper we investigate the approximation power of these sampling operators for functions belonging to periodic Sobolev spaces of dominating mixed smoothness $S_p^r W(\mathbb{T}^d)$, see the appendix for a definition.

The present article should be understood as a continuation of investigations done in [16] by Sickel and the author. There we derived sharp estimates of Smolyak's algorithm in the framework of Besov spaces with dominating mixed smoothness on $\mathbb{T}^d$ and continued earlier work of Smolyak [17], Temlyakov [18, 20], Sickel [13, 14] and Wasilkowski, Woźniakowski [25]. The main focus lies on the optimal approximate recovery of functions, measured by the quantity ρ_M (defined in Section 6), which can be considered as a counterpart to the linear widths, where we restrict to linear sampling operators with rank $\leq$ M. The main result stated in Corollary 5 is a new upper bound for ρ_M for the spaces $S_p^r W(\mathbb{T}^d)$, given by

$$\rho_M(S_p^r W(\mathbb{T}^d), L_p(\mathbb{T}^d)) \leq c M^{-r} (\log M)^{(d-1)(r+1/2)}$$

in the case $1 < p < \infty$ and $r > \max(1/p, 1/2)$. This relation extends the corresponding assertion in [16, Cor. 7] to a greater range of p, i.e. $1 < p < \infty$, and improves the result in [20, IV.5] by $(d-1)/2$ in the power of the logarithm.

The paper is organised as follows. Section 2 introduces the Smolyak algorithm in a rather general frame. In Section 3 we specify the sampling operators, for which we will consider Smolyak's algorithm and collect all necessary properties. We still act in a very general setting, so let us refer to the examples given in Section 5. Afterwards we turn to the main part of the paper, the proof of error estimates in the L_p-metric. The main proof technique differs completely from the strategy applied in the Besov case, see [16]. We mainly employ Lizorkin's multiplier theorem, the theory of periodic Triebel-Lizorkin spaces and their behaviour under complex interpolation (see the appendix for details) to derive results for the spaces $S_p^r W(\mathbb{T}^d)$. The method is based on what has been done in [14] for two dimensions. After discussing several concrete examples in Section 5 we point out the consequences for the problem of optimal recovery in Section

6, which contains the main result stated above. Section 7 contains the proofs. In the appendix we collect the necessary definitions and properties of Besov, Sobolev and Triebel-Lizorkin spaces as well as Fourier multipliers and complex interpolation.
Some remarks concerning notation: The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ denote the real numbers, complex numbers, natural numbers, natural numbers including 0 and the integers. The natural number d is always reserved for the dimension of the underlying Euclidean space. We use $\mathbb{T}^d$ as usual for the d-dimensional torus represented in $\mathbb{R}^d$ by $[0, 2\pi]^d$. The symbol I will be reserved for identity operators (we do not indicate the space where I is considered, hoping this will be clear from the context). We shall write $a \asymp b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1}\, a \le b \le c\, a\,.$$

Constants will change their value from line to line. Sometimes we indicate this by adding subscripts. In case a constant will represent a fixed value for the paper we shall use capital letters like $C_1, C_2, \ldots$. Finally, if $x \in \mathbb{R}^d$ then $|x|$ is used for the Euclidean distance (norm in ℓ_2^d) and $|x|_1$ denotes the norm in ℓ_1^d.

2 The Smolyak Algorithm

This section contains the definition of Smolyak's algorithm in its full generality.
Let $d \ge 2$ and let X and Y be Banach spaces such that $X, Y \hookrightarrow L_1(\mathbb{T})$. Further we assume that $P_1, \ldots, P_d : X \to Y$ are continuous linear operators. Then we define its tensor product $P_1 \otimes \ldots \otimes P_d$ to be the linear operator such that:

$$(P_1 \otimes \, \ldots \, \otimes P_d)(e^{ik_1 \cdot} \cdot \, \ldots \, \cdot e^{ik_d \cdot})(x_1, ..., x_d) := \prod_{\ell=1}^{d} P_\ell(e^{ik_\ell \cdot})(x_\ell)$$

$x_\ell \in \mathbb{T}, \quad k_\ell \in \mathbb{Z}, \quad \ell = 1, \ldots, d$. Formally this operator is defined on trigonometric polynomials only. If X is either $L_p(\mathbb{T})$, $1 \le p < \infty$, or if $X = C(\mathbb{T})$, then, because of the density of trigonometric polynomials, there exists a unique continuous extension of $P_1 \otimes \, \ldots \, \otimes P_d$ to either $L_p(\mathbb{T}^d)$ or $C(\mathbb{T}^d)$, respectively. For this extension we shall use the same symbol.
Let either $L_j : L_p(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \le p < \infty$, or $L_j : C(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \le p \le \infty$, $j \in \mathbb{N}_0$, be a sequence of continuous linear operators, denoted by L. Then we put

$$\Delta_j(L) := \begin{cases} L_j - L_{j-1} & \text{if} \quad j \in \mathbb{N}\,, \\ L_0 & \text{if} \quad j = 0\,. \end{cases}$$

Definition 1 *Let $m \in \mathbb{N}_0$. The Smolyak-Algorithm $A(m,d,\vec{L})$ relative to the d sequences $L^1 := (L^1_j)_{j=0}^\infty, \ldots, L^d := (L^d_j)_{j=0}^\infty$, is the linear operator*

$$A(m,d,\vec{L}) := \sum_{j_1+\ldots+j_d \le m} \Delta_{j_1}(L^1) \otimes \ldots \otimes \Delta_{j_d}(L^d)\,.$$

Remark 1 *Originally introduced in [17] there are now hundreds of references dealing with this construction. A few basics and some references can be found in [9] and [25]. In particular the following formula is proved in [25]:*

$$A(m,d,\vec{L}) = \sum_{m-d+1 \le |j|_1 \le m} (-1)^{m-|j|_1} \binom{d-1}{m-|j|_1} L^1_{j_1} \otimes \ldots \otimes L^d_{j_d}\,. \tag{1}$$

3 Sampling on Sparse Grids

In this article we shall restrict mainly to sequences $(L_j)_j$ of special linear sampling operators using an equidistant sampling grid. For any $j \in \mathbb{N}_0$ there exist a natural number N_j and a trigonometric polynomial $\Lambda_j : \mathbb{T} \to \mathbb{C}$ such that

$$L_j f(x) = \sum_{\ell=1}^{N_j} f(x_\ell^{N_j})\, \Lambda_j(x - x_\ell^{N_j})\,, \quad f \in C(\mathbb{T})\,, \quad x_\ell^{N_j} = 2\pi \frac{\ell}{N_j}\,, \tag{2}$$

with

$$\Lambda_j(x) = \frac{1}{N_j} \sum_{k \in \mathbb{Z}^d} \gamma_j(k) e^{ikx}\,. \tag{3}$$

Let us recall the following property concerning the Fourier coefficients of such sampling operators applied to a continuous function f. The following result concerning the Fourier coefficients $c_k(L_j f)$, $k \in \mathbb{Z}$, see (43), is well known

$$c_k(L_j f) = \frac{\gamma_j(k)}{N_j} \sum_{\ell=1}^{N_j} f(x_\ell^{N_j}) e^{-ikx_\ell^{N_j}}\,. \tag{4}$$

If f is a trigonometric polynomial, then even the following holds

$$c_k(L_j f) = \gamma_j(k) \sum_{w \in \mathbb{Z}} c_{k+wN_j}(f)\,. \tag{5}$$

Therefore this gives

$$\begin{aligned} L_j\, f(x) &= \sum_{w \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \gamma_j(k) c_{k+wN_j}(f) e^{ikx} \\ &= \sum_{w \in \mathbb{Z}} e^{-iwN_j x} \sum_{k \in \mathbb{Z}} \gamma_j(k - wN_j) c_k(f) e^{ikx} \end{aligned} \tag{6}$$

for trigonometric polynomials f. Next we fix the following hypotheses to collect some additional properties.

$\big(H_1(\lambda)\big)$ The operators L_j reproduce trigonometric polynomials with degree at most $\lambda 2^j$, precisely

$$L_j(e^{ik\cdot})(t) = e^{ikt}\,, \quad t \in \mathbb{T}\,, \quad |k| \le \lambda\, 2^j\,, \quad k \in \mathbb{Z}\,, \quad j \in \mathbb{N}_0\,.$$

$\big(H_1'(\lambda)\big)$ For every Λ_j exists a positive number A_j such that such that the following holds

$$\gamma_j(k) = \begin{cases} 1 & : \quad |k| \le \lambda 2^j \\ 0 & : \quad |k| > A_j \end{cases} \quad , \quad \lambda 2^j + A_j < N_j\,.$$

Remark 2 *Formula (5) implies for the function* $f(x) = e^{imx}$, $m \in \mathbb{N}_0$, *the relation*

$$c_k(L_j(e^{im\cdot})) = \gamma_j(k) \cdot \begin{cases} 1 & : \quad \textit{if } N_j \textit{ divides } k - m \\ 0 & : \quad \textit{otherwise} \end{cases} \quad , \quad k \in \mathbb{Z} \quad , \tag{7}$$

which yields immediately the implication

$$H_1'(\lambda) \implies H_1(\lambda)\,.$$

$\big(H_2(p,r)\big)$ There exists a positive constant $c_{p,r}$ such that

$$\sup_{j=0,1,\ldots} 2^{jr}\, \|\, I - L_j\, | B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| = c_{p,r} < \infty\,.$$

(H_3) The sequences $\big\{\gamma_j(k)\big\}_{k\in\mathbb{Z}}$ are uniformly bounded in variation, i.e. the quantity

$$C := \sup_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{Z}} |\gamma_j(k) - \gamma_j(k-1)|$$

is finite.

(H_4) We assume the existence of positive constants $C_1 < C_2$ and C_3 such that

$$C_1\, 2^j \le N_j \le C_2\, 2^j \quad \text{and} \quad \Big| J_{N_{j+1}} \setminus \bigcup_{k=1}^{j} J_{N_k} \Big| \ge C_3\, 2^{j+1}\,, \qquad j \in \mathbb{N}. \tag{8}$$

(H_5) Here we assume

$$N_0 < N_1 < \ldots N_j < N_{j+1} < \ldots$$

and the nestedness of the sampling grids J_{N_j}, $j = 0, 1, 2, \ldots$, i.e.

$$J_{N_0} \subset J_{N_1} \subset \ldots \subset J_{N_j} \subset J_{N_{j+1}} \subset \ldots$$

Remark 3 *Hypothesis* (H_5) *is satisfied if and only if all numbers* N_{j+1}/N_j, $j \in \mathbb{N}_0$, *belong to* $\mathbb{N}$, *i.e.* N_j *divides* N_{j+1}. *This is a consequence of the use of equidistant sampling grids.*

We shall say that $\vec{L}$ satisfies the hypothesis (H_n) if each sequence L^i, $i = 1, \ldots, d$, satisfies (H_n). Let us collect some properties of related $A(m, d, \vec{L})$. For the proofs we refer to [16, Lem. 4,5,6].
The following Lemma holds also in a more general context. The crucial point is (H_1).

Lemma 1 *Let*

$$H(m,d,\lambda) := \Big\{ \ell \in \mathbb{Z}^d : \exists u_1, ..., u_d \in \mathbb{N}_0 \quad s.t. \quad |\ell_k| \le 2^{u_k}\lambda \text{ and } \sum_{k=1}^{d} u_k = m \Big\} \tag{9}$$

be a dyadic hyperbolic cross. Suppose that the vector $\vec{L}$ *of sequences of sampling operators of the form (2) satisfies* $(H_1(\lambda))$ *for some* $\lambda > 0$. *Then*

$$A(m,d,\vec{L})\, e^{ik\cdot} = e^{ik\cdot}\,, \quad k \in H(m,d,\lambda)\,. \tag{10}$$

The Smolyak algorithm associated to sequences of sampling operators, defined in (2), uses the sampling grid

$$\mathcal{G}(m,d,\vec{L}) := \bigcup_{m-d+1 \le |j|_1 \le m} J_{N_{j_1}} \times ... \times J_{N_{j_d}}\,. \tag{11}$$

This is a simple consequence of (1). If hypothesis (H_4) is fulfilled then it holds the following relation for the grid size $|\mathcal{G}(m,d,\vec{L})|$

$$\min(C_1, C_3)^d S(m,d) \le |\mathcal{G}(m,d,\vec{L})| \le (2C_2)^d\, S(m,d)\,, \quad m \in \mathbb{N}_0\,, \tag{12}$$

where

$$S(m,d) = \sum_{j=0}^{d-1} \binom{d-1}{j} 2^{m-j} \binom{m}{j}\,. \tag{13}$$

This implies the asymptotic behaviour

$$|\mathcal{G}(m,d,\vec{L})| \asymp 2^m m^{d-1}\,. \tag{14}$$

We call the grids $\mathcal{D}(m,d,\vec{L})$ sparse because their cardinality is growing only with a logarithmic order with respect to d.

4 The Rate of Convergence

We will study the approximation power of Smolyak's algorithm for functions taken from periodic Sobolev spaces with dominating mixed smoothness $S^r_pW(\mathbb{T}^d)$, where $1 < p < \infty$ and $r > 1/p$. The approximation error is measured in the $L_p(\mathbb{T}^d)$-metric. The main result is sharp estimate obtained as a special case of a corresponding result for Triebel-Lizorkin spaces $S^r_{p,q}F(\mathbb{T}^d)$, stated in Proposition 1 and 2 ($q = 2$, see Lemma 7 and 8) . In this sense the scale of Triebel-Lizorkin spaces with dominating mixed smoothness acts as a vehicle for our purpose. Proposition 1 contains the estimate from above, obtained by complex interpolation of two corner results, which are stated in Lemma 2 and 3, see Section 7. One of them can be proved directly using Lizorkin's multiplier theorem, see Lemma 6 together with Remark 10. The other one is a consequence of a corresponding result for the Besov scale. See [16, Thm. 1] and Remark 4 for details. We only consider the special situation $\vec{L} = (L, ..., L)$ to avoid technical difficulties. To indicate this, we write $A(m, d, L)$ instead of $A(m, d, \vec{L})$. But nevertheless the general case can be proved analogously.

Proposition 1 *Let $d \geq 1$ and assume that (p, q) belongs to the set $[1,\infty) \times (1,\infty] \cup \{(1, 1)\}$. Let further $r > \max(1/p, 1/q)$. The sequence $L = \{L_j\}_{j=0}^\infty$ is given by (2) and supposed to satisfy $\big(H_2(u, s)\big)$ for all $1 < u < \infty$, $s > 1/u$ as well as $H_2(p, s)$ for all $s > 1/p$ and $\big(H_1'(\lambda)\big)$ for a certain $\lambda > 0$. Moreover we assume hypothesis (H_3). Then it exists a constant $C > 0$ such that*

$$\|I - A(m, d, L) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \leq C2^{-rm}m^{(d-1)(1-1/q)} \qquad (15)$$

holds for all $m \in \mathbb{N}_0$.

Remark 4 *In [16, Thm. 1] a corresponding assertion was proved in the framework of Besov spaces with dominating mixed smoothness (see (47) for a definition). Hence, in the case $1 \leq p < \infty$, $p \leq q \leq \infty$ and $r > 1/p$ relation (15) is an immediate consequence of the embedding (48). We will use this fact during the proof of Proposition 1, see Lemma 2.*

The estimate from below is covered by the following theorem.

Proposition 2 *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$, $r > 1/p$ and $L = \{L_j\}_{j=0}^\infty$ given by (2). If $(H_1'(\lambda))$ holds for a certain $\lambda > 0$ and (H_4) and (H_5) are satisfied then it exists a constant $C > 0$ such that*

$$\|I - A(m, d, L) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \geq C\,2^{-rm}m^{(d-1)(1-1/q)}\,, \quad m \in \mathbb{N}\,.$$

In the case $1 < p < \infty$ the corresponding behaviour of $I - A(m,d,L)$ for Sobolev spaces of dominating mixed smoothness can be derived as a special case of Proposition 1 and Proposition 2 for $q = 2$, see Lemma 7 and 8. It turns out that the results from [16] can be extended to $1 < p < \infty$. Let us formulate a combination of the Propositions 1 and 2 for the Sobolev space scale.

Theorem 1 *Let $d \geq 1$ and $1 < p < \infty$. Let further $r > \max(1/p, 1/2)$. The sequence $L = \{L_j\}_{j=0}^{\infty}$ is given by (2) and supposed to satisfy $\big(H_2(u,s)\big)$ for all $1 < u < \infty$, $s > 1/u$, $\big(H_1'(\lambda)\big)$ for a certain $\lambda > 0$ and (H_3). Moreover we assume hypothesis (H_4) and (H_5). Then we have*

$$\|I - A(m,d,L) : S_p^r W(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-rm} m^{(d-1)/2} . \tag{16}$$

5 Examples

Let us give several examples of sampling operators on the torus and associated Smolyak algorithms to inspire the results of the previous section with life.

5.1 Interpolation on the Torus

In this Paragraph we give a short survey about certain aspects of trigonometric interpolation.

5.1.1 Interpolation with the Dirichlet Kernel

Let

$$\mathcal{D}_m(t) := \sum_{|k| \leq m} e^{ikt} , \qquad t \in \mathbb{T}, \quad m \in \mathbb{N}_0 ,$$

be the Dirichlet kernel and let

$$I_m f(t) := \frac{1}{2m+1} \sum_{\ell=0}^{2m} f(t_\ell)\, \mathcal{D}_m(t - t_\ell) , \qquad t_\ell = \frac{2\pi\ell}{2m+1} , \tag{17}$$

which is in some sense the discrete counterpart of the classical Fourier partial sum

$$S_m f(t) := \sum_{k=-m}^{m} c_k(f)\, e^{ikt} = \frac{1}{2\pi} \int\limits_{\mathbb{T}} f(t) \mathcal{D}_m(x - t)\, dt , \tag{18}$$

assigned to periodic integrable function $f \in L_1(\mathbb{T})$.
Then $I_m f$ is the unique trigonometric polynomial of degree less than or equal to m which interpolates f at the nodes t_ℓ. Let us shortly mention an important result

concerning the classical case of trigonometric interpolation. The following is known, see [4, 5, 18, 20, 12].

Proposition 3 *Let $1 < p < \infty$ and let $r > 1/p$. Then we have*

$$\| I - I_n \,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r} \,. \tag{19}$$

5.1.2 Interpolation with De La Valée-Poussin Means

For $0 < \mu < 1/2$ we consider the functions

$$\Lambda_\mu(t) := 2\,\frac{\sin(t/2)\,\sin(\mu t)}{\mu\, t^2}\,, \qquad t \in \mathbb{R}\,. \tag{20}$$

Then the Fourier transform is given by

$$\mathcal{F}\Lambda_\mu(\xi) = \sqrt{2\pi}\begin{cases} 1 & \text{if} \quad |\xi| \le \frac{1}{2} - \mu\,, \\ \frac{1}{2\,\mu}\left(\frac{1}{2} + \mu - |\xi|\right) & \text{if} \quad \frac{1}{2} - \mu < |\xi| < \frac{1}{2} + \mu\,, \\ 0 & \text{if} \quad \frac{1}{2} + \mu \le |\xi|\,, \end{cases} \tag{21}$$

i.e. a piecewise linear function. Furthermore we assign for $n \in \mathbb{N}$ the 2π-periodic, continuous function

$$\Lambda^\pi_{\mu,n}(t) := \sum_{\ell\in\mathbb{Z}} \Lambda_\mu(nt + 2\pi\ell n)\,, \qquad t \in \mathbb{R}\,, \tag{22}$$

which represents a periodic fundamental interpolant satisfying

$$c_\ell(\Lambda^\pi_{\mu,n}) = \frac{1}{n\sqrt{2\pi}}\,\mathcal{F}\Lambda_\mu(\ell/n)\,, \qquad \ell \in \mathbb{Z}\,, \quad n \in \mathbb{N}\,. \tag{23}$$

Similar to (17) we define the interpolation operator for $f \in C(\mathbb{T})$

$$I(\Lambda^\pi_{\mu,n},\cdot)f(t) = \sum_{\ell=1}^{n} f(t^n_\ell)\Lambda^\pi_n(t - t^n_\ell) \quad , \quad t^n_\ell = 2\pi\frac{\ell}{n}\,. \tag{24}$$

Proposition 4 *Let $0 < \mu < 1/2$. Let further $1 \le p \le \infty$ and $r > 1/p$. Then we have*

$$\| I - I(\Lambda^\pi_{\mu,n},\cdot)\,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r}\,.$$

Proof Let us refer to [15] for the estimate from above in case $1 < p < \infty$ and to Remark 5 for the estimate from below. The modifications for the limiting cases $p = 1$ and $p = \infty$ are straightforward. For full details we refer to [23]. ∎

Remark 5 *Linear widths. For two Banach spaces X, Y such that $X \hookrightarrow Y$ we define*

$$\lambda_n(I, X, Y) := \inf \Big\{ \| \, I - L \, | \mathcal{L}(X,Y) \| : \quad L \in \mathcal{L}(X,Y), \ \mathrm{rank}\, L \le n \Big\} .$$

Since the operator $I(\Lambda^{\pi}_{\mu,n}, \cdot)$ has rank $\le n$ we obtain

$$\lambda_n(I, B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \le \| \, I - I(\Lambda^{\pi}_{\mu,n}, \cdot) \, | \mathcal{L}(B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \, \| .$$

Because of $\lambda_n(I, B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \asymp n^{-r}$, where $1 \le p \le \infty$ and $r > 0$, cf. e.g. [20, 1.4], it is clear that our interpolation operator yields optimal in the order of approximation.

Remark 6 *In contrast to our treatment Temlyakov [20, 1.6] considered the sequence of sampling operators*

$$\mathcal{R}_n f(t) := \frac{1}{4n} \sum_{\ell=1}^{4n} f(t^{4n}_\ell) \, v_{2n-1}(t - t^{4n}_\ell) \tag{25}$$

and proved that these operators also satisfy

$$\| \, I - \mathcal{R}_n \, | B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T}) \| \asymp n^{-r} ,$$

if $1 \le p \le \infty$ and let $r > 1/p$.

5.2 Associated Smolyak Algorithms

First we consider the Smolyak algorithm associated to the interpolation with de la Vallée-Poussin means, cf. Subsection 5.1.2. We put

$$A(m,d,\mu) := A(m,d,L) , \qquad L_j := I(\Lambda^{\pi}_{\mu,2^j}, \, \cdot \,) , \quad j \in \mathbb{N}_0 . \tag{26}$$

As an immediate consequence of Proposition 4 we obtain that L satisfies $(H_2(u,s))$ for every admissible pair (u,s). A simple calculation shows that $(H_1'(\lambda))$ is satisfied with $\lambda = 1/2 - \mu$. Since $I(\Lambda^{\pi}_{\mu,2^j}, \, \cdot \,)$ uses function values from the standard grid J_{2^j} also (H_4) and (H_5) are fulfilled. Obviously also H_3 holds true, which follows from (21) together with (23). Altogether Propositions 1 and 2 yield the following.

Corollary 1 *Let $(p,q) \in [1,\infty) \times (1,\infty] \cup \{(1,1)\}$, $r > \max(1/p, 1/q)$ and $0 < \mu < 1/2$. Then*

$$\| \, I - A(m,d,\mu) \, | S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \asymp m^{(d-1)(1-1/q)} \, 2^{-mr} , \qquad m \in \mathbb{N}_0$$

and if $r > \max(1/p, 1/2)$ then

$$\| \, I - A(m,d,\mu) \, | S^r_p W(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \asymp m^{(d-1)/2} 2^{-mr} , \qquad m \in \mathbb{N}_0 .$$

As a second example we consider Smolyak's algorithm associated to the interpolation operators $\mathcal{R}_n$, cf. (25) or [20, 1.6]. Putting

$$A(m,d,\mathcal{R}) := A(m,d,L) \quad , \quad \text{where} \quad L_j := \mathcal{R}_{2^j}\,, j \in N_0\,,$$

we obtain with exactly the same arguments as in the previous case:

Corollary 2 *Let* $(p,q) \in [1,\infty) \times (1,\infty] \cup \{(1,1)\}$, $r > \max(1/p, 1/q)$ *and* $0 < \mu < 1/2$. *Then*

$$\| I - A(m,d,\mathcal{R}) \,| S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)(1-1/q)}\, 2^{-mr}\,, \qquad m \in \mathbb{N}_0$$

and if $r > \max(1/p, 1/2)$ *then*

$$\| I - A(m,d,\mathcal{R}) \,| S^r_{p}W(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)/2} 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

The last example of a sparse grid sampling operator is the following. We put

$$A(m,d,D) := A(m,d,L)\,, \qquad L_j := I_{2^j}\,, \quad j \in \mathbb{N}_0\,,$$

cf. (17). Here (H_5) is not satisfied and hence Proposition 2 is not applicable in the stated form. With specific modifications of the family of testfunctions used in the proof, see (40), it is also possible to obtain a sharp estimate from below, see [14] for the bivariate case. Nevertheless Proposition 1 gives the following.

Corollary 3 *Let* $1 < p < \infty$, $1 < q < \infty$, *and* $r > \max(1/p, 1/q)$. *Then there is a constant* $c > 0$ *such that*

$$\| I - A(m,d,D) \,| S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(1-1/q)}\, 2^{-mr}\,, \qquad m \in \mathbb{N}_0$$

and if $r > \max(1/p, 1/2)$ *then*

$$\| I - A(m,d,D) \,| S^r_{p}W(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)/2} 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

For sake of completeness we also pay some attention to Smolyak's algorithm applied to the sequence of dyadic Fourier partial sums, see (18). We denote by $A(m,d,S)$ the operator $A(m,d,L)$ associated to the sequence

$$L_j := S_{2^j}\,, \qquad j \in \mathbb{N}_0\,.$$

Of course Definition 1 makes also sense in this situation. This operator and its approximation properties have been studied very intensively in the former Soviet Union, for instance by Bugrov, Galeev, Nikol'skaya, Nikol'skij, Romanyuk, Temlyakov and Dinh Dung. However, in contrast to the previous considerations, the operator $A(m,d,S)$ is of convolution type and leads to better results, namely:

Proposition 5 *Let $1 < p < \infty$, $1 < q < \infty$ and $r > 0$. Then we have*

$$\|I - A(m,d,S) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-mr} m^{(d-1)(1/2-1/q)_+} \quad , \quad m \in \mathbb{N}\,.$$

This implies immediately the following assertion for Sobolev spaces of dominating mixed smoothness, already obtained by Temlyakov, see for instance [20, Thm. III.3.2].

Theorem 2 *Let $1 < p < \infty$ and $r > 0$. Then it holds*

$$\|\, I - A(m,d,S)\,|S^r_p W(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

6 Optimal Recovery of Functions

In this section we study the question of optimal recovery of a function from a finite number of function values. In order to define the quantity ρ_M, mentioned in the introduction, we introduce the following framework.
Let

$$\Psi_M(f,\xi)(x) := \sum_{j=1}^{M} f(\xi^j)\, \psi_j(x)$$

denote a general sampling operator for a class F of continuous, periodic functions defined on $\mathbb{T}^d$, where

$$\xi := \left\{\xi^1, ..., \xi^M\right\}, \qquad \xi^i \in \mathbb{T}^d\,, \quad i = 1, 2, \dots, M\,,$$

is a fixed set of sampling points and $\psi_j : \mathbb{T}^d \to \mathbb{C}$, $j = 1, \dots, M$, are fixed, continuous, periodic functions. Then the quantity

$$\rho_M(F, L_p(\mathbb{T}^d)) := \inf_{\xi} \inf_{\psi_1,\dots,\psi_M} \sup_{\|f|F\|\le 1} \|\, f - \Psi_M(f,\xi)\,|L_p(\mathbb{T}^d)\|$$

measures the optimal rate of approximate recovery of the functions taken from F. We are interested in the case, when $F = S^r_{p,q}F(\mathbb{T}^d)$, $1 \le p,q \le \infty$, $r > 1/p$. Observe that the operator $A(m,d,\mu)$, see (26), uses $M = M(m,d) \asymp 2^m\, m^{d-1}$ function values from its argument, see (14). Therefore $m \le c \log M$ with some c independent of m and hence

$$2^{-rm} m^{(d-1)(1-1/q)} \le M^{-r} (c \log M)^{(d-1)(r+1-1/q)}\,.$$

On the basis of Corollary 1 (upper bound), connections between entropy numbers and linear widths (see Remark 5 and the monograph [3, 1.3]) and the results from [24] we obtain the following.

Corollary 4 *Let* $(p,q) \in [1,\infty) \times (1,\infty] \cup \{(1,1)\}$, $r > \max(1/p, 1/q)$. *Then there exist two positive constants* c_1 *and* c_2 *such that for all* $M \in \mathbb{N}$

$$\begin{aligned} c_1\, M^{-r} (\log\, M)^{(d-1)(r+1/2-1/q)_+} &\le \rho_M(S^r_{p,q}F(\mathbb{T}^d), L_p(\mathbb{T}^d)) \\ &\le c_2\, M^{-r} (\log M)^{(d-1)(r+1-1/q)}\,. \end{aligned} \tag{27}$$

For Sobolev spaces this reads as follows.

Corollary 5 *Let* $1 < p < \infty$ *and* $r > \max(1/p, 1/2)$. *Then there exist positive constants* c_1, c_2 *such that*

$$c_1\, M^{-r} (\log M)^{(d-1)r} \;\le\; \rho_M(S^r_p W(\mathbb{T}^d), L_p(\mathbb{T}^d)) \le c_2\, M^{-r} (\log M)^{(d-1)(r+1/2)}\,, \quad M \in \mathbb{N}$$

holds.

Remark 7 (i) *The Smolyak algorithm uses samples of a very specific inner structure of the function to be approximated. Corollary 4 and 5 tell us that allowing arbitrary sets of sampling points of the same cardinality we can not do much better. The difference is at most* $(\log M)^{(d-1)/2}$.

7 Proofs

7.1 Proof of Proposition 1

We split the proof into two Lemmas in order to connect their assertions by complex interpolation.

Lemma 2 *Let* $1 \le p < \infty$, $p \le q \le \infty$ *and* $r > 1/p$. *Let further* $L = \{L_j\}_{j=0}^\infty$ *be given by (2) and satisfy* $\big(H_2(p,s)\big)$ *for* $1/p < s \le r$ *and* $\big(H_1(\lambda)\big)$ *for a certain* $\lambda > 0$. *Then it exists a constant* $C > 0$ *such that*

$$\| I - A(m,d,L) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \le C\, 2^{-rm} m^{(d-1)(1-1/q)}\,. \tag{28}$$

Proof Since $p \le q$ the assertion is a direct consequence of the embedding $S^r_{p,q}F(\mathbb{T}^d) \hookrightarrow S^r_{p,q}B(\mathbb{T}^d)$, see (48), and Theorem 1 in [16]. There we proved

$$\| I - A(m,d,L) : S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \asymp 2^{-mr} m^{(d-1)(1-1/q)}$$

under much weaker assumptions. ■

The next lemma gives the same conclusion as the previous one even in the case $1 < p, q < \infty$, but the price we have to pay is the very strong restriction $r > 1$.

Lemma 3 *Let* $1 < p, q < \infty$ *and* $r > 1$. *Let* $L = \{L_j\}_{j=0}^{\infty}$ *be given by (2) and satisfy* $\big(H_1'(\lambda)\big)$ *for a certain* $\lambda > 0$, $\big(H_2(p,s)\big)$ *for* $1 < s \le r$ *as well as* (H_3). *Then (28) holds for* $m \in \mathbb{N}_0$.

Proof The first part of the proof is similar to the corresponding part in the proof of Theorem 1 in [16]. Because of dealing with $1 < p, q < \infty$ the decomposition (53) of $f \in S^r_{p,q}F(\mathbb{T}^d)$ into the pieces $\widetilde{f_\ell}$ is sufficient. Let us fix a natural number n_λ such that $2^{-n_\lambda} \le \lambda$. Now we suppose that m is larger than dn_λ. Further we put $s_m := m - dn_\lambda \ge 0$ (we drop the parameter λ in all other notations). Let $I_0^m := [0, s_m]$ and $I_1^m := (s_m, \infty)$, respectively. For $b = (b_1, \dots b_d)$, $b_i \in \{0,1\}$, $i = 1, \dots, d$, we define

$$Q_b^m := \{\ell \in \mathbb{N}_0^d :\ \ell_n \in I_{b_n}^m,\ n = 1, \dots, d,\ |\ell|_1 > s_m\}\,.$$

This leads to the decomposition

$$f(x) = h(x) + \sum_{b \in \{0,1\}^d} f^b(x)\ ,$$

where

$$f^b(x) := \sum_{\ell \in Q_b^m} \widetilde{f_\ell}(x)\,.$$

The function $h(x)$ is a trigonometric polynomial given by

$$h(x) := \sum_{|\ell|_1 \le s_m} \widetilde{f_\ell}(x)\,.$$

Observe $c_k(h) \ne 0$ implies $c_k(\widetilde{f_\ell}) \ne 0$ for some $\ell \in \mathbb{Z}^d$ satisfying $|\ell|_1 \le s_m$. Hence $|k_n| \le 2^{\ell_n} \le 2^{\ell_n + n_\lambda}\lambda$ for $n = 1, ..., d$. Therefore $k \in H(m, d, \lambda)$ and consequently $A(m, d, L)\, h = h$ follows, see Lemma 1.

Step 1. Estimation (first part). By means of the invariance of h under the application of $A(m, d, L)$ we find

$$\|\, f - A(m, d, L) f \,|L_p(\mathbb{T}^d)\| \le \sum_{b \in \{0,1\}^d} \|\, f^b - A(m, d, L) f^b \,|L_p(\mathbb{T}^d)\|\,.$$

Obviously, there exists a number $M_\ell \in \mathbb{N}$, such that the trigonometric polynomial $\widetilde{f_\ell}$ has all its harmonics in the hyperbolic cross $H(M_\ell, d, \lambda)$. For $M_\ell > |\ell|_1 + dn_\lambda$

Lemma 1 implies

$$\begin{aligned} f^b - A(m,d,L)f^b &= \sum_{\ell\in Q_b^m} \widetilde{f}_\ell - A(m,d,L)\widetilde{f}_\ell \\ &= \sum_{\ell\in Q_b^m} A(M_\ell,d,L)\widetilde{f}_\ell - A(m,d,L)\widetilde{f}_\ell \\ &= \sum_{\ell\in Q_b^m} \sum_{m<|j|_1\le M_\ell} \Big(\bigotimes_{n=1}^d \Delta_{j_n}\Big)\widetilde{f}_\ell \\ &= \sum_{\ell\in Q_b^m} \sum_{j\in\Lambda_\ell^m} \Big(\bigotimes_{n=1}^d \Delta_{j_n}\Big)\widetilde{f}_\ell\,, \end{aligned} \tag{29}$$

where

$$\Lambda_\ell^m := \Big\{ j=(j_1,...,j_d): \quad |j|_1 > m, \quad 0 \le j_n \le \ell_n + n_\lambda,\, n=1,...,d \Big\}\,.$$

Now and in the sequel we denote by $|\cdot|$ also the cardinality of a set, for instance denotes $|\Lambda_\ell^m|$ the cardinality of the set Λ_ℓ^m . In order to keep the notation simple we used Δ_{j_n} instead of $\Delta_{j_n}(L)$, $j_n = 0,1,2,....$ The last step here is a consequence of $(H_1'(\lambda))$, the definition of the tensor product and the choice of M_ℓ. In the case $|b|_1 > 1$ we argue analogous to the corresponding part of the proof of [16, Thm. 1]. Let us recall the basic ideas used there. We shall use Lemma 19 in [16] to exploit the tensor product structure. Choosing r_0 such that $1 < r_0 < r$ we obtain

$$\Big\| \Big(\bigotimes_{n=1}^d \Delta_{j_n}\Big)\widetilde{f}_\ell \Big| L_p(\mathbb{T}^d)\Big\| \le \| \widetilde{f}_\ell \,| S^{r_0}_{p,p}B(\mathbb{T}^d)\| \prod_{n=1}^d \| \Delta_{j_n} \,| B^{r_0}_{p,p}(\mathbb{T}) \to L_p(\mathbb{T})\|\,.$$

Hypothesis $(H_2(p,r_0))$ and the triangle inequality give

$$\Big\| \Big(\bigotimes_{n=1}^d \Delta_{j_n}\Big)\widetilde{f}_\ell \Big| L_p(\mathbb{T}^d)\Big\| \le c_1\, 2^{-mr_0}\, \| \widetilde{f}_\ell \,| S^{r_0}_{p,p}B(\mathbb{T}^d)\|\,,$$

Furthermore

$$\| \widetilde{f}_\ell \,| S^{r_0}_{p,p}B(\mathbb{T}^d)\| \le\; c_2\, 2^{r_0|\ell|_1}\, \| \widetilde{f}_\ell \,| L_p(\mathbb{T}^d)\|\,.$$

Altogether we obtain

$$\begin{aligned} &\| f^b - A(m,d,L)f^b \,| L_p(\mathbb{T}^d)\| \le c_3 \sum_{\ell\in Q_b^m} 2^{r_0(|\ell|_1 - m)}\, |\Lambda_\ell^m|\; \| \widetilde{f}_\ell \,| L_p(\mathbb{T}^d)\| \\ &= c_3 2^{-r_0 m} \sum_{\ell\in Q_b^m} 2^{(r_0-r)|\ell|_1}\, |\Lambda_\ell^m\,|\; 2^{r|\ell|_1}\, \| \widetilde{f}_\ell \,| L_p(\mathbb{T}^d)\| \\ &\le c_3 \|f|S^r_{p,\infty}B(\mathbb{T}^d)\| \Big(2^{-r_0 m} \sum_{\ell\in Q_b^m} 2^{(r_0-r)|\ell|_1}\, |\Lambda_\ell^m\,| \Big)\,. \end{aligned} \tag{30}$$

Obviously it holds

$$2^{-mr_0}\sum_{\ell\in Q_b^m} 2^{(r_0-r)|\ell|_1}\,|\Lambda_\ell^m| \le 2^{-mr_0}\prod_{n=1}^{d}\Big(\sum_{\ell_n\in I_{b_n}^m} 2^{(r_0-r)\ell_n}(\ell_n+1+n_\lambda)\Big)\,,$$

where

$$\begin{aligned}\sum_{\ell_n=s_m+1}^{\infty} 2^{(r_0-r)\ell_n}(\ell_n+1+n_\lambda) &= c_4\,2^{m(r_0-r)}\sum_{u=0}^{\infty} 2^{(r_0-r)u}\,(u+m-dn_\lambda+2)\\ &\le c_4\,2^{m(r_0-r)}\sum_{u=0}^{\infty} 2^{(r_0-r)u}\,(u+m+2)\\ &\le c_5\,2^{m(r_0-r)}\,m\end{aligned}$$

and

$$\sum_{\ell_n=0}^{s_m} 2^{(r_0-r)\ell_n}(\ell_n+1+n_\lambda) \le m\sum_{u=0}^{\infty} 2^{(r_0-r)u} \le c_6\,m\,.$$

Altogether this leads to

$$\begin{aligned}&\|\, f_b - A(m,d,L)f_b\,|L_p(\mathbb{T}^d)\|\\ \le\;& c_7\,\|\, f\,|S^r_{p,\infty}B(\mathbb{T}^d)\|\,2^{-mr}m^d 2^{m(r_0-r)(|b|_1-1)}\\ \le\;& c_8\,2^{-mr}\,\|\, f\,|S^r_{p,\infty}B(\mathbb{T}^d)\|\,.\end{aligned}\tag{31}$$

Step 2. In the case $|b|_1\le 1$ we go back to (29) and derive further

$$|f^b - A(m,d,L)f^b| \le \sum_{\ell\in Q_b^m}\sum_{j\in\Lambda_\ell^m}\sum_{u\in U_j}\Big|\Big(\bigotimes_{n=1}^{d} L_{u_n}\Big)\widetilde{f}_\ell\Big|\,,\tag{32}$$

where

$$U_j = \big\{(u_1,...,u_n)\in\mathbb{N}_0^d : u_k = j_k \text{ or } u_k = j_k-1\big\}\,.$$

Of course, the cardinality $|U_j|$, $j\in\mathbb{N}_0^d$, is less or equal to 2^d . Next we use (6) to obtain

$$\Big|\Big(\bigotimes_{k=1}^{d} L_{u_k}\Big)\widetilde{f}_\ell(x)\Big| \le \sum_{w\in\mathbb{Z}^d}\Big|\underbrace{\sum_{k\in\mathbb{Z}^d}\gamma_{u_1}(k_1-w_1N_{u_1})\cdot\ldots\cdot\gamma_{u_d}(k_d-w_dN_{u_d})c_k(\widetilde{f}_\ell)e^{ikx}}_{=:f_{\ell,u,w}(x)}\Big|\,.$$

Inserted into (32) and taking the L_p-norm afterwards this leads to

$$\|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\| \le \Big\|\sum_{\ell\in Q_b^m}\sum_{j\in\Lambda_\ell^m}\sum_{u\in U_j}\sum_{w\in\mathbb{Z}^d}|f_{\ell,u,w}(x)|\Big|L_p(\mathbb{T}^d)\Big\|\,.$$

Let us restrict the range for w in the corresponding sum. We do not need the entire $\mathbb{Z}^d$ here. Suppose that the trigonometric polynomial $f_{\ell,u,w}(x)$ does not vanish. This implies, because of $(H_1'(\lambda))$ holds, the existence of a number $k \in \mathbb{Z}^d$ such that

$$|k_i - w_i N_{u_i}| \le A_{u_i} \quad \text{and} \quad |k_i| \le 2^{\ell_i} \quad , \quad i = 1, ..., d \,. \tag{33}$$

Now again hypothesis $H_1'(\lambda)$ comes into play. The first condition in (33) implies for all $i = 1, ..., d$

$$\begin{aligned} |k_i| &\ge |w_i| N_{u_i} - A_{u_i} \\ &\ge |w_i| N_{u_i} - (N_{u_i} - \lambda 2^{u_i}) \\ &\ge C_1(|w_i| - 1) 2^{u_i} \,. \end{aligned}$$

Because of $u \in U_j$ and $j \in \Lambda_\ell^m$ there exists a number $\eta > 0$ such that for $|w_i| \ge 2^{\ell_i - u_i + \eta} > 1$ holds

$$C_1(|w_i| - 1) 2^{u_i} > 2^{\ell_i} \,.$$

This means, the condition $|w_i| \ge 2^{\ell_i - u_i + \eta}$ implies $|k_i| > 2^{\ell_i}$. Altogether (33) leads to the following index set for w:

$$W_{\ell,u} := \Big\{ (w_1, ..., w_d) \in \mathbb{Z}^d : |w_i| \le 2^{\ell_i - u_i + \eta}, i = 1, ..., d \Big\} \quad , \quad \text{for some } \eta > 0 \,. \tag{34}$$

We go further in estimating $\|f^b - A(m,d,L) f^b | L_p(\mathbb{T}^d)\|$ by Hölder's inequality using $1/q + 1/q' = 1$ and obtain

$$\begin{aligned} \|f^b - A(m,d,L) f^b | L_p(\mathbb{T}^d)\| \le \Big(\sum_{\ell \in Q_b^m} \sum_{j \in \Lambda_\ell^m} \sum_{u \in U_j} \sum_{w \in W_{\ell,u}} 2^{-r|\ell|_1 q'} |\Lambda_\ell^m|^{q'/q} |W_{u,\ell}|^{q'/q} \Big)^{1/q'} \times \\ \times \Big\| \Big(\sum_{\ell \in Q_b^m} \sum_{j \in \Lambda_\ell^m} \sum_{u \in U_j} \sum_{w \in W_{\ell,u}} 2^{r|\ell|_1 q} |\Lambda_\ell^m|^{-1} |W_{\ell,u}|^{-1} |f_{\ell,u,w}(x)|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^d) \Big\| \,. \end{aligned}$$

Because of (H_3), Remark 10 can be considered in this situation. Lizorkin's multiplier theorem, see Lemma 6, can be applied and we obtain from the previous estimate

$$\begin{aligned} \|f^b - A(m,d,L) f^b | L_p(\mathbb{T}^d)\| \le c_9 \Big(\sum_{\ell \in Q_b^m} \sum_{j \in \Lambda_\ell^m} \sum_{u \in U_j} 2^{-r|\ell|_1 q'} |\Lambda_\ell^m|^{q'/q} |W_{u,\ell}|^{q'/q+1} \Big)^{1/q'} \times \\ \times \Big\| \Big(\sum_{\ell \in Q_b^m} \sum_{j \in \Lambda_\ell^m} \sum_{u \in U_j} \sum_{w \in W_{\ell,u}} 2^{r|\ell|_1 q} |\Lambda_\ell^m|^{-1} |W_{\ell,u}|^{-1} |\widetilde{f}_\ell(x)|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^d) \Big\| \,. \end{aligned}$$

This is a very comfortable situation since we can compute each sum one after another inside the second factor. Taking into account $q'/q + 1 = q'$ we obtain

$$\begin{aligned} \|f^b - A(m,d,L) f^b | L_p(\mathbb{T}^d)\| \le & c_{10} \Big\| \Big(\sum_{\ell \in Q_b^m} 2^{r|\ell|_1 q} |\widetilde{f}_\ell(x)|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^d) \Big\| \times \\ & \times \Big(\sum_{\ell \in Q_b^m} \sum_{j \in \Lambda_\ell^m} \sum_{u \in U_j} 2^{-r|\ell|_1 q'} |\Lambda_\ell^m|^{q'/q} |W_{u,\ell}|^{q'} \Big)^{1/q'} \end{aligned}$$

It remains to estimate the second factor in the previous inequality. We start by estimating the size of the index sets Λ_ℓ^m and $W_{\ell,u}$. Because of $u_i \geq j_i - 1$ and $|j|_1 > m$ the definition of $W_{\ell,u}$, see (34), implies

$$|W_{\ell,u}| \leq 2^{|\ell|_1 - |j|_1 + d(\eta+3)} \leq 2^{|\ell|_1 - m + d(\eta+3)} . \tag{35}$$

On the basis of

$$\Lambda_\ell^m \subset \Big[m - \sum_{\substack{n=1\\n\neq 1}}^{d} (\ell_n + n_\lambda), \ell_1 + n_\lambda\Big] \times \cdots \times \Big[m - \sum_{\substack{n=1\\n\neq d}}^{d} (\ell_n + n_\lambda), \ell_d + n_\lambda\Big]$$

we derive

$$|\Lambda_\ell^m| \leq \big(|\ell|_1 + dn_\lambda + 1 - m\big)^d . \tag{36}$$

Thus it holds

$$\|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\| \leq c_{11} \, \|f|S^r_{p,q}F(\mathbb{T}^d)\| \Big(\sum_{\ell \in Q_b^m} 2^{-r|\ell|_1 q'} |\Lambda_\ell^m|^{q'} 2^{(|\ell|_1 - m)q'} \Big)^{1/q'} .$$

Without loss of generality we may assume that $b_1 = |b|_1$. The index transform $u := |\ell|_1 + dn_\lambda + 1 - m$ yields

$$\begin{aligned}
&\Big(\sum_{\ell \in Q_b^m} 2^{-r|\ell|_1 q'} |\Lambda_\ell^m|^{q'} 2^{(|\ell|_1 - m)q'} \Big)^{1/q'} \\
&\leq \Big(\sum_{\ell \in Q_b^m} 2^{-r|\ell|_1 q'} \big(|\ell|_1 + dn_\lambda + 1 - m\big)^{dq'} 2^{(|\ell|_1 - m)q'} \Big)^{1/q'} \\
&\leq c_{12} \Big(\sum_{\ell_2,\ldots,\ell_d = 0}^{s_m} \sum_{u=0}^{\infty} 2^{-r(u+m)q'} u^{dq'} 2^{uq'} \Big)^{1/q'} \\
&= c_{12} \, 2^{-mr} \Big(\sum_{\ell_2,\ldots,\ell_d = 0}^{s_m} \sum_{u=0}^{\infty} 2^{u(1-r)q'} u^{dq'} \Big)^{1/q'} .
\end{aligned}$$

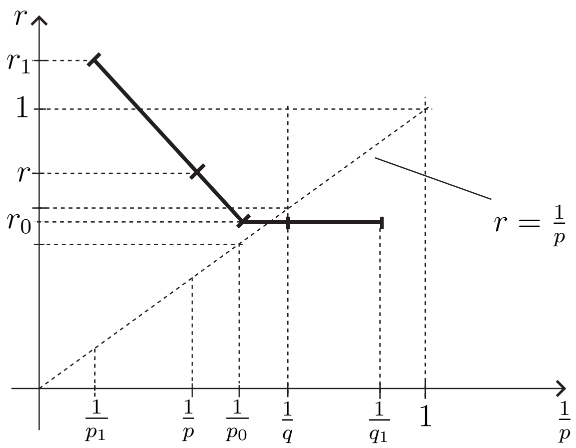
Finally the condition $r > 1$ enters the stage. It is needed to ensure the finiteness of the series over u. Altogether everything leads to

$$\begin{aligned}
\|f^b - A(m,d,L)f^b|L_p(\mathbb{T}^d)\| &\leq c_{13} \, 2^{-mr} m^{(d-1)/q'} \|f|S^r_{p,q}F(\mathbb{T}^d)\| \\
&= c_{13} \, 2^{-mr} m^{(d-1)(1-1/q)} \|f|S^r_{p,q}F(\mathbb{T}^d)\| .
\end{aligned}$$

Summing up over b by using (31) together with Lemma 4 this gives the proposed estimate for $\|I - A(m,d,L) : S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\|$. ■

Proof of Proposition 1

Having Lemma 2 and Lemma 3 it remains to prove (15) for $1 < q < p < \infty$ and $1/q < r \le 1$. We use complex interpolation and the result stated in Lemma 12. Furthermore we shall use the relations stated in Lemmata 2 and 3 as corner information to interpolate the norm of the operator $I - A(m,d,L)$. To do this we have to construct proper parameter triples (p_0, q_0, r_0) and (p_1, q_1, r_1) satisfying on the one hand the conditions of Lemma 2 and Lemma 3, respectively, where the triple (p,q,r) is an intermediate tuple on the other hand. The figure below gives an idea what we have to manage.



Step 1. We start by choosing p_0 such that

$$\frac{1}{p} < \frac{1}{p_0} < \frac{q}{p(q-1)+q} < \frac{1}{q}\,. \tag{37}$$

This is possible because

$$\frac{1}{p} = \frac{q}{pq} = \frac{q}{p(q-1)+p} \underset{(p>q)}{<} \frac{q}{p(q-1)+q} \underset{(p>q)}{<} \frac{q}{q^2-q+q} = \frac{1}{q}\,.$$

Furthermore we choose r_0 such that

$$\frac{1}{p_0} < r_0 < r - \frac{1/q - 1/p_0}{1-1/q}(1-r) \le r\,. \tag{38}$$

This is possible because of

$$0 \le \frac{1/q - 1/p_0}{1-1/q}(1-r) < \frac{1/q - 1/p_0}{1-r}(1-r) < r - 1/p_0\,.$$

This choice has the following consequences

(a)

$$\frac{1-r}{r-r_0} < \frac{1-1/q}{1/q-1/p_0}$$

(b) and

$$\frac{1/p_0 - 1/p}{1/p} < \frac{1/q - 1/p_0}{1 - 1/q} \quad .$$

Let us give a short comment on (b). Having (37) we conclude

$$\frac{1}{p_0} < \frac{q}{p(q-1)+q} = \frac{q-1+1}{p(q-1)+q} = \frac{1+\frac{1}{q-1}}{p+\frac{q}{q-1}}$$

which implies (b). The inequalities in (a) and (b) allow us to choose a number $1 < q_1 < q$ such that it holds simultaneously

$$\frac{1-r}{r-r_0} < \frac{1/q_1 - 1/q}{1/q - 1/p_0} \quad \text{and} \quad \frac{1/p_0 - 1/p}{1/p} < \frac{1/q - 1/p_0}{1/q_1 - 1/q} .$$

And finally this implies the existence of $r_1 > 1$, $1/p_1 < 1/p$ and $0 < \vartheta < 1$ such that

$$\frac{1-\vartheta}{\vartheta} = \frac{r_1 - r}{r - r_0} = \frac{1/q_1 - 1/q}{1/q - 1/p_0} \quad \text{and} \quad \frac{\vartheta}{1-\vartheta} = \frac{1/p_0 - 1/p}{1/p - 1/p_1} = \frac{1/q - 1/p_0}{1/q_1 - 1/q} .$$

And therefore with $q_0 := p_0$ it follows

$$(r, 1/p, 1/q) = (1-\vartheta)\big(r_0, 1/p_0, 1/q_0\big) + \vartheta\big(r_1, 1/p_1, 1/q_1\big) , \tag{39}$$

where $r_0 > 1/p_0, 1 < p_0 = q_0 < \infty$ and $r_1 > 1$, $1 < p_1, q_1 < \infty$.

Step 2. Lemma 2 now gives

$$\|I - A(m,d,L) : S^{r_0}_{p_0,q_0}F(\mathbb{T}^d) \to L_{p_0}(\mathbb{T}^d)\| \le c_1 2^{-mr_0} m^{(d-1)(1-1/q_0)} .$$

Additionally Lemma 3 implies

$$\|I - A(m,d,L) : S^{r_1}_{p_1,q_1}F(\mathbb{T}^d) \to L_{p_1}(\mathbb{T}^d)\| \le c_2 2^{-mr_1} m^{(d-1)(1-1/q_1)} .$$

We finish the proof by applying Lemma 12, Lemma 9, (57) and (39) to obtain

$$\begin{aligned}
&\|I - A(m,d,L) : S^{r}_{p,q}F(\mathbb{T}^d) \to L_{p}(\mathbb{T}^d)\| \\
&\le \ \|I - A(m,d,L) : S^{r_0}_{p_0,q_0}F \to L_{p_0}\|^{1-\vartheta} \cdot \|I - A(m,d,L) : S^{r_1}_{p_1,q_1}F \to L_{p_1}\|^{\vartheta} \\
&\le c_3\, 2^{-mr_0(1-\vartheta)} m^{(d-1)(1-1/q_0)(1-\vartheta)} \cdot 2^{-mr_1\vartheta} m^{(d-1)(1-1/q_0)\vartheta} \\
&= c_3\, 2^{-mr} m^{(d-1)(1-1/q)} .
\end{aligned}$$

■

7.2 Proof of Proposition 2

This proof can be transferred almost word by word from the proof of [16, Thm. 3]. See also Theorem 7 there, where Sobolev spaces ($q = 2$) are considered. Let me recall the basic ideas. The estimate from above follows directly from Theorem 1. For the estimate from below we construct the following lacunary test function

$$f_m(x_1, ..., x_d) = \sum_{\substack{u_k \geq d \\ |u|=m}} e^{iN_{u_1}x_1 + ... + iN_{u_d}x_d} \,. \tag{40}$$

With help of (5) and (H_5) we find in this situation

$$\Big|c_0(A(m,d,L)\, f_m)\Big| = \Big|\sum_{\substack{u_k \geq d \\ |u|=m}} (-1)^{d-1}\Big| \asymp m^{d-1} \,.$$

and therefore $\|f_m - A(m,d,L)f_m|L_p(\mathbb{T}^d)\| \geq c_1\, m^{d-1}$. Taking

$$\|f_m|S^r_{p,q}F(\mathbb{T}^d)\| \leq c_2 2^{rm} m^{(d-1)/q}$$

into account, which is a consequence of (46) and (H_4), the desired estimate from below follows. ■

7.3 Proof of Proposition 5

We apply the same strategy as used in the proof of [16, Thm. 5]. Let us recall the basic ideas.

Step 1. Estimate from above. Starting from the decomposition

$$f - A(m,d,S)f = \sum_{\ell \in Q_0^m} \widetilde{f}_\ell + \sum_{\ell \in Q_1^m} \widetilde{f}_\ell \,,$$

where

$$Q_1^m \;:=\; \Big\{\ell \in \mathbb{N}_0^d : \quad \exists k \quad \text{s.t.} \quad \ell_k > m\Big\}, \tag{41}$$

$$Q_0^m \;:=\; \Big\{\ell \in \mathbb{N}_0^d : \quad \ell_k \leq m\,,\ k = 1, \ldots, d, \quad \text{and} \quad |\ell|_1 > m\Big\} \tag{42}$$

we find

$$\begin{aligned} \|\sum_{\ell \in Q_1^m} \widetilde{f}_\ell\,|L_p(\mathbb{T}^d)\| \;&\leq\; \sum_{\ell \in Q_1^m} \|\,\widetilde{f}_\ell\,|L_p(\mathbb{T}^d)\| \\ &\leq\; c\,\|\,f|S^r_{p,\infty}B(\mathbb{T}^d)\|\,\Big(\sum_{\ell \in Q_1^m} 2^{-r|\ell|_1}\Big) \\ &\leq\; c' 2^{-rm}\|f|S^r_{p,q}F(\mathbb{T}^d)\|\,. \end{aligned}$$

So it remains to consider

$$\| \sum_{\ell \in Q_1^m} \widetilde{f}_\ell \, |L_p(\mathbb{T}^d)\| \asymp \Big\| \Big(\sum_{\ell \in Q_0^m} |\widetilde{f}_\ell(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^d) \Big\| \,,$$

see (54) and Remark 12 . We distinguish two cases, i.e. $1 < q \leq 2$ and $2 < q < \infty$, and proceed using standard arguments.

Step 2. Estimate from below. The family of testfunctions, defined by

$$f_m(x) = \sum_{|j|_1 = m} e^{ip_j \cdot x} \quad , \quad m \in \mathbb{N} \,,$$

where p_j is the center of the according rectangle $\mathcal{P}_j$ (see (51)), turn out to be useful for $q > 2$. Observing that $A(m,d,S) = \sum\limits_{|\ell|_1 \leq m} \widetilde{f}_\ell$ we have $A(m,d,S)\, f_{m+1} = 0$ and therefore by Lemma 8

$$\|f_{m+1} - A(m,d,S) f_{m+1} | L_p(\mathbb{T}^d)\| = \|f_{m+1} | L_p(\mathbb{T}^d)\| \asymp m^{(d-1)/2}$$

and

$$\|f_{m+1} | S^r_{p,q} F(\mathbb{T}^d\| \asymp 2^{rm} m^{(d-1)/q} \,.$$

which proves the claim. In the case $q \leq 2$ we test with

$$g_m(x) = e^{ip_{(m,0,\ldots,0)} x} \quad , \quad m \in \mathbb{N} \,.$$

This gives also $A(m,d,S) g_{m+1} = 0$ and furthermore $\|g_{m+1} | L_p(\mathbb{T}^d)\| = c$ as well as $\|g_{m+1} | S^r_{p,q} F(\mathbb{T}^d)\| \asymp 2^{rm}$.

■

7.4 Proof of Corollary 4 and 5

We use the connection between entropy and approximation numbers. For a definition of these quantities we refer, e.g., to [3], [22] or [24]. Let $e_n(I,X,Y)$ denote the n-th dyadic entropy number of the embedding operator I which maps the Banach space X into the Banach space Y and let $\lambda_n(I,X,Y)$ denote the n-th approximation number (linear width, see Remark 5) of this embedding. Then trivially $\lambda_M \leq \rho_M$ and furthermore $e_n \leq c\, \lambda_n$ under certain weak conditions on X and Y which are satisfied in our context, see Theorem 1.3.3 in [3]. So, entropy numbers can be used as well for deriving lower bounds of ρ_M. The estimates

$$e_M(I, S^r_{p,q} F(\mathbb{T}^d), L_p(\mathbb{T}^d)) \asymp M^{-r} (\log M)^{(d-1)(r + \frac{1}{2} - \frac{1}{q})_+} \,.$$

are known, at least in a situation very close to ours. For (non-periodic) function spaces on domains it has been proved in [24]. This can be transferred to the periodic situation. In our case it is enough to construct a bounded linear extension operator from $S^r_{p,q}F((-1,1)^d)$ to $S^r_{p,q}F(\mathbb{T}^d)$ and to apply the multiplicativity of entropy numbers, see [3, 1.3.1]. We omit details and refer to [2] where a similar situation is investigated. In the case of Sobolev spaces (Corollary 5) we refer also to [1] and [18]. ■

8 Appendix - Function spaces

Let $D(\mathbb{T}^d)$ be the collection of all infinitely differentiable, complex-valued and in each component 2π-periodic functions equipped with the topology generated by

$$\| f \|_\alpha := \sup_{x\in\mathbb{T}^d} |D^\alpha f(x)| \,, \qquad \alpha \in \mathbb{N}_0^d \,.$$

The elements of the topological dual $D'(\mathbb{T}^d)$ (equipped with the weak topology) we call periodic distributions. In this general setting the kth Fourier coefficient of $f \in D'(\mathbb{T}^d)$ is defined by

$$c_k(f) = (2\pi)^{-d} f(e^{-ikx}) \quad , \quad k \in \mathbb{Z}^d \,.$$

All function spaces considered here in this paper will be continuously embedded into $D'(\mathbb{T}^d)$.

8.1 Sobolev Spaces on the Torus

Let $1 \le p \le \infty$ and r be a natural number. A measurable periodic function $f : \mathbb{T}^d \to \mathbb{C}$ belongs to the periodic Lebesgue space $L_p(\mathbb{T}^d)$, if the quantity

$$\|f|L_p(\mathbb{T}^d)\| = \int\limits_{\mathbb{T}^d} |f(x)|^p \, dx.$$

is finite (usual modifications if $p = \infty$). Moreover we denote by $C(\mathbb{T}^d)$ the collection of all periodic and continuous functions. Every function $f \in L_1(\mathbb{T}^d)$ can be considered as a periodic distribution by the well known interpretation. Using this we derive

$$c_k(f) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x)\, e^{-ikx}\, dx \quad , \quad k \in \mathbb{Z}^d \,. \tag{43}$$

The formal Fourier series Sf of f is given by

$$Sf(x) := \sum_{k\in\mathbb{Z}^d} c_k(f)\, e^{ikx} \,.$$

Here $kx = \sum_{\ell=1}^d k_\ell\, x_\ell$, $k = (k_1, \ldots , k_d)$, $x = (x_1, \ldots , x_d)$.
The Sobolev space $W_p^r(\mathbb{T})$ is defined via

$$\| \, f \, |W_p^r(\mathbb{T})\| := \sum_{k=0}^{r} \| \, D^k f | L_p(\mathbb{T})\| < \infty \ ,$$

where $D^k f$, $k = 0, 1, .., r$, has to be understood in the weak sense. Our general reference for these spaces is [11, Chapt. 3]. For general $r \geq 0$ and $1 < p < \infty$ we replace $D^r f$ by

$$\sum_{k \in \mathbb{Z}} c_k(f)(1 + |k|^2)^{r/2} e^{ikx}$$

and define fractional Sobolev spaces by using

$$\| \, f \, |W_p^r(\mathbb{T})\| := \Big\| \sum_{k \in \mathbb{Z}} c_k(f)(1 + |k|^2)^{r/2} e^{ikx} \Big| L_p(\mathbb{T})\Big\| \ . \tag{44}$$

8.2 Besov Spaces on the Torus

Let us introduce the scale of Besov spaces on the torus using Fourier analytical tools. We make use of a smooth dyadic decomposition of unity.
Let $C_0^\infty(\mathbb{R})$ denote the set of all compactly supported, complex-valued and infinite differentiable functions on the real line and Φ the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R})$ satisfying

(i) supp $\varphi_0 \subset \{x : |x| \leq 2\}$,

(ii) supp $\varphi_j \subset \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\} \quad , \quad j = 1, 2, ...,$

(iii) $\forall \, \ell \in \mathbb{N}_0$ we have $\sup\limits_{x,j} 2^{j\ell} \, |\varphi_j^{(\ell)}(x)| \leq c_\ell < \infty$,

(iv) $\sum\limits_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}$.

Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $r > 0$. Then $f \in L_p(\mathbb{T})$ belongs to $B_{p,q}^r(\mathbb{T})$ if

$$\| \, f \, |B_{p,q}^r(\mathbb{T})\| := \Big\| \Big(\sum_{j=0}^{\infty} 2^{jrq} \Big| \sum_{k \in \mathbb{Z}} \varphi_j(k) \, c_k(f) \, e^{ikt} \Big|^q \Big)^{1/q} \Big| L_p(\mathbb{T}) \Big\| < \infty$$

with the usual modification in the case $\max(p, q) = \infty$. Different elements of Φ lead to equivalent norms. For this and other properties we refer again to [11, Chapt. 3].

8.3 Sobolev Spaces of Dominating Mixed Smoothness

If r is a natural number and $1 \le p \le \infty$, then the Sobolev space $S^r_pW(\mathbb{T}^d)$ of dominating mixed smoothness of order r is defined as the collection of all $f \in L_p(\mathbb{T}^d)$ such that

$$D^\alpha f \in L_p(\mathbb{T}^d)\,, \qquad \alpha = (\alpha_1, \ldots \alpha_d)\,, \quad 0 \le \alpha_\ell \le r\,, \quad \ell = 1, \ldots , d\,.$$

Derivatives have again to be understood in the weak sense. For general $r > 0$ and $1 < p < \infty$ one may use

$$\sum_{k\in\mathbb{Z}^d} c_k(f)(1 + |k_1|^2)^{r/2} \ldots (1 + |k_d|^2)^{r/2} \, e^{ikx} \in L_p(\mathbb{T}^d)\,.$$

In case $r \in \mathbb{N}$ this leads to an equivalent characterisation. For $r \in \mathbb{N}$ we endow these classes with the norm

$$\| \, f \, |S^r_pW(\mathbb{T}^d)\| := \sum_{\alpha\le r} \| \, D^\alpha f \, |L_p(\mathbb{T}^d)\|\,,$$

for $r > 0$, $r \notin \mathbb{N}$, and $1 < p < \infty$ we shall use

$$\| \, f \, |S^r_pW(\mathbb{T}^d)\| := \Big\| \sum_{k\in\mathbb{Z}^d} c_k(f)(1 + |k_1|^2)^{r/2} \ldots (1 + |k_d|^2)^{r/2} \, e^{ikx} \Big| L_p(\mathbb{T}^d)\Big\|\,.$$

8.4 Triebel-Lizorkin Spaces of Dominating Mixed Smoothness

Smooth dyadic decompositions of unity on $\mathbb{R}$ can be used to construct decompositions on $\mathbb{R}^d$ by means of tensor products. Let $\ell = (\ell_1, \ldots , \ell_d) \in \mathbb{N}_0^d$. Then we put

$$\varphi_\ell(x) := \varphi_{\ell_1}(x_1) \cdot \ \ldots \ \cdot \varphi_{\ell_d}(x_d)\,.$$

Hence

$$\sum_{\ell\in\mathbb{N}_0^d} \varphi_\ell(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d\,.$$

As an abbreviation we shall use

$$f_\ell(x) := \sum_{k\in\mathbb{Z}^d} c_k(f)\, \varphi_\ell(k)\, e^{ikx}\,, \qquad x \in \mathbb{T}^d, \quad \ell \in \mathbb{N}_0^d\,, \tag{45}$$

which results in

$$f = \sum_{\ell\in\mathbb{N}_0^d} f_\ell\,,$$

at least in the sense of periodic distributions.

Let $\varphi \in \Phi$, $1 \le p < \infty$, $1 \le q \le \infty$ and $r > 0$. Then the Triebel-Lizorkin space

$S^r_{p,q}F(\mathbb{T}^d)$ of dominating mixed smoothness is the collection of all functions $f \in L_p(\mathbb{T}^d)$ such that

$$\| f \,|S^r_{p,q}F(\mathbb{T}^d)\| := \Big\| \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{|\ell|_1 rq} \, |f_\ell(x)|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^d) \Big\| < \infty \,. \tag{46}$$

These classes are Banach spaces independent of the chosen system Φ (in the sense of equivalent norms), cf. [11, Chapt. 2,3] for $d = 2$. Let us further recall the definition of Besov spaces of dominating mixed smoothness, normed by

$$\| f \,|S^r_{p,q}B(\mathbb{T}^d)\| := \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{r|\ell|_1 q} \, \| f_\ell \,|L_p(\mathbb{T}^d)\|^q \Big)^{1/q}. \tag{47}$$

Below we shall recall a few facts concerning elementary embeddings between these classes. Let $1 \le p < \infty$, $1 \le q \le \infty$. Then

$$S^r_{p,\min(p,q)}B(\mathbb{T}^d) \hookrightarrow S^r_{p,q}F(\mathbb{T}^d) \hookrightarrow S^r_{p,\max(p,q)}B(\mathbb{T}^d)\,, \tag{48}$$

see [11, 2.2.3]. Here the nonperiodic case (spaces on $\mathbb{R}^d$) is treated for $d = 2$. The necessary modifications for the periodic setting will be published elsewhere.

Lemma 4 *Let $1 \le p < \infty$ and $1 \le q \le \infty$. Let $r > 1/p$. Then*

$$S^r_{p,q}F(\mathbb{T}^d) \hookrightarrow S^r_{p,\infty}B(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$$

holds.

Remark 8 *Let us refer to a similar assertion in [16, Lem. 20]. A proof of the nonperiodic counterpart of $S^{1/p}_{p,1}B(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$ can be found in [11, 2.4.1].*

8.5 Fourier Multipliers

First of all we need some spaces of functions on $\mathbb{R}^d$. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform and its inverse on $L_2(\mathbb{R}^d)$, respectively. Let $\kappa \ge 0$. Then a function $f \in L_2(\mathbb{R}^d)$ belongs to $S^\kappa_2 H(\mathbb{R}^d)$ if

$$\| f \,|S^\kappa_2 H(\mathbb{R}^d)\| := \Big(\int_{\mathbb{R}^d} (1+|\xi_1|^2)^\kappa \, \ldots \, (1+|\xi_1|^2)^\kappa \, |\mathcal{F}(\xi)|^2 \, d\xi \Big)^{1/2} < \infty \,.$$

Furthermore we define the quantity

$$\Big\| (g_j)_j \,|L_p(\mathbb{T}^d, \ell_q) \Big\| := \Big(\int_{\mathbb{T}^d} \Big(\sum_{j \in \mathbb{Z}^d} |g_j(x)|^q \Big)^{p/q} dx \Big)^{1/p},$$

where $1 \le p \le \infty$, $1 \le q \le \infty$ and the functions $g_j : \mathbb{T}^d \to \mathbb{C}$ are supposed to be Lebesgue measurable. $L_p(\mathbb{T}^d, \ell_q)$ denotes the corresponding Banach space.
Let $(b^j)_j$ be a sequence in $(0,\infty)^d$ and let $\Lambda = (\Lambda_j)_j$ be a sequence of subsets of $\mathbb{Z}^d$ s.t.

$$\Lambda_j \subset \{\ell \in \mathbb{Z}^d : \quad |\ell_i| \le b_i^j\,, \quad i = 1,\ldots,d\}\,, \qquad j \in \mathbb{N}_0^d\,.$$

We say that a sequence $(g_j)_j$ of trigonometric polynomials belongs to $L_p^\Lambda(\mathbb{T}^d, \ell_q)$ if

$$\Big\| (g_j)_j \,|L_p(\mathbb{T}^d, \ell_q)\Big\| < \infty$$

and

$$c_k(g_j) = 0 \qquad \text{for all} \quad k \notin \Lambda_j\,, \quad j \in \mathbb{N}_0^d\,.$$

Lemma 5 *Let $1 \le p < \infty$, $1 \le q \le \infty$ and let*

$$\kappa > \frac{1}{\min(p,q)} + \frac{1}{2}$$

If $(M_j)_j$ is a sequence in $S_2^\kappa H(\mathbb{R}^d)$, then there exists a constant c such that

$$\Big\| \sum_{k\in\mathbb{Z}^d} M_j(k)\, c_k(g_j)\, e^{ikx} \Big| L_p(\mathbb{T}^d, \ell_q)\Big\| \le c \sup_{j\in\mathbb{N}_0^d} \|M_j(b^j\,\cdot\,)\,|S_2^\kappa H(\mathbb{R}^d)\|\; \|\,g_j\,|L_p(\mathbb{T}^d, \ell_q)\|$$

holds for all $(g_j)_j \in L_p^\Lambda(\mathbb{T}^d, \ell_q)$. Here c neither depends on $(g_j)_j$ nor on $(M_j)_j$.

Remark 9 *A nonperiodic counterpart of Lemma 5 is proved in [11, 1.8.3]. The proof in the periodic situation is similar. Details will be published elsewhere.*

8.5.1 Lizorkin's Multiplier Theorem

Let us recall the definition of signed and complex measures. For what follows we mainly refer to [7, A.6]. Here $(X,\mathcal{F})$ denotes a measurable space. A mapping $\mu : \mathcal{F} \to \mathbb{R} \cup \{-\infty,\infty\}$ is called a signed measure if and only if the following two conditions are satisfied

(i) $\mu(\emptyset) = 0\,,$

(ii)$\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \sum\limits_{n=1}^{\infty} \mu(A_n) \quad , \quad$ for $A_i \cap A_j = \emptyset$, $i \ne j$.

Analogously we define complex measures. The variation of a signed (complex) measure is given by the positive measure

$$|\mu|(E) = \sup\Big\{\sum_{k=1}^{n} |\mu(A_k)| : A_k \in \mathcal{F} \text{ pairwise disjoint }, \bigcup_{k=1}^{n} A_k = E\Big\}.$$

The number $|\mu|(X)$ is called total variation of the signed (complex) measure μ on the space X. Now we are able to state Lizorkin's multiplier theorem. As measurable space we consider now $(\mathbb{R}^d, \mathcal{R}^d)$, where $\mathcal{R}^d$ denotes the usual Borel σ-algebra.

Lemma 6 Let $1 < p, q < \infty$. Let further $M = \{M_j(x)\}_{j=0}^{\infty} \subset L_\infty(\mathbb{R}^d)$ a sequence of functions satisfying:

(i) There are finite complex measures μ_j, $j = 0, 1, ...$ on $(\mathbb{R}^d, \mathcal{R}^d)$ such that

$$M_j(x_1, ..., x_d) = \mu_j((-\infty, x_1] \times ... \times (-\infty, x_d]) \,.$$

(ii) The measures μ_j, $j = 0, 1, 2...$ provide uniformly bounded total variation on $\mathbb{R}^d$, i.e.

$$|\mu_j|(\mathbb{R}^d) \le C_M \quad , \quad j = 0, 1, 2....$$

(iii) The functions $M_j(x)$, $j = 0, 1, 2...$ are continuous in all points $k \in \mathbb{Z}^d$.

Under this conditions there exists a positive constant $C(p, q, d)$ such that

$$\Big\| \sum_{k \in \mathbb{Z}^d} M_j(k) c_k(f_j) e^{ik\cdot x} \Big| L_p(\mathbb{T}^d, \ell_q) \Big\| \le C \cdot C_M \| f_j | L_p(\mathbb{T}^d, \ell_q) \|$$

holds for all $\{f_j\}_{j=0}^{\infty} \in L_p(\mathbb{T}^d, \ell_q)$

Proof The nonperiodic counterpart is due to Lizorkin, cf. [6]. The lemma follows by applying Theorem 3.4.2 in [11]. ■

Remark 10 *Consider a compactly supported piecewise linear continuous function $\gamma : \mathbb{R} \to \mathbb{C}$, uniquely defined by the nodes $\{(k, \gamma(k)) : k \in \mathbb{Z}\}$, and its weak derivative denoted by $\gamma'(x)$, which exists of course as a piecewise constant step function. Consequently the corresponding complex measure μ, which satisfies $\mu((-\infty, x]) = \gamma(x)$, $x \in \mathbb{R}$, is given by*

$$\mu(A) = \int_A \gamma'(x)\, dx \quad , \quad A \in \mathcal{R} \,.$$

Then we find

$$|\mu|(\mathbb{R}) = \sum_{k \in \mathbb{Z}} |\gamma(k) - \gamma(k-1)| \,. \tag{49}$$

Let us now consider a sequence $\gamma = \{\gamma_j(x)\}_{j=0}^{\infty}$ of such functions, where $\{\mu_j\}_{j=0}^{\infty}$ denotes the sequence of corresponding measures. Additionally we assume the uniform boundedness of (49), i.e.

$$\sup_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}} |\gamma_j(k) - \gamma_j(k-1)| =: C_\gamma < \infty \,.$$

Their tensor product ($u = (u_1, ..., u_d) \in \mathbb{N}_0^d$)

$$M_u(x_1, ..., x_d) := \gamma_{u_1}(x_1) \cdot ... \cdot \gamma_{u_d}(x_d) \quad , \quad x = (x_1, ..., x_d) \in \mathbb{R}^d \quad ,$$

can be written as follows

$$\begin{aligned} M_u(x_1,...,x_d) &= \mu_{u_1}((-\infty,x_1])\cdot ... \cdot \mu_{u_d}((-\infty,x_d]) \\ &= (\mu_{u_1}\otimes ... \otimes \mu_{u_d})((-\infty,x_1]\times ... \times (-\infty,x_d])\,, \end{aligned}$$

where $\mu_{u_1}\otimes ... \otimes \mu_{u_d}$ *denotes the product measure of* $\mu_{u_1},...,\mu_{u_d}$. *An easy calculation, using the Jordan decomposition of a signed measure, cf. for instance [7, A.6], and (49) give*

$$|\mu_{u_1}\otimes ... \otimes \mu_{u_d}|(\mathbb{R}^d) \le (4C_\gamma)^d\,.$$

Finally Lemma 6 implies

$$\Big\| \sum_{k\in\mathbb{Z}^d} c_k(f_u) M_u(k) e^{ikx} \Big| L_p(\mathbb{T}^d,\ell_q)\Big\| \le C\cdot (4C_\gamma)^d \|f_u|L_p(\mathbb{T}^d,\ell_q)\|\,,$$

for all systems $f=\{f_u\}_{u\in\mathbb{N}_0^d}\in L_p(\mathbb{T}^d,\ell_q)$, *where* C *depends only on* p,q *and* d.

8.6 Further Littlewood-Paley Characterisations

As isotropic Sobolev spaces also the classes $S^r_pW(\mathbb{T}^d)$ allow a Littlewood-Paley characterisation.

Lemma 7 *Let* $1<p<\infty$ *and* $r\ge 0$. *Let* f_ℓ *be as in (45). Then*

$$\|\, f\,|S^r_pW(\mathbb{T}^d)\| \asymp \Big\| \Big(\sum_{\ell\in\mathbb{Z}^d} 2^{2r|\ell|_1} |f_\ell(x)|^2\Big)^{1/2} \Big| L_p(\mathbb{T}^d)\Big\| \tag{50}$$

holds for all $f\in L_p(\mathbb{T}^d)$.

Remark 11 *For* $r=0$ *this can be found in Nikol'skij [8, 1.5.2/(13)]. For* $r>0$ *one has to use a lifting property, we refer to [11, 2.2.6] for the nonperiodic counterpart.*

Now we turn to the Lizorkin representation of Triebel-Lizorkin as well as Sobolev spaces. We need a special covering of $\mathbb{R}^d$. We put

$$\begin{aligned} P_0 &:= [-1,1]\,, \qquad P_j := \{x\in\mathbb{R}^d:\ 2^{j-1}<|x|\le 2^j\}\,, \quad j\in\mathbb{N}\,, \\ \mathcal{P}_j &:= P_{j_1}\times \ldots \times P_{j_d}\,, \qquad j\in\mathbb{N}_0^d\,. \end{aligned} \tag{51}$$

Then

$$\mathbb{R}^d = \bigcup_{j\in\mathbb{N}_0^d} \mathcal{P}_j \qquad \text{and} \qquad \mathcal{P}_j\cap\mathcal{P}_\ell=\emptyset \quad \text{if} \quad j\neq \ell\,. \tag{52}$$

Hence, with

$$\widetilde{f}_\ell(x) := \sum_{k\in\mathcal{P}_\ell} c_k(f)\, e^{ikx}\,, \qquad x\in\mathbb{T}^d,\ \ell\in\mathbb{N}_0^d\,, \tag{53}$$

we find

$$f = \sum_{\ell \in \mathbb{N}_0^d} \widetilde{f}_\ell$$

(convergence in the sense of periodic distributions), compare with (45).

Lemma 8 *Let $1 < p < \infty$, $1 < q < \infty$, and $r > 0$. Then*

$$\| f \,|S^r_p W(\mathbb{T}^d)\| \asymp \Big\| \Big(\sum_{\ell \in \mathbb{Z}^d} 2^{2r|\ell|_1} \,|\widetilde{f}_\ell(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^d) \Big\| \,, \tag{54}$$

and

$$\| f \,|S^r_{p,q} F(\mathbb{T}^d)\| \asymp \Big\| \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{r|\ell|_1 q} \,|\, \widetilde{f}_\ell\,(x)\,|^q \Big)^{1/q} \Big| L_p(\mathbb{T}^d) \Big\|, \tag{55}$$

holds for all $f \in L_p(\mathbb{T})$.

Remark 12 *The Lemma can be proved by making use of Lemma 6, in combination with Lemma 5 and Lemma 7. (54) holds even in the case $r = 0$, where we replace $S^r_p W(\mathbb{T}^d)$ by $L_p(\mathbb{T}^d)$.*

8.7 Complex Interpolation

8.7.1 Abstract Background

We briefly describe the complex interpolation method following [21]. Let A_0, A_1 be Banach spaces. Then (A_0, A_1) is said to be an interpolation couple if there exists a linear Hausdorff space $\mathcal{A}$ such that $A_0, A_1 \hookrightarrow \mathcal{A}$. For two interpolation couples (A_0, A_1) and (B_0, B_1) we denote by
$\mathcal{L}((A_0, A_1), (B_0, B_1))$ the collection of all linear operators $T : A_0 + A_1 \to B_0 + B_1$ such that the restrictions $T|_{A_i} : A_i \to B_i$, $i = 0, 1$, are continuous.
The class $\mathcal{F} = \mathcal{F}(A_0, A_1)$ is the collection of all vector valued continuous functions $f : \bar{S} \to A_0 + A_1$, which are additionally analytic on $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\} \subset \mathbb{C}$. This class equipped with the norm

$$\|f|\mathcal{F}\| = \max \Big\{ \sup_{t \in \mathbb{R}} \|f(it)|A_0\| \ , \ \sup_{t \in \mathbb{R}} \|f(1 + it)|A_1\| \Big\}$$

is a Banach space. Finally, the interpolation space $[A_0, A_1]_\vartheta$ is defined via

$$[A_0, A_1]_\vartheta := \{a \in A_0 + A_1 : \text{ it exists } f \in \mathcal{F} \text{ such that } a = f(\vartheta)\}$$

and equipped with the norm

$$\|a|[A_0, A_1]_\vartheta\| := \inf_{f \in \mathcal{F}\,,\, f(\vartheta) = a} \|f|\mathcal{F}\| \,. \tag{56}$$

Let $T \in \mathcal{L}((A_0, A_1), (B_0, B_1))$ then it turns out that

(i)

$$T|_{[A_0,A_1]_\vartheta} : [A_0, A_1]_\vartheta \to [B_0, B_1]_\vartheta$$

is continuous and moreover

(ii)

$$\|T : [A_0, A_1]_\vartheta \to [B_0, B_1]_\vartheta\| \le \|T : A_0 \to B_0\|^{1-\vartheta} \|T : A_1 \to B_1\|^{\vartheta}. \tag{57}$$

In order to apply this method to function spaces, let us mention the retraction and coretraction concept. For two Banach spaces A and B we call a linear continuous mapping $R \in \mathcal{L}(A, B)$ retraction if there exists an $S \in \mathcal{L}(B, A)$ such that $R \circ S = id_B$. We call S the corresponding coretraction. Let $R \in \mathcal{L}((A_0, A_1), (B_0, B_1))$ and $S \in \mathcal{L}((B_0, B_1), (A_0, A_1))$ be retraction and coreatraction in $\mathcal{L}(A_0, B_0)$ as well as in $\mathcal{L}(A_1, B_1)$. Then it holds

$$\|f|\,[B_0, B_1]_\vartheta\| \asymp \|Sf|\,[A_0, A_1]_\vartheta\| \quad , \quad f \in [B_0, B_1]_\vartheta\,. \tag{58}$$

8.7.2 Preliminaries

The interpolation method applied to weighted sequence spaces of type $\ell_p^\sigma(A_j)$, defined in the sense of [21, 1.18.1/2], where $A = \{A_j\}_j$ is a sequence of Banach spaces, yields in the case $1 \le p_0, p_1 \le \infty$, $\sigma_0, \sigma_1 \in \mathbb{R}$ the following formula

$$[\ell_{p_0}^{\sigma_0}(A_j), \ell_{p_1}^{\sigma_1}(B_j)]_\vartheta = \ell_p^\sigma([A_j, B_j]_\vartheta)\,, \tag{59}$$

where $0 < \vartheta < 1$ and

$$(1/p, \sigma) = (1 - \vartheta)(1/p_0, \sigma_0) + \vartheta(1/p_1, \sigma_1)\,.$$

Secondly we need a result concerning $L_p(A)$-spaces of A-valued functions, where A is a Banach space, stated in [21, 1.18.4] .

Lemma 9 *Let $1 \le p_0, p_1 < \infty$, $0 < \vartheta < 1$ and $\frac{1}{p} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}$. Then*

$$[L_{p_0}(A), L_{p_1}(B)]_\vartheta = L_p([A, B]_\vartheta)\,. \tag{60}$$

Proof A proof can be found in [21, 1.18.4]. ■

As done in [10] for the case $d = 2$ we introduce the sequence spaces

$$\ell_p^\sigma(\mathbb{N}_0^d) = \Big\{ a = \{a_\ell\}_{\ell \in \mathbb{N}_0^d} \Big| \, a_\ell \in \mathbb{C} \text{ and } \|a|\ell_p^\sigma\| = \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{\sigma|\ell|_1 p} |a_\ell|^p \Big)^{1/p} < \infty \Big\}\,,$$

and obtain the following interpolation result.

Lemma 10 *Let* $1 \le p_0, p_1 \le \infty$, $\mu, \sigma, \nu \in \mathbb{R}^d$, $0 < \vartheta < 1$ *such that*

$$(1/p, \mu) = (1-\vartheta)(1/p_0, \sigma) + \vartheta(1/p_1, \nu)$$

then

$$[\ell^{\sigma}_{p_0}(\mathbb{N}^d_0), \ell^{\nu}_{p_1}(\mathbb{N}^d_0)]_{\vartheta} = \ell^{\mu}_{p}(\mathbb{N}^d_0)\,.$$

Proof We prove this formula by iterating (59) .

8.7.3 Interpolation of Triebel-Lizorkin Spaces with Dominating Mixed Smoothness

We construct proper retractions and coretractions (see (58) to apply the results from the previous paragraph observing that

$$\|f|S^r_{p,q}F(\mathbb{T}^d)\| = \|f_\ell|L_p(\mathbb{T}^d, \ell^r_q)\|\,,$$

where f_ℓ , $\ell \in \mathbb{N}^d_0$, comes from (45) . Let us fix a system $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi(\mathbb{R})$. According to this we define the system

$$\psi_j(x) = \varphi_{j-1}(x) + \varphi_j(x) + \varphi_{j+1}(x) \quad , \quad x \in \mathbb{R}\,,$$

where we put $\varphi_{-1} \equiv 0$. Of course we have

$$\varphi_j(x)\psi_j(x) = \varphi_j(x) \quad , \quad j = 0, 1, 2, ...$$

Moreover we need

$$\varphi_\ell(x_1, ...x_d) = \varphi_{\ell_1}(x_1) \cdot ... \cdot \varphi_{\ell_d}(x_d) \quad \text{and} \quad \psi_\ell(x_1, ..., x_d) = \psi_{\ell_1}(x_1) \cdot ... \cdot \psi_{\ell_d}(x_d)\,,$$

with $x = (x_1, ..., x_d) \in \mathbb{T}^d$ and $\ell = (\ell_1, ..., \ell_d) \in \mathbb{N}^d_0$. We start with defining the mapping S_φ by

$$S_\varphi\, f(x) = \Big\{ \sum_{k \in \mathbb{Z}^d} \varphi_\ell(k) c_k(f) e^{ik\cdot x} \Big\}_{\ell \in \mathbb{N}^d_0} \quad , \quad f \in D'(\mathbb{T}^d)\,.$$

Let further the mapping R_ψ be defined through

$$R_\psi\, g = R_\psi(g_\ell)_{\ell \in \mathbb{N}^d_0} = \sum_{\ell \in \mathbb{N}^d_0} \Big(\sum_{k \in \mathbb{Z}^d} \psi_\ell(k) c_k(g_\ell) e^{ikx} \Big) \quad , \quad g = (g_\ell)_\ell \subset D'(\mathbb{T}^d)\,,$$

provided that the right-hand side makes sense. A simple consequence is the following. Let us start with $f \in D'(\mathbb{T}^d)$. Then $R_\psi(S_\varphi f)$ makes sense and we obtain

$$\begin{aligned} R_\psi(S_\varphi f) &= \sum_{\ell\in\mathbb{N}_0^d} \psi_\ell(k) \left(\sum_{k\in\mathbb{Z}^d} \varphi_\ell(k)\hat{f}(k)e^{ik\cdot x} \right) \\ &= \sum_{\ell\in\mathbb{N}_0^d} \left(\sum_{k\in\mathbb{Z}^d} \psi_\ell(k)\varphi_\ell(k)\hat{f}(k)e^{ik\cdot x} \right) \\ &= \sum_{\ell\in\mathbb{N}_0^d} \left(\sum_{k\in\mathbb{Z}^d} \varphi_\ell(k)\hat{f}(k)e^{ik\cdot x} \right) \\ &= f. \end{aligned}$$

The following result is also not difficult to see

Lemma 11 *If $1 \le p < \infty$, $1 \le q \le \infty$ and $r > 0$ then*

$$\begin{aligned} S_\varphi &: S^r_{p,q}F(\mathbb{T}^d) \to L_p(\mathbb{T}^d, \ell^r_q) \\ R_\psi &: L_p(\mathbb{T}^d, \ell^r_q) \to S^r_{p,q}F(\mathbb{T}^d) \end{aligned}$$

are bounded linear operators.

Proof Obviously $S_\varphi \in \mathcal{L}(S^r_{p,q}F(\mathbb{T}^d), L_p(\mathbb{T}^d))$ because of

$$\|S_\varphi f|L_p(\mathbb{T}^d, \ell^r_q)\| = \|f|S^r_{p,q}F(\mathbb{T}^d)\|\,.$$

Let $g = (g_\ell)_\ell \in L_p(\mathbb{T}^d, \ell^r_q)$. Of course g_ℓ, for fixed $\ell \in \mathbb{N}_0^d$, belongs to $L_p(\mathbb{T}^d)$. This leads to

$$\begin{aligned} \|R_\psi g|S^r_{p,q}F(\mathbb{T}^d)\|_\varphi &= \|R_\psi\{g_\ell\}_\ell|S^r_{p,q}F(\mathbb{T}^d)\|_\varphi \\ &= \Big\| \sum_{\ell\in\mathbb{N}_0^d} \Big(\sum_{k\in\mathbb{Z}^d} \varphi_u(k)\psi_\ell(k)c_k(g_\ell)e^{ik\cdot x} \Big) \Big| L_p(\mathbb{T}^d, \ell^r_q) \Big\| \\ &= \Big\| \sum_{\substack{|u_j-\ell_j|\le 2 \\ j=1,\dots,d}} \Big(\sum_{k\in\mathbb{Z}^d} \varphi_u(k)\psi_\ell(k)c_k(g_\ell)e^{ik\cdot x} \Big) \Big| L_p(\mathbb{T}^d, \ell^r_q) \Big\| \\ &\le \sum_{\ell\in([-2,2]\cap\mathbb{Z})^d} \Big\| \sum_{k\in\mathbb{Z}^d} (\underbrace{\varphi_u \cdot \psi_{u+\ell}}_{=:\Phi_{u,\ell}})(k)c_k(g_{u+\ell})e^{ikx} \Big| L_p(\mathbb{T}^d, \ell^r_q) \Big\|. \end{aligned}$$

Because of

$$\sum_{k\in\mathbb{Z}^d} \Phi_{u,\ell}(k)c_k(g_{u+\ell})e^{ikx} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \mathcal{F}^{-1}\Phi_{u,\ell}(y)g_{u+\ell}(x-y)\,dy \quad , \quad x \in \mathbb{T}^d\,,$$

the generalised Minkowski's inequality gives

$$\begin{aligned}\|R_\psi g|S^r_{p,q}F(\mathbb{T}^d)\| &\le c_1 \sum_{\ell\in([-2,2]\cap\mathbb{Z})^d} \int_{\mathbb{R}^d} \|\mathcal{F}^{-1}\Phi_{u,\ell}(y)g_{u+\ell}(x-y)|L_p(\mathbb{T}^d,\ell^r_q)\|\,dy\\ &= c_1 \sum_{\ell\in([-2,2]\cap\mathbb{Z})^d} \int_{\mathbb{R}^d} |\mathcal{F}^{-1}\Phi_{u,\ell}(y)|\cdot\|g_{u+\ell}(x)|L_p(\mathbb{T}^d,\ell^r_q)\|\,dy\\ &= c_1 \sum_{\ell\in([-2,2]\cap\mathbb{Z})^d} \|g_{u+\ell}(x)|L_p(\mathbb{T}^d,\ell^r_q)\| \int_{\mathbb{R}^d} |\mathcal{F}^{-1}\Phi_{u,\ell}(y)|\,dy.\end{aligned}$$

And finally the homogeneity of the Fourier transform implies

$$\int_{\mathbb{R}^d} |\mathcal{F}^{-1}\Phi_{u,\ell}(y)|\,dy \le c_2 < \infty\,,$$

where c_2 is independent of ℓ and u. And therefore

$$\|R_\psi g|S^r_{p,q}F(\mathbb{T}^d)\| \le c_3\|g_u|L_p(\mathbb{T}^d),\ell^r_q\|\,.$$

■

The final result of this subsection reads as follows

Lemma 12 *Let $1\le p_0,p_1<\infty$, $1\le q_0,q_1\le\infty$, $r_0,r_1>0$, $0<\vartheta<1$, $\frac{1}{p}=\frac{1-\vartheta}{p_0}+\frac{\vartheta}{p_1}$, $\frac{1}{q}=\frac{1-\vartheta}{q_0}+\frac{\vartheta}{q_1}$ and $r=(1-\vartheta)r_0+\vartheta r_1$. Then it holds the complex interpolation formula*

$$[S^{r_0}_{p_0,q_0}F(\mathbb{T}^d),S^{r_1}_{p_1,q_1}F(\mathbb{T}^d)]_\vartheta = S^r_{p,q}F(\mathbb{T}^d)\,.$$

Proof We make use of Lemma 9, the previously defined mappings S_φ and R_ψ and (58), where $(A_0,A_1)=\big(L_{p_0}(\mathbb{T}^d,\ell^{r_0}_{q_0}),L_{p_1}(\mathbb{T}^d,\ell^{r_1}_{q_1})\big)$ and $(B_0,B_1)=\big(S^{r_0}_{p_0,q_0}F(\mathbb{T}^d),S^{r_1}_{p_1,q_1}F(\mathbb{T}^d)\big)$. This gives

$$\begin{aligned}\|f\,|\,[S^{r_0}_{p_0,q_0}F(\mathbb{T}^d),\,S^{r_1}_{p_1,q_1}F(\mathbb{T}^d)]_\vartheta\,\| &\asymp \|S_\varphi f\,|\,[L_{p_0}(\mathbb{T}^d,\ell^{r_0}_{q_0}),\,L_{p_1}(\mathbb{T}^d,\ell^{r_1}_{q_1})]_\vartheta\,\|\\ &\asymp \|S_\varphi f\,|\,L_p(\mathbb{T}^d,[\ell^{r_0}_{q_0},\ell^{r_1}_{q_1}]_\vartheta)\|\\ &\asymp \|S_\varphi f\,|\,L_p(\mathbb{T}^d,\ell^r_q)\|\\ &\asymp \|f|S^r_{p,q}F(\mathbb{T}^d)\|_\varphi\,.\end{aligned}$$

■

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