The Smolyak Algorithm, Sampling on Sparse Grids and Function Spaces of Dominating Mixed Smoothness

Winfried Sickel and Tino Ullrich

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Friedrich-Schiller-University Jena
Mathematical Institute
Faculty of Mathematics and Computer Science
D-07737 Jena, Germany,
e-mail: sickel@minet.uni-jena.de, ullricht@minet.uni-jena.de

Abstract

We investigate the rate of convergence in $\| \cdot |L_p\|$, $1 \le p \le \infty$, of the d-dimensional Smolyak algorithm, associated to a sequence of sampling operators, in the framework of periodic Sobolev and Besov spaces with dominating mixed smoothness.

Key Words. Trigonometric interpolation, sampling operators, blending operators, Smolyak algorithm, sparse grids, rate of convergence, function spaces of dominating mixed smoothness, approximate recovery.

AMS subject classifications. 41A25, 41A63, 42B99, 46E35.

Abbreviated title: Smolyak algorithm

Corresponding author: Winfried Sickel
Friedrich-Schiller-University Jena,
Mathematical Institute
Faculty of Mathematics and Computer Science
D-07737 Jena, Germany,
e-mail: sickel@minet.uni-jena.de

1 Introduction

Let $(a_j^1)_{j=0}^\infty, \ldots , (a_j^d)_{j=0}^\infty$ be convergent sequences of complex numbers. The respective limits are denoted by $a^1, \ldots , a^d$. In addition we put $a_{-1}^\ell = 0$, $\ell = 1, \ldots , d$. Then $a^\ell = \sum_{j=0}^\infty (a_j^\ell - a_{j-1}^\ell)$ and hence

$$a^1 \cdot \ldots \cdot a^d = \sum_{j_1,\ldots,j_d=0}^\infty \prod_{\ell=1}^d (a_{j_\ell}^\ell - a_{j_\ell-1}^\ell) \, .$$

It has been the idea of Smolyak [31] to use the sequence

$$\sum_{j_1+\ldots+j_d \le m} \prod_{\ell=1}^d (a_{j_\ell}^\ell - a_{j_\ell-1}^\ell) \, , \qquad m = 0, 1, \ldots ,$$

to approximate the product $a^1 \cdot \ldots \cdot a^d$. Now, if

$$a_j^1 = a_j^2 = \ldots = a_j^d = L_j f(x) \, , \qquad x \in \mathbb{T} \, ,$$

where L_j denotes a sampling operator with respect to a certain set $\mathcal{T}_j$ of sample points, then the suggested approximation procedure results in an operator which uses samples from a sparse grid in $\mathbb{T}^d$ only, cf. Section 2 for details. Furthermore, the sequence of sampling operators constructed in such a way, should yield good approximations of tensor products $f_1 \otimes \ldots \otimes f_d$ of functions $f_\ell : \mathbb{T} \to \mathbb{C}$. In this paper we investigate the approximation power of these sampling operators for functions belonging to periodic Sobolev and Besov spaces of dominating mixed smoothness, see the Appendix for a definition. Let $A(\mathbb{T}^d)$ be either a Sobolev or a Besov space of dominating mixed smoothness on $\mathbb{T}^d$. Then the norm in these classes is a cross-norm, i.e.

$$\| \, f_1 \otimes \ldots \otimes f_d \, | A(\mathbb{T}^d) \| = \prod_{\ell=1}^d \| \, f_\ell \, | A(\mathbb{T}) \| \, .$$

Hence, the function spaces under consideration here, are sufficiently close to the tensor product of function spaces defined on $\mathbb{T}$. Based on the approximation power of L_j on the torus $\mathbb{T}$ we shall derive sharp estimates for the order of convergence of Smolyak's algorithm on $\mathbb{T}^d$.

The present article continues investigations of the approximation properties of trigonometric interpolation with respect to uniform grids, see [15, 16, 32, 34, 26], where we now study the d-variate situation with respect to a sparse grid. More precisely, we investigate the rate of convergence of the Smolyak algorithm (applied to a sampling operator) for functions belonging to a Besov space of dominating mixed smoothness. This continues earlier work of Dinh Dung [10], Smolyak [31], Temlyakov [32], Wasilkowski,

Woźniakowski [40] and one of the authors [27, 28]. It turns out that the Smolyak algorithm applied to a sampling operator yields a worst case within a wider class of Smolyak algorithms. In particular, the Smolyak algorithm applied to the partial sum of the Fourier series behaves better in approximation order than the Smolyak algorithm with respect to a sampling operator, see Subsection 2.2.2 for details. Let us mention that the algorithm applied to the partial sum of the Fourier series results in approximation from dyadic hyperbolic crosses, a subject, widely treated in the literature, see e.g. [1],[2], [3], [5], [7], [9], [13], [17], [18], [20], [23], [24], [29], [31], [34], [35] and [40].
The paper is organised as follows. To begin with we recall the construction of the Smolyak algorithm (Subsection 2.1), discuss a few more or less elementary properties of it and then we formulate our main results on the approximation power of this algorithm in a rather general frame (Subsection 2.2). Also consequences of our estimates for the problem of optimal recovery (sampling numbers) are discussed. Afterwards a few examples are presented. Section 4 contains the proofs. The definitions and a few properties of Sobolev and Besov spaces of dominating mixed smoothness will be recalled in the Appendix.
Notation. As usual, $\mathbb{N}$ is reserved for the natural numbers, by $\mathbb{N}_0$ we denote the natural numbers including 0 and by $\mathbb{Z}$ the set of all integers. Let $\mathbb{T}$ denote the torus, represented in $\mathbb{R}$ by the interval $\mathbb{T} = [0, 2\pi]$, where opposite points are identified. The symbol I is reserved for identity operators (we do not indicate the space where I is considered, hoping this will be clear from the context). I_n and $I(\Lambda^{\pi}_{\mu,n}, \cdot)$ denote special sampling operators defined in Subsection 2.3. We also use the notation $a \asymp b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1}\, a \le b \le c\, a\, .$$

Constants will change their value from line to line, indicated by adding subscripts. Sometimes a constant will represent a fixed value for the paper. This is indicated by capital letters like $C_1, C_2, \ldots$. Finally, if $x \in \mathbb{R}^d$ then $|x|$ is used for the Euclidean distance (norm in ℓ_2^d) and $|x|_1$ denotes the norm in ℓ_1^d, respectively.

2 Approximation on Sparse Grids

$\mathbb{R}^d$ denotes the Euclidean d-space and $\mathbb{Z}^d$ means those elements in $\mathbb{R}^d$ having integer components. The symbol $\mathbb{T}^d$ is used for the d-dimensional torus represented in $\mathbb{R}^d$ by

$[0,2\pi]^d$. For $f \in L_1(\mathbb{T}^d)$ and $k \in \mathbb{Z}^d$ we put

$$c_k(f) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x)\, e^{-ikx}\, dx\,.$$

Then the Fourier partial sum $S_n f$ and the formal Fourier series Sf of f is given by

$$S_n f(x) = \sum_{|k|\le n} c_k(f) e^{ikx} \quad \text{and} \quad Sf(x) := \sum_{k\in\mathbb{Z}^d} c_k(f)\, e^{ikx}\,,$$

respectively. Here $kx = \sum_{\ell=1}^d k_\ell\, x_\ell$, $k = (k_1, \ldots, k_d)$, $x = (x_1, \ldots, x_d)$.

2.1 The Smolyak Algorithm

2.1.1 The Definition and General Properties

We fix an integer $d \ge 2$. Let $X, Y \hookrightarrow L_1(\mathbb{T})$ be Banach spaces of periodic functions on $\mathbb{T}$ containing all trigonometric polynomials. Further we assume that $P_1, \ldots, P_d : X \to Y$ are continuous linear operators. Then we define its tensor product $P_1 \otimes \ldots \otimes P_d$ to be the linear operator such that:

$$(P_1 \otimes \ldots \otimes P_d)(e^{ik_1\cdot} \cdot \ldots \cdot e^{ik_d\cdot})(x_1, ..., x_d) := \prod_{\ell=1}^{d} P_\ell(e^{ik_\ell\cdot})(x_\ell)$$

$x_\ell \in \mathbb{T}$, $\quad k_\ell \in \mathbb{Z}$, $\quad \ell = 1, \ldots, d$. Formally, this operator is defined on trigonometric polynomials only. If X is either $L_p(\mathbb{T})$, $1 \le p < \infty$, or if $X = C(\mathbb{T})$, then, because of the density of trigonometric polynomials, there exists a unique continuous extension of $P_1 \otimes \ldots \otimes P_d$ to either $L_p(\mathbb{T}^d)$ or $C(\mathbb{T}^d)$, respectively. For this extension we shall use the same symbol.

Let either $L_j : L_p(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \le p < \infty$, or $L_j : C(\mathbb{T}) \to L_p(\mathbb{T})$, $1 \le p \le \infty$, $j \in \mathbb{N}_0$, be a sequence of continuous linear operators, denoted by L. Then we put

$$\Delta_j(L) := \begin{cases} L_j - L_{j-1} & \text{if} \quad j \in \mathbb{N}\,, \\ L_0 & \text{if} \quad j = 0\,. \end{cases}$$

Definition 1 *Let $m \in \mathbb{N}_0$. The Smolyak-Algorithm $A(m, d, \vec{L})$ of order m relative to the d sequences $L^1 := (L^1_j)_{j=0}^\infty, \ldots, L^d := (L^d_j)_{j=0}^\infty$, is the linear operator*

$$A(m, d, \vec{L}) := \sum_{j_1+\ldots+j_d \le m} \Delta_{j_1}(L^1) \otimes \ldots \otimes \Delta_{j_d}(L^d)\,.$$

Remark 1 *Originally introduced in* [31] *there are now hundreds of references dealing with this construction. A few basics and some references can be found in* [22] *and* [40]. *In particular, the following formula is proved in* [40]*:*

$$(1) \qquad A(m, d, \vec{L}) = \sum_{m-d+1 \le |j|_1 \le m} (-1)^{m-|j|_1} \binom{d-1}{m-|j|_1} L^1_{j_1} \otimes \ldots \otimes L^d_{j_d}\,.$$

This will be used later on.

2.1.2 Some Properties of Sampling Operators

In this article we shall restrict ourselves to sequences $(L_j)_j$ of linear operators having some additional properties. Recall, the symbol I is reserved for identity operators. Let $1 \le p \le \infty$ and $r > 0$ be fixed. We suppose

(H1) For any $j \in \mathbb{N}_0$ we have $L_j \in \mathcal{L}(L_p(\mathbb{T}), L_p(\mathbb{T}))$;
or alternatively

(H1') For any $j \in \mathbb{N}_0$ we have $L_j \in \mathcal{L}(C(\mathbb{T}), L_p(\mathbb{T}))$.

(H2) There exists a positive number λ such that

$$L_j(e^{ik\cdot})(t) = e^{ikt}\,, \qquad t \in \mathbb{R}\,, \quad |k| \le \lambda\, 2^j\,, \quad k \in \mathbb{Z}\,, \quad j \in \mathbb{N}_0\,.$$

(H3) For all $0 < s \le r$ there exists a positive constant $C_0(s)$ such that

$$\sup_{j=0,1,\ldots} \; 2^{js}\, \| \, I - L_j \, | B^s_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T}) \| = C_0(s) < \infty\,;$$

or alternatively

(H3') For all $1/p < s \le r$ there exists a positive constant $C_0(s)$ such that

$$\sup_{j=0,1,\ldots} \; 2^{js}\, \| \, I - L_j \, | B^s_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T}) \| = C_0(s) < \infty\,.$$

We shall say that $\vec{L}$ satisfies the hypothesis (Hn) if each sequence L^i, $i = 1, \ldots, d$, satisfies (Hn).

Remark 2 *Some comments on these restrictions: The hypotheses (H1)and (H3) will be used in connection with convolution operators L_j. For sampling operators we need to have function values at certain points. So we require the continuity of the underlying functions. This leads to the hypotheses (H1') and (H3'). Furthermore, the hypothesis (H2) is convenient for us but probably not necessary.*

We collect some properties of related sequences $A(m, d, \vec{L})$.

Lemma 1 *Let*

$$H(m,d,\lambda) := \Big\{ \ell \in \mathbb{Z}^d : \; \exists u_1, \ldots, u_d \in \mathbb{N}_0 \quad s.t. \quad |\ell_k| \le 2^{u_k}\lambda \text{ and } \sum_{k=1}^{d} u_k = m \Big\}$$

be a dyadic hyperbolic cross. Suppose that $\vec{L}$ satisfies (H2) for some $\lambda > 0$. Then

$$A(m, d, \vec{L})\, e^{ik\cdot} = e^{ik\cdot}\,, \quad k \in H(m, d, \lambda)\,.$$

Remark 3 *A special case of Lemma 1 can be found in* [32].

Of particular interest for us are sampling operators. Here we shall work with the following additional hypothesis:

(H4) For any $j \in \mathbb{N}_0$ there exist a natural number N_j, fixed continuous functions $\psi_1^j, \ldots, \psi_{N_j}^j : \mathbb{T} \to \mathbb{C}$ and sampling points $\mathcal{T}_j = \{t_1^j, \ldots, t_{N_j}^j\}$ such that

$$L_j f(x) = \sum_{\ell=1}^{N_j} f(t_\ell^j)\, \psi_\ell^j(x)\,, \quad f \in C(\mathbb{T})\,, \quad x \in \mathbb{T}. \tag{2}$$

Furthermore, we assume the existence of two positive constants C_1 and C_2 such that

$$C_1\, 2^j \le N_j \le C_2\, 2^j\,, \qquad j \in \mathbb{N}\,.$$

Sometimes we also need the assumption **(H5)**: We suppose

$$\Big|\mathcal{T}_{n+1} \setminus \bigcup_{j=0}^{n} \mathcal{T}_j\Big| \ge C_3\, 2^{n+1}\,, \qquad n \in \mathbb{N}\,,$$

with some constant $0 < C_3 \le C_2$.

Let $\vec{L}$ consist of sequences of sampling operators. The set of sampling points used by L_j^i will be denoted by $\mathcal{T}_j^i$. Then we put

$$\mathcal{G}(m, d, \vec{L}) := \bigcup_{m-d+1 \le |j|_1 \le m} \mathcal{T}_{j_1}^1 \times \ldots \times \mathcal{T}_{j_d}^d \tag{3}$$

By (1) the operator $A(m, d, \vec{L})$ uses only samples from the grid $\mathcal{G}(m, d, \vec{L})$. Sometimes we shall concentrate on the following situation: the operators $L_j^1, \ldots, L_j^d$ are using function values from the grid

$$J_{N_j} := \left\{ t_\ell = \frac{2\pi\ell}{N_j} \,:\, \ell \in \mathbb{Z},\, -N_j/2 \le \ell < N_j/2 \right\}, \qquad j = 0, 1 \ldots . \tag{4}$$

In case $N_j = 2^j$ the grid used by $A(m, d, \vec{L})$ is given by

$$\mathcal{G}(m,d) := \Big\{ \Big(\frac{2\pi\ell_1}{2^{j_1}}, \ldots, \frac{2\pi\ell_d}{2^{j_d}} \Big) : \\ -2^{j_i - 1} \le \ell_i < 2^{j_i - 1}, i = 1, \ldots, d, \quad m - d + 1 \le |j|_1 \le m \Big\}$$

which we will call *standard grid* in the sequel.

Lemma 2 (i) *The cardinality $S(m,d)$ of the standard grid $\mathcal{G}(m,d)$ is given by*

$$S(m,d) = \sum_{j=0}^{d-1} \binom{d-1}{j} 2^{m-j} \binom{m}{j}\,,$$

where we put $\binom{m}{j} := 0$ *in case* $m < j$.

(ii) *The hypothesis (H4) should be fulfilled. Then the cardinality* $|\mathcal{G}(m,d,\vec{L})|$ *of the grid* $\mathcal{G}(m,d,\vec{L})$ *satisfies*

$$|\mathcal{G}(m,d,\vec{L})| \le (2C_2)^d\, S(m,d)\ .$$

(iii) *If the hypotheses (H4) and (H5) are fulfilled then*

$$\min(C_1,C_3)^d S(m,d) \le |\mathcal{G}(m,d,\vec{L})| \le (2C_2)^d\, S(m,d)\ , \quad m\in\mathbb{N}_0\ ,$$

holds true.

Remark 4 *Obviously, for fixed* d *we have*

$$S(m,d) \asymp 2^m\, m^{d-1}\,, \qquad m \in \mathbb{N}\,.$$

We call the grids $\mathcal{G}(m,d,\vec{L})$ *sparse because their cardinality is growing only in a logarithmic order with respect to* d.

Lemma 3 *The hypothesis (H4) should be fulfilled. In addition to (2) we assume that*

$$L_j^i f(t) = f(t)\,, \qquad t\in \mathcal{T}_j^i\,, \qquad i=1,\ldots,d\,,$$

for all $j\le m$ *and all* $f\in C(\mathbb{T})$*. If the grids* $\mathcal{T}_j^i$ *are nested, i.e.*

$$\mathcal{T}_j^i \subset \mathcal{T}_{j+1}^i\,, \qquad i=1,2,\ldots,d\,, \quad j\in\mathbb{N}_0\,,$$

then $A(m,d,\vec{L})$ *interpolates on* $\mathcal{G}(m,d,\vec{L})$*, more precisely*

$$A(m,d,\vec{L})f(x) = f(x)\,, \qquad x\in\mathcal{G}(m,d,\vec{L})\,, \quad f\in C(\mathbb{T}^d)\,.$$

2.2 The Approximation Power of Smolyak's Algorithm and Besov Spaces of Dominating Mixed Smoothness

We study the approximation power of the Smolyak algorithms for two different classes of operators, sampling operators and (mainly for comparison reasons) convolution operators.

2.2.1 Smolyak's Algorithm and Sampling Operators

This subsection contains the main results of this paper. As mentioned in the Introduction we will study the approximation power of the Smolyak algorithm for functions taken from Besov spaces of dominating mixed smoothness $S^r_{p,q}B(\mathbb{T}^d)$, see Subsection 5.4 for a definition. Our first result is the following general estimate for sampling operators. Here we shall use that functions from $S^r_{p,q}B(\mathbb{T}^d)$ with $r > 1/p$ have a continuous representative, see Lemma 7.

Theorem 1 *Let $1 \le p, q \le \infty$ and $r > 1/p$. Let further $\vec{L}$ satisfy the hypotheses (H1'), (H2), and (H3'). Then there exists a constant $c > 0$ such that*

$$\| I - A(m,d,\vec{L}) \,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(1-1/q)}\, 2^{-mr}$$

holds for all $m \in \mathbb{N}_0$.

In case of sampling operators which use samples from (4) the obtained estimates are unimprovable. We consider operators of the following structure

$$(5) \qquad L^i_j f(t) = \sum_{-N_j/2 \le \ell < N_j/2} f(t^j_\ell)\, \Lambda_j(t - t^j_\ell)\,, \qquad i = 1, \ldots, d\,, \quad f \in C(\mathbb{T})\,,$$

with some sequence of functions $\Lambda_j \in C(\mathbb{T})$ and $t^j_\ell \in J_{N_j}$, $j \in \mathbb{N}_0$, see (4).

Theorem 2 *Let $1 \le p, q \le \infty$ and $r > 1/p$. Let $\vec{L}$ be given by (5) and satisfy the hypotheses (H1'), (H2), (H3') and (H4). Furthermore, we assume $1 < N_{j+1}/N_j \in \mathbb{N}$. Then the relation*

$$\| I - A(m,d,\vec{L}) \,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)(1-1/q)}\, 2^{-mr} \quad , \qquad m \in \mathbb{N}\,,$$

holds true.

Remark 5 *Let L^i_j be given by*

$$L^i_j := \mathcal{R}_{2^j}\,, \qquad j \in \mathbb{N}_0\,, \quad i = 1, \ldots, d\,,$$

cf. Remark 13 below. For the corresponding algorithm $A(m,d,\vec{L})$ and $q = \infty$ Theorem 2 has been proved by Temlyakov in [34, Chapt. 4, Thm. 5.1].

Using a simple bump function argument we can show that starting with sampling on non equidistant points does not help to improve the approximation properties of the related Smolyak algorithm, at least if $q = 1$.

Theorem 3 *Let $1 \le p \le \infty$ and $r > 1/p$. Let $\vec{L}$ satisfy the hypotheses (H1'), (H2), (H3'), (H4) and (H5). Then*

$$\|I - A(m,d,\vec{L})|S^r_{p,1}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-rm} \quad , \quad m \in \mathbb{N}\,,$$

is valid.

2.2.2 Smolyak's Algorithm and Convolution Operators

Two other examples of Smolyak algorithms are of interest for us, mainly to have a comparison with the results from Subsection 2.2.1. Let us mention that most of the results of this subsection are well-known. So this part of the paper can be understood as a rough survey to this subject. For that reason no proofs will be given (except Theorem 6) but all details may be found in [30]. A few further references will be given below.

First we consider

$$L_j^i f(t) := S_{2^j} f(t) = \sum_{k=-2^j}^{2^j} c_k(f)\, e^{ikt}\,, \qquad j \in \mathbb{N}_0\,, \quad i = 1, \ldots d\,.$$

To indicate this special choice we write $A(m,d,S)$ instead of $A(m,d,\vec{L})$. In the second case we choose ψ as in Subsection 5.4.3 and define for $f \in L_1(\mathbb{T})$

$$L_j^i f(t) := \sum_{k\in\mathbb{Z}} \psi(2^{-j}k)\, c_k(f)\, e^{ikt}, \qquad t \in \mathbb{T}\,, \quad j \in \mathbb{N}_0\,, \quad i = 1, \ldots d.$$

We shall call them smooth de la Vallée-Poussin means. This time we write $A(m,d,\psi)$ instead of $A(m,d,\vec{L})$. These operators are related to those decompositions of the functions as used in connection with the definition (characterization) of the considered function spaces.

Lemma 4 *For $f \in L_1(\mathbb{T}^d)$ we have*

$$(6) \qquad A(m,d,S)f(x) = \sum_{k\in H(m,d,1)} c_k(f)\, e^{ikx} \quad \textit{and} \quad A(m,d,\psi)f(x) = \sum_{|j|_1 \le m} f_j^{\psi}(x)\,,$$

cf. (28). Furthermore

$$(7) \qquad \operatorname{rank} A(m,d,S)\,,\ \operatorname{rank} A(m,d,\psi) \ \asymp\ m^{d-1}\, 2^m\,, \qquad m \in \mathbb{N}\,.$$

Lemma 4 makes clear that both, $A(m,d,S)$ as well as $A(m,d,\psi)$, are related to approximation with respect to hyperbolic crosses. These operators $A(m,d,S)$, $A(m,d,\psi)$ have better approximation properties than sampling operators.

Theorem 4 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r > 0$. Then the relation*

$$\begin{aligned} &\| I - A(m,d,S)\, |S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \\ &\asymp \begin{cases} m^{(d-1)(\frac{1}{p}-\frac{1}{q})}\, 2^{-mr} & \textit{if} \quad 1 < p \le 2 \textit{ and } p \le q \le \infty\,, \\ m^{(d-1)(\frac{1}{2}-\frac{1}{q})}\, 2^{-mr} & \textit{if} \quad 2 < p < \infty \textit{ and } 2 < q \le \infty\,, \\ 2^{-mr} & \textit{otherwise}\,, \end{cases} \end{aligned}$$

holds true.

Remark 6 *In the periodic setting Theorem 4 with $q = \infty$ was known for a long time, cf. Bugrov* [5] *($p = 2$), Mityagin* [19]*, Nikol'skaya* [20] *and Temlyakov* [34, Thm. 3.3.3]*. The case $q \neq \infty$ has been treated in Dinh Dung* [9]*, Galeev* [13] *and Romanyuk* [23]*. For the nonperiodic case we refer to Lizorkin, Nikol'skij* [18]*, Bazarkhanov* [2, 3]*, and* [24]*. The analogous problem for spaces defined on the unit cube and spline approximation has been treated by Kamont* [17]*. Many times the problem has been investigated in connection with best approximation from hyperbolic crosses. So this remark applies also with respect to Theorem 5 and Corollary 1 below.*

Theorem 4 can be extended to the limiting cases $p = 1$ and $p = \infty$ by switching from $A(m, d, S)$ to $A(m, d, \psi)$.

Theorem 5 *Let $1 \le p \le \infty$, $1 \le q \le \infty$ and $r > 0$. Then the relation*

$$\| I - A(m,d,\psi) \, | S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d) \| \asymp \begin{cases} m^{(d-1)(\frac{1}{p}-\frac{1}{q})}\, 2^{-mr} & \text{if } \; 1 \le p \le 2 \text{ and } \; p \le q \le \infty\,, \\ m^{(d-1)(\frac{1}{2}-\frac{1}{q})}\, 2^{-mr} & \text{if } \; 2 < p < \infty \text{ and } \; 2 < q \le \infty\,, \\ m^{(d-1)(1-\frac{1}{q})}\, 2^{-mr} & \text{if } \; p = \infty \text{ and } \; 1 \le q \le \infty\,, \\ 2^{-mr} & \text{otherwise}\,, \end{cases}$$

holds for $m \in \mathbb{N}_0$.

Remark 7 *We compare the multi-dimensional situation with the univariate one by considering the quantities*

$$\begin{aligned} E^S_m(r,p,q) &:= \|I - A(m,d,S) : S^r_{p,q}B(\mathbb{T}^d) \mapsto L_p(\mathbb{T}^d)\| \\ E^\psi_m(r,p,q) &:= \|I - A(m,d,\psi) : S^r_{p,q}B(\mathbb{T}^d) \mapsto L_p(\mathbb{T}^d)\|. \\ E^{\vec{L}}_m(r,p,q) &:= \|I - A(m,d,\vec{L}) : S^r_{p,q}B(\mathbb{T}^d) \mapsto L_p(\mathbb{T}^d)\| \end{aligned}$$

Here $\vec{L}$ stands for a sequence of sampling operators as in Theorem 2. There are at least three new phenomena in the multi-dimensional case.

- *The sampling operators $A(m,d,\vec{L})$ and the Fourier partial sum operator $A(m,d,S)$ do not have the same approximation power. This has to be compared with Remark 12 and Propositions 1, 2 below.*

- *Observe that the case $p = \infty$ plays a particular role. Obviously, for fixed $p \ge 2$ and m sufficiently large we have*

  $$\frac{E^\psi_m(r,\infty,q)}{E^\psi_m(r,p,q)} \asymp \begin{cases} m^{\frac{d-1}{2}} & \text{if } \; 2 < q \le \infty\,, \\ m^{(d-1)(1-\frac{1}{q})} & \text{if } \; 1 \le q \le 2\,. \end{cases}$$

  So if p is approaching infinity we have a jump in the order of convergence except $q = 1$.

- *Let $1 < p < \infty$. The microscopic index q of the Besov space enters the order of approximation for all operators we have considered here. Let $\vec{L}$ be as in Theorem 2. Then also the ratio*

$$\frac{E_m^{\vec{L}}(r,p,q)}{E_m^{S}(r,p,q)} \asymp \begin{cases} m^{(d-1)(1-\frac{1}{p})} & \text{if} \quad 1 < p \le 2 \text{ and } \; p \le q \le \infty\,, \\ m^{\frac{d-1}{2}} & \text{if} \quad 2 < p < \infty \text{ and } \; 2 < q \le \infty\,, \\ m^{(d-1)(1-\frac{1}{q})} & \text{if} \quad 1 < p < \infty \text{ and } \; 1 \le q \le \min(2,p)\,, \end{cases}$$

depends on q. For the limiting situations we replace $A(m,d,S)$ by $A(m,d,\psi)$ and obtain

$$\frac{E_m^{\vec{L}}(r,1,q)}{E_m^{\psi}(r,1,q)} \asymp 1 \qquad \textit{as well as} \qquad \frac{E_m^{\vec{L}}(r,\infty,q)}{E_m^{\psi}(r,\infty,q)} \asymp 1\,.$$

Best Approximation with Respect to Hyperbolic Crosses

We define for $1 \le p,q \le \infty$ and $r > 0$

$$\mathcal{E}_m(S^r_{p,q}B(\mathbb{T}^d))_p := \sup_{\|f|S^r_{p,q}B\|=1} \inf \Big\{ \| f - g\,|L_p(\mathbb{T}^d)\| \;:\; g \quad \text{is a trigonometric polynomial s.t.} \quad c_k(g) = 0 \quad \text{for all} \quad k \notin H(m,d,1) \Big\}$$

Now Theorem 5 yields the upper bound in the following corollary.

Corollary 1 *Let $1 \le p \le \infty$, $1 \le q \le \infty$ and $r > 0$. Then*

$$\mathcal{E}_m(S^r_{p,q}B(\mathbb{T}^d))_p \asymp \| \, I - A(m,d,\psi)\,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \tag{8}$$

holds for $m \in \mathbb{N}_0$.

Remark 8 (i) *The given estimates for $\mathcal{E}_m(f, L_p(\mathbb{T}^d))$ do not characterize the classes $S^r_{p,q}B(\mathbb{T}^d)$ in $L_p(\mathbb{T}^d)$, see* [24] *for more details in this direction. Function spaces defined by best approximation from hyperbolic crosses have been investigated in* [8], [18] *and* [24].

(ii) *In view of (8) it is interesting to notice that*

$$\|A(m,d,\psi):\, L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)\| \asymp \|A(m,d,\psi):\, L_\infty(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d)\| \asymp m^{d-1}\,,$$

see [34, Lem. 3.1.2], *whereas for $1 < p < \infty$ one has*

$$\sup_{m=1,2,\ldots} \|A(m,d,S):\, L_p(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| < \infty\,,$$

and

$$\sup_{m=1,2,\ldots} \|A(m,d,\psi):\, L_p(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| < \infty\,,$$

see e.g. [21, 1.5.2]. *Hence, the estimate from below of $\mathcal{E}_m(S^r_{p,q}B(\mathbb{T}^d))_p$ in case $1 < p < \infty$ is more or less an obvious consequence of Theorem 4, whereas the proof in cases $p = 1$ and $p = \infty$ requires different arguments, see e.g.* [34] *or* [30].

We finish this subsection with a counterpart of Theorem 1 by replacing the hypotheses (H1') and (H3') by (H1) and (H3), respectively.

Theorem 6 *Let $1 \le p, q \le \infty$ and $r > 0$. Let further $\vec{L}$ satisfy the hypotheses (H1), (H2), and (H3). Then there exists a constant $c > 0$ such that*

$$\| I - A(m, d, \vec{L}) \,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(1-1/q)}\, 2^{-mr}$$

holds for all $m \in \mathbb{N}_0$.

Remark 9 *Theorems 1 and 6 generalize the results obtained in* [27] *in various directions. In* [27] *we have investigated only the bivariate case and restricted us to $1 < p < \infty$. In addition, the admissible operators $A(m, d, \vec{L})$ are more general now.*

2.3 Examples

We shall treat two types of examples. For a more general treatment we refer to [30]. First we consider classical trigonometric interpolation.

2.3.1 Trigonometric Interpolation

Let

$$\mathcal{D}_m(t) := \sum_{|k|\le m} e^{ikt}\,, \qquad t \in \mathbb{T}, \quad m \in \mathbb{N}_0\,,$$

be the Dirichlet kernel of order m and let

$$I_m f(t) := \frac{1}{2m+1} \sum_{\ell=0}^{2m} f(t_\ell)\, \mathcal{D}_m(t - t_\ell)\,, \qquad t_\ell = \frac{2\pi\ell}{2m+1}\,. \tag{9}$$

Here f is assumed to be 2π-periodic. Then I_m is the unique trigonometric polynomial of degree less than or equal to m which interpolates f at the nodes t_ℓ. Then the following is known, see [15, 16, 32, 34, 26].

Proposition 1 *Let $1 < p < \infty$ and let $r > 1/p$. Then we have*

$$\| I - I_m \,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp m^{-r}\,.$$

We put

$$A(m, d, D) := A(m, d, \vec{L})\,, \qquad L^i_j := I_{2^j}\,, \qquad i = 1, \ldots d\,, \quad j \in \mathbb{N}_0\,,$$

cf. (9). Obviously, the hypotheses (H1'), (H2), (H3'), (H4) and (H5) are satisfied in this situation. However, the sequence of grids $\mathcal{G}(m, d, D)$ does not coincide with our standard grid. So Theorem 2 is not applicable. Nevertheless Theorem 1 and Theorem 3 give the following.

Corollary 2 *Let $1 < p < \infty$, $1 \le q \le \infty$, and $r > 1/p$. Then there is a constant $c > 0$ such that*

$$\| I - A(m,d,D)\, |S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \le c\, m^{(d-1)(1-1/q)}\, 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

If $q = 1$ we have

$$\| I - A(m,d,D)\, |S^r_{p,1}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

Furthermore, with

$$\mathcal{G}^*(m,d) := \Big\{ \Big(\frac{2\pi \ell_1}{2^{j_1+1}+1}, \ldots , \frac{2\pi \ell_d}{2^{j_d+1}+1}\Big) : \\ 0 \le \ell_i \le 2^{j_i+1}, i = 1, \ldots , d, \quad m-d+1 \le |j|_1 \le m \Big\}\,,$$

the operator $A(m,d,D)$ uses function values from $\mathcal{G}^*(m,d)$, but in general $A(m,d,D)$ is not interpolating.

Remark 10 (i) *From Lemma 2, Remark 4 and Lemma 4 it follows that $A(m,d,\mu)$, $A(m,d,D)$ as well as $A(m,d,S)$ and $A(m,d,\psi)$ are all of the same complexity. However, from Theorems 2, 4, 5 it follows that they are not of the same efficiency.*
(ii) *To see that $A(m,d,D)$ is not interpolating it is sufficient to consider the operator $A(1,2,D)$ applied to the function $f(x_1,x_2) = e^{i3x_1}$ at the point $\big(2\pi/3, 0\big) \in \mathcal{G}^*(1,2)$. Of course, the reason for this consists in*

$$\Big\{ \frac{2\pi\ell}{2^j+1} : \quad 0 \le \ell \le 2^j \Big\} \not\subset \Big\{ \frac{2\pi\ell}{2^{j+1}+1} : \quad 0 \le \ell \le 2^{j+1} \Big\}\,.$$

2.3.2 Interpolation with de la Vallée-Poussin Means

For $0 < \mu < 1/2$ we consider the functions

$$\Lambda_\mu(t) := 2\, \frac{\sin(t/2)\, \sin(\mu t)}{\mu\, t^2}\,, \qquad t \in \mathbb{R}\,. \tag{10}$$

Then the Fourier transform is given by

$$\mathcal{F}\Lambda_\mu(\xi) = \sqrt{2\pi} \begin{cases} 1 & \text{if} \quad |\xi| \le \frac12 - \mu\,, \\ \frac{1}{2\mu}\big(\frac12 + \mu - |\xi|\big) & \text{if} \quad \frac12 - \mu < |\xi| < \frac12 + \mu\,, \\ 0 & \text{if} \quad \frac12 + \mu \le |\xi|\,, \end{cases}$$

i.e. a piecewise linear function. It is not difficult to see that

$$\Lambda^\pi_{\mu,n}(t) := \sum_{\ell\in\mathbb{Z}} \Lambda_\mu(nt + 2\pi\ell n)\,, \qquad t \in \mathbb{R}\,,$$

is a continuous 2π-periodic fundamental interpolant with respect to the grid J_n, see (4) with n in place of 2^j. The Fourier coefficients of these functions $\Lambda_{\mu,n}$ are given by

$$c_\ell(\Lambda_n^\pi) = \frac{1}{n\sqrt{2\pi}}\,\mathcal{F}\Lambda(\ell/n)\,, \qquad \ell \in \mathbb{Z}\,, \quad n \in \mathbb{N}\,.$$

We define

$$I(\Lambda^\pi_{\mu,n}, f)(t) := \sum_{-n/2\le \ell < n/2} f(t_\ell)\,\Lambda^\pi_{\mu,n}(t-t_\ell)\,, \qquad t \in \mathbb{T}\,, \quad t_\ell \in J_n\,.$$

Proposition 2 *Let $0 < \mu < 1/2$ and Λ_μ be defined as in (10). Let further $1 \le p \le \infty$ and $r > 1/p$. Then we have*

$$\|\, I - I(\Lambda^\pi_{\mu,n}, \cdot)\,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r}\,.$$

Remark 11 *A proof of the estimate from above can be found in* [29]*, at least if $1 < p < \infty$. The necessary modifications, to include the limiting cases, are straightforward. For full details we refer to* [37]*. The estimate from below can be deduced from the behaviour of the linear widths (approximation numbers) of the embeddings $B^r_{p,\infty}(\mathbb{T}) \hookrightarrow L_p(\mathbb{T})$, see Remark 12 below.*

Remark 12 *Linear widths. For two Banach spaces X, Y such that $X \hookrightarrow Y$ we define*

$$\lambda_n(I,X,Y) := \inf\Big\{\|\, I - L\,|\mathcal{L}(X,Y)\| \; : \quad L \in \mathcal{L}(X,Y),\ \mathrm{rank}\, L \le n\Big\}\,.$$

Since our operator $I(\Lambda^\pi_n, \cdot)$ has rank $\le n$ we obtain

$$\lambda_n(I, B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \le \|\, I - I(\Lambda^\pi_n, \cdot)\,|\mathcal{L}(B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T}))\,\|\,.$$

Since $\lambda_n(I, B^r_{p,\infty}(\mathbb{T}), L_p(\mathbb{T})) \asymp n^{-r}$, cf. e.g. [34, 1.4]*, it is clear that our interpolation operators yield optimal in the order of approximation.*

Remark 13 *Let*

$$v_{2n-1}(t) := \frac{1}{n}\sum_{j=n}^{2n-1} D_j(t)\,, \qquad t \in \mathbb{R}\,, \quad n \in \mathbb{N}\,,$$

denote the de la Vallée-Poussin kernels of odd order. Then

$$c_k(v_{2n-1}) = \begin{cases} 1 & \text{if} \quad |k| \le n\,, \\ 2\,(1 - |k|/(2n)) & \text{if} \quad n < |k| < 2n\,, \\ 0 & \text{if} \quad |k| \ge 2n\,. \end{cases}$$

From the formula of the Fourier coefficients of $\Lambda^\pi_{\mu,n}$ above we conclude the identity

$$\Lambda^\pi_{\mu,3n} = \frac{v_{2n-1}}{3n}\,, \qquad \mu = \frac{1}{6}\,, \quad n \in \mathbb{N}\,.$$

Hence

$$I(\Lambda^\pi_{\mu,3n}, f)(t) = \frac{1}{3n} \sum_{-(3n)/2 \le \ell < (3n)/2} f(t_\ell)\, v_{2n-1}(t - t_\ell)\,, \qquad t_\ell \in J_{3n}\,, \quad \mu = \frac{1}{6}\,.$$

In contrast to our treatment Temlyakov [34, 1.6] *considered the sequence of sampling operators*

$$\mathcal{R}_n f(t) := \frac{1}{4n} \sum_{-2n \le \ell < 2n} f(t_\ell)\, v_{2n-1}(t - t_\ell)\,, \qquad t_\ell \in J_{4n}\,,$$

and proved that these operators also satisfy

$$\|\, I - \mathcal{R}_n \,|B^r_{p,\infty}(\mathbb{T}) \to L_p(\mathbb{T})\| \asymp n^{-r}\,,$$

if $1 \le p \le \infty$ and $r > 1/p$.

Now we turn to the related Smolyak algorithm. We put

(11) $$A(m,d,\mu) := A(m,d,\vec{L})\,, \qquad L^i_j := I(\Lambda^\pi_{\mu,2^j}, \,\cdot\,)\,, i = 1, \ldots d\,, \quad j \in \mathbb{N}_0\,.$$

As an immediate consequence of Proposition 2 we obtain that $\vec{L}$ satisfies (H1') and (H3'). A simple calculation shows that (H2) is satisfied with $\lambda = 1/2 - \mu$. Since $I(\Lambda^\pi_{\mu,2^j}, \,\cdot\,)$ uses function values from the standard grid J_{2^j} also (H4) and (H5) are fulfilled. Altogether Theorem 2 and Lemma 3 yield the following.

Corollary 3 *Let $1 \le p, q \le \infty$, $r > 1/p$, and $0 < \mu < 1/2$. Then*

$$\|\, I - A(m,d,\mu) \,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)(1-1/q)}\, 2^{-mr}\,, \qquad m \in \mathbb{N}_0\,.$$

Furthermore, with

$$\mathcal{G}(m,d) := \Big\{ \Big(\frac{2\pi \ell_1}{2^{j_1}}, \ldots, \frac{2\pi \ell_d}{2^{j_d}} \Big) : \\ -2^{j_i - 1} \le \ell_i < 2^{j_i - 1}, i = 1, \ldots, d, \quad m - d + 1 \le |j|_1 \le m \Big\}\,,$$

the operator $A(m,d,\mu)$ interpolates on $\mathcal{G}(m,d)$, i.e.

$$A(m,d,\mu) f(x) = f(x)\,, \qquad x \in \mathcal{G}(m,d)\,,$$

for all $f \in C(\mathbb{T}^d)$.

2.4 Optimal Recovery of Functions from Besov Spaces of Dominating Mixed Smoothness

Let

$$\Psi_M(f,\xi)(x) := \sum_{j=1}^{M} f(\xi^j)\,\psi_j(x)$$

denote a general sampling operator for a class F of continuous, periodic functions defined on $\mathbb{T}^d$, where

$$\xi := \Big\{\xi^1,...,\xi^M\Big\}\,, \qquad \xi^i \in \mathbb{T}^d\,, \quad i = 1,2,\ldots,M\,,$$

is a fixed set of sampling points and $\psi_j : \mathbb{T}^d \to \mathbb{C}$, $j = 1,\ldots,M$, are fixed, continuous, periodic functions. Then the quantity

$$\rho_M(F, L_p(\mathbb{T}^d)) := \inf_{\xi}\ \inf_{\psi_1,\ldots,\psi_M}\ \sup_{\|f|F\|\le 1}\ \|\, f - \Psi_M(f,\xi)\,|L_p(\mathbb{T}^d)\|$$

measures the optimal rate of approximate recovery of the functions taken from F. We are interested in the case, when $F = S^r_{p,q}B(\mathbb{T}^d)$, $1 \le p,q \le \infty$, $r > 1/p$. Observe that the operator $A(m,d,\mu)$, see (11), uses $M = M(m,d) \asymp 2^m\, m^{d-1}$ function values from its argument, see Remark 4. Therefore $m \le c \log M$ with some c independent of m and hence

$$2^{-rm} m^{(d-1)(1-1/q)} \le M^{-r}(c\,\log\, M)^{(d-1)(r+1-1/q)}\,.$$

In view of Theorem 2 this implies the upper bound given below.

Corollary 4 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $r > 1/p$. Then there exist positive constants c_1 and c_2 such that for all $M \in \mathbb{N}$*

$$\begin{aligned} c_1\, M^{-r} (\log M)^{(d-1)r} \eta(M,d,p,q) \ &\le\ \rho_M(S^r_{p,q}B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \\ &\le\ c_2\, M^{-r}(\log M)^{(d-1)(r+1-1/q)}\,, \end{aligned}$$

where

$$\eta(M,d,p,q) := \begin{cases} (\log M)^{(d-1)(\frac{1}{2}-\frac{1}{q})} & \textit{if} \quad 2 \le p,q\,, \\ (\log M)^{(d-1)(\frac{1}{p}-\frac{1}{q})} & \textit{if} \quad 1 < p < 2 \quad \textit{and} \quad p \le q\,, \\ 1 & \textit{otherwise}\,. \end{cases} \tag{12}$$

Remark 14 (i) *The Smolyak algorithm uses samples of a very specific structure. Corollary 4 tells us that allowing arbitrary sets of sampling points of the same cardinality we can not do much better. The difference is at most $(\log M)^{(d-1)/2}$ if $1 < p < \infty$.*

(ii) *In case* $q = \infty$ *Temlyakov proved the estimate from above in Corollary 4, cf.* [32] *and* [34, 4.5]*; see also Dinh Dung* [10, Thm. 4.2]*. The given estimates from below are a consequence of the known behaviour of related Kolmogorov numbers, cf. e.g. Galeev* [14]*.*

3 Sobolev Spaces of Dominating Mixed Smoothness and the Problem of Best Recovery

By using $S^r_pW(\mathbb{T}^d) \hookrightarrow S^r_{p,2}B(\mathbb{T}^d)$, $1 < p \le 2$, see (26), one can derive immediately some consequences for the quality of the approximation by $A(m,d,\vec{L})$. Surprisingly, the conclusions obtained in that way, are unimprovable. The counterpart of Theorem 2 reads as follows.

Theorem 7 *Let* $1 < p \le 2$ *and* $r > 1/p$*. Let* $\vec{L}$ *be given by (5) with* $1 < N_{j+1}/N_j \in \mathbb{N}, j \in \mathbb{N}_0$*, and satisfy the hypotheses (H1'), (H2), (H3') and (H4). Then*

$$\| I - A(m,d,\vec{L}) \,|S^r_pW(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)/2}\, 2^{-mr}\,, \qquad m \in \mathbb{N}\,,$$

holds true.

In case of approximation from the hyperbolic cross, see Theorem 4, we find

Theorem 8 *If* $1 < p \le 2$ *and* $r > 0$ *then*

$$\| I - A(m,d,S) \,|S^r_pW(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| \asymp m^{(d-1)(\frac{1}{p}-\frac{1}{2})}\, 2^{-mr}\,, \qquad m \in \mathbb{N}\,,$$

holds true.

Remark 15 *In a forthcoming paper one of the authors will prove optimal estimates in case* $2 < p < \infty$*. This approach will be based on a different method, see* [28] *for the bivariate situation.*

Probably the most interesting consequence of Theorem 1, applied to the algorithm $A(m,d,D)$, is given by the following corollary which was the main motivation of our research.

Corollary 5 *Let* $1 < p \le 2$ *and* $r > 1/p$*. Then two positive constants* c_1*,* c_2 *exist such that*

$$c_1\, M^{-r} (\log M)^{(d-1)r} \;\le\; \rho_M(S^r_pW(\mathbb{T}^d), L_p(\mathbb{T}^d)) \le c_2\, M^{-r} (\log M)^{(d-1)(r+1/2)}$$

is valid for all $M \in \mathbb{N}$*.*

4 Proofs

To begin with we discuss several types of test functions which will be used later on. Then we continue with proofs of the statements in Subsections 2.1.2 and 2.2.2. After these preparations we shall proof our main results stated in Subsection 2.2.

4.1 Proof of the Lemmata in Subsection 2.1

4.1.1 Proof of Lemma 1

We consider the linear operators

$$T := \sum_{0\le j_1\le m} \ldots \sum_{0\le j_d\le m} \bigotimes_{k=1}^{d} \Delta^k_{j_k} \qquad \text{and} \qquad R := \sum_{|j|_1\ge m+1} \bigotimes_{k=1}^{d} \Delta^k_{j_k}\,,$$

where we put $\Delta^k_{j_k} := \Delta_{j_k}(L^k)$. Then $A(m,d,\vec{L}) = T - R$. Since $\sum_{j=0}^{m} \Delta^k_j = L^k_m$ we obtain

$$T = \bigotimes_{k=1}^{d} L^k_m\,, \qquad m \in \mathbb{N}_0\,.$$

Obviously, if $\ell \in H(m,d,\lambda)$, i.e. $|\ell_u| \le \lambda\, 2^m$ for all $1 \le u \le d$, then

$$\big(Te^{i\ell\,\cdot}\big)(x) = \prod_{u=1}^{d} (L^u_m e^{i\ell_u \cdot})(x_u) = e^{i\ell x}\,, \qquad x \in \mathbb{T}^d\,,$$

because of (H2). It remains to prove $Re^{i\ell\,\cdot} \equiv 0$. Let $j = (j_1,...,j_d)$ be such that $|j|_1 \ge m+1$. Because of $\ell \in H(m,d,\lambda)$ there exist nonnegative integers u_k, $k = 1,...,d$, satisfying $\sum_{k=1}^{d} u_k = m$ and $|\ell_k| \le 2^{u_k}\lambda$. Thanks to $|j|_1 \ge m+1 > m$ there is at least one component j_k of j with $j_k > u_k$. It follows

$$|\ell_k| \le 2^{u_k}\,\lambda \le 2^{j_k-1}\,\lambda < 2^{j_k}\,\lambda\,.$$

Hence, using again (H2), we find

$$\Delta^k_{j_k} e^{i\ell_k t} = L^k_{j_k} e^{i\ell_k t} - L^k_{j_k-1} e^{i\ell_k t} = e^{i\ell_k t} - e^{i\ell_k t} = 0\,, \qquad t \in \mathbb{T}\,.$$

By definition of R this proves the claim. ■

4.1.2 Proof of Lemma 2

The proof is more or less elementary, we refer to [30] for details.

4.1.3 Proof of Lemma 3

The proof is similar to the proof of Lemma 1. Observe that the nestedness of the grids $\mathcal{T}_j^i$ implies

$$\mathcal{G}(m,d,\vec{L}) = \bigcup_{|j|_1 \le m} \mathcal{T}_{j_1}^1 \times \ldots \times \mathcal{T}_{j_d}^d = \bigcup_{|j|_1 = m} \mathcal{T}_{j_1}^1 \times \ldots \times \mathcal{T}_{j_d}^d .$$

We employ the same notation and decomposition of $A(m,d,\vec{L}) = T - R$ as in proof of Lemma 1. Since L_m^i interpolates on $\mathcal{T}_m^i$ the operator T interpolates on $\mathcal{T}_m^1 \times \ldots \times \mathcal{T}_m^d$. Hence, it is enough to prove

$$Rf(x) = 0 \qquad \text{for all} \quad x \in \mathcal{T}_{k_1}^1 \times \ldots \times \mathcal{T}_{k_d}^d , \qquad |k|_1 = m ,$$

and all $f \in C(\mathbb{T}^d)$. We shall prove even more, namely

$$\Big(\Delta_{j_1}^1 \otimes \ldots \otimes \Delta_{j_d}^d\Big) f(x) = 0 , \qquad x \in \mathcal{T}_{k_1}^1 \times \ldots \times \mathcal{T}_{k_d}^d , \quad |k|_1 = m ,$$

$f \in C(\mathbb{T}^d)$ and $|j|_1 > m$.
Let j, $|j|_1 > m$, k, $|k|_1 = m$ and $x \in \mathcal{G}(m,d,\vec{L})$ be given. For $f \in C(\mathbb{T}^d)$ and $1 \le u \le d$ we put $g_u(t) := f(x_1, \ldots, x_{u-1}, t, x_{u+1}, \ldots, x_d)$, $t \in \mathbb{T}$. Furthermore, there exists at least one component u such that $k_u < j_u$. But this implies $L_{j_u}^u g_u(x_u) = L_{j_u-1}^u g_u(x_u)$ which proves the claim. ■

4.1.4 Proof of Lemma 4

Step 1. We prove (6) for $A(m,d,S)$. Since for $j \in \mathbb{N}_0^d$

$$\big(\Delta_{j_1} \otimes ... \otimes \Delta_{j_d}\big) f(x) = \sum_{k \in \mathcal{P}_j} c_k(f) e^{ikx} \qquad \text{and} \qquad H(m,d,1) = \bigcup_{|j|_1 \le m} \mathcal{P}_j ,$$

Definition 1 leads to the desired representation. For the second identity observe

$$(\Delta_\ell) f(t) = \sum_{k \in \mathbb{Z}} \Big(\psi(2^{-\ell}k) - \psi(2^{-\ell+1}k)\Big)\, c_k(f)\, e^{ikt} , \qquad t \in \mathbb{T} , \quad \ell \in \mathbb{N} .$$

Step 2. Proof of (7). Step 1 yields that both operators map their arguments to trigonometric polynomials with respect to the hyperbolic cross $H(m,d,1)$ and $H(m+d,d,1)$, respectively. Hence (7) is a consequence of the elementary formulas

$$2^d\, S(m,d) \le |H(m,d,1)| \le 3^d\, S(m,d) ,$$

see e.g. [30] for full details. ■

4.2 Proof of Theorems 1 and 6

Before we start to prove these theorems we need a preparation.

4.2.1 Tensor Products of Operators

One of our tools consists in the following estimate for tensor product operators.

Lemma 5 *Let $1 \le p \le \infty$ and $r > 0$. Suppose $P_j \in \mathcal{L}(B^r_{p,p}(\mathbb{T}), L_p(\mathbb{T}))$, $j = 1, \ldots, d$. Then*

$$\| P_1 \otimes \ldots \otimes P_d \, f \,|L_p(\mathbb{T}^d)\| \le \Big(\prod_{j=1}^d \| P_j \,|B^r_{p,p}(\mathbb{T}) \to L_p(\mathbb{T})\| \Big) \, \| f \,|S^r_{p,p}B(\mathbb{T}^d)\|$$

holds for all trigonometric polynomials f.

Proof Let $f = \sum_{k\in\mathbb{Z}^d} c_k(f)\, e^{ikx}$ be a trigonometric polynomial. We define $k = (k_1, k')$, $k_1 \in \mathbb{Z}$, $k' \in \mathbb{Z}^{d-1}$, $x = (x_1, x')$, $x_1 \in \mathbb{T}$, $x' \in \mathbb{T}^{d-1}$, and

$$g_{k_1}(x') := \sum_{k'\in\mathbb{Z}^{d-1}} c_k(f) \Big(\prod_{n=2}^d P_n(e^{ik_n\,\cdot})(x_n) \Big), \qquad x' \in \mathbb{R}^{d-1}, \quad k_1 \in \mathbb{Z}.$$

Then

$$\| (P_1 \otimes \ldots \otimes P_d) f \,|L_p(\mathbb{T}^d)\|^p = \int_{T^{d-1}} \Big\| P_1\Big(\sum_{k_1\in\mathbb{Z}} g_{k_1}(x')\, e^{ik_1\cdot} \Big)(x_1) \Big| L_p(\mathbb{T}, x_1) \Big\|^p dx'$$

$$\le \| P_1 \|^p \int_{T^{d-1}} \Big\| \Big(\sum_{k_1\in\mathbb{Z}} g_{k_1}(x')\, e^{ik_1\cdot} \Big)(x_1) \Big| B^r_{p,p}(\mathbb{T}, x_1) \Big\|^p dx'. \tag{13}$$

Now, let $(\varphi_j)_j \in \Phi$ be an appropriate decomposition of unity, see Subsection 5.2. Then

$$\Big\| \Big(\sum_{k_1\in\mathbb{Z}} g_{k_1}(x')\, e^{ik_1x_1} \Big) \Big| B^r_{p,p}(\mathbb{T}) \Big\|^p = \sum_{j_1=0}^\infty 2^{j_1rp} \Big\| \sum_{k_1\in\mathbb{Z}} \varphi_{j_1}(k_1)\, g_{k_1}(x')\, e^{ik_1x_1} \Big| L_p(\mathbb{T}, x_1) \Big\|^p$$

$$= \sum_{j_1=0}^\infty 2^{j_1rp} \Big\| P_2\Big(\sum_{k\in\mathbb{Z}^d} c_k(f)\, \varphi_{j_1}(k_1)\, e^{ik_1x_1} \Big(\prod_{n=3}^d P_n(e^{ik_n\,\cdot})(x_n) \Big)\, e^{ik_2\cdot} \Big)(x_2) \Big| L_p(\mathbb{T}, x_1) \Big\|^p$$

This identity will be inserted into (13). Then we interchange the order of integration and proceed as above:

$$\begin{aligned}
\| (P_1 \otimes \ldots \otimes P_d) f \,|L_p(\mathbb{T}^d)\|^p &\le \| P_1 \|^p \sum_{j_1=0}^\infty 2^{j_1rp} \int_{\mathbb{T}^{d-1}} \int_0^{2\pi} \\
&\quad \Big| P_2\Big(\sum_{k\in\mathbb{Z}^d} c_k(f)\, \varphi_{j_1}(k_1)\, e^{ik_1x_1} \Big(\prod_{n=3}^d P_n(e^{ik_n\,\cdot})(x_n) \Big)\, e^{ik_2\cdot} \Big)(x_2) \Big|^p dx_2\, dx_1\, dx_3 \ldots dx_d \\
&\le \| P_1 \|^p \ldots \| P_d \|^p \\
&\quad \times \sum_{j\in\mathbb{N}_0^d} 2^{r|j|_1p} \int_{\mathbb{T}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k(f)\, \varphi_{j_1}(k_1)\, e^{ik_1x_1} \ldots \varphi_{j_d}(k_d)\, e^{ik_dx_d} \Big|^p dx_1 \ldots dx_d \\
&= \| P_1 \|^p \ldots \| P_d \|^p \, \| f \,|S^r_{p,p}B(\mathbb{T}^d)\|^p.
\end{aligned}$$

This proves the claim. ■

4.2.2 Proof of Theorems 1 and 6

The proofs of Theorem 1 and Theorem 6 do not differ very much. So we shall prove both theorems simultaneously.

Step 1. The aim of this first step consists in a description of the decomposition we are going to use. First of all recall the decomposition (25) of $f \in S^r_{p,q}B(\mathbb{T}^d)$ into the pieces f_ℓ. Because of $r > 0$ ($r > 1/p$) we have convergence in $L_p(\mathbb{T}^d)$ ($C(\mathbb{T}^d)$), see Lemma 7. At next we need to fix a natural number n_λ such that $2^{-n_\lambda} \leq \lambda$. Now we suppose that m is larger than $d(n_\lambda + 1)$. Further we put $s_m := m - d(n_\lambda + 1) \geq 0$ (we drop the parameter λ in all other notations). Let $I^m_0 := [0, s_m]$ and $I^m_1 := (s_m, \infty)$, respectively. For $b = (b_1, \ldots b_d)$, $b_i \in \{0, 1\}$, $i = 1, \ldots, d$, we define

$$Q^m_b := \{\ell \in \mathbb{N}^d_0 : \; \ell_n \in I^m_{b_n},\, n = 1, \ldots, d,\; |\ell|_1 > s_m\}\,,$$

This leads to the decomposition

$$f(x) = h(x) + \sum_{b \in \{0,1\}^d} f^b(x)\;,$$

where

$$f^b(x) := \sum_{\ell \in Q^m_b} f_\ell(x)\,.$$

The function $h(x)$ is a trigonometric polynomial given by

$$h(x) := \sum_{|\ell|_1 \leq s_m} f_\ell(x)\,.$$

Observe $c_k(h) \neq 0$ implies $c_k(f_\ell) \neq 0$ for some $\ell \in \mathbb{Z}^d$ satisfying $|\ell|_1 \leq s_m$. Hence $|k_n| \leq 2^{\ell_n + 1} \leq 2^{\ell_n + 1 + n_\lambda}\lambda$ for $n = 1, ..., d$. Therefore $k \in H(m, d, \lambda)$ and consequently $A(m, d, \vec{L})\, h = h$ follows, see Lemma 1.

Step 2. Estimation (first part). By means of the invariance of h under the application of $A(m, d, \vec{L})$ we find

$$\|\, f - A(m, d, \vec{L}) f \,|L_p(\mathbb{T}^d)\| \leq \sum_{b \in \{0,1\}^d} \|\, f^b - A(m, d, \vec{L}) f^b \,|L_p(\mathbb{T}^d)\|\,.$$

Obviously, there exists a number $M_\ell \in \mathbb{N}$, such that the trigonometric polynomial f_ℓ has all its harmonics in the hyperbolic cross $H(M_\ell, d, \lambda)$. For $M_\ell > |\ell|_1 + d(1 + n_\lambda)$

Lemma 1 implies

$$
\begin{aligned}
\| f_\ell - A(m,d,\vec{L}) f_\ell \,|L_p(\mathbb{T}^d)\| &= \| A(M_\ell,d,\vec{L}) f_\ell - A(m,d,\vec{L}) f_\ell \,|L_p(\mathbb{T}^d)\| \\
&= \Big\| \sum_{m<|j|_1\le M_\ell} \Big(\bigotimes_{n=1}^{d} \Delta^n_{j_n}\Big) f_\ell \,\Big| L_p(\mathbb{T}^d)\Big\| \\
&= \Big\| \sum_{j\in\Lambda^m_\ell} \Big(\bigotimes_{n=1}^{d} \Delta^n_{j_n}\Big) f_\ell \,\Big| L_p(\mathbb{T}^d)\Big\| ,
\end{aligned}
$$

where

$$
\Lambda^m_\ell := \Big\{ j=(j_1,...,j_d): \quad |j|_1 > m, \quad j_n \le \ell_n + 1 + n_\lambda,\, n=1,...,d \Big\}.
$$

In order to keep the notation simple we used again $\Delta^n_{j_n}$ instead of $\Delta_{j_n}(L^n)$, $n=1,...,d$, $j_n = 0,1,2,....$ The last step here is a consequence of (H2), the definition of the tensor product and the choice of M_ℓ. We continue by using Lemma 5. Let us choose r_0 such that $0 < r_0 < r$ (this condition has to be replaced by $1/p < r_0 < r$ in the case of Theorem 1). Then

$$
\Big\| \Big(\bigotimes_{n=1}^{d} \Delta^n_{j_n}\Big) f_\ell \,\Big| L_p(\mathbb{T}^d)\Big\| \le \| f_\ell \,| S^{r_0}_{p,p}B(\mathbb{T}^d)\| \prod_{n=1}^{d} \| \Delta^n_{j_n} \,| B^{r_0}_{p,p}(\mathbb{T}) \to L_p(\mathbb{T})\| .
$$

Using hypothesis (H3), the triangle inequality and (24) this gives

$$
\Big\| \Big(\bigotimes_{n=1}^{d} \Delta^n_{j_n}\Big) f_\ell \,\Big| L_p(\mathbb{T}^d)\Big\| \le c_1\, 2^{-mr_0}\, \| f_\ell \,| S^{r_0}_{p,p}B(\mathbb{T}^d)\| .
$$

Furthermore, we estimate

$$
\begin{aligned}
\| f_\ell \,| S^{r_0}_{p,p}B(\mathbb{T}^d)\| &\le \Big(\sum_{\substack{|j_k-\ell_k|\le 1\\ k=1,...,d}} 2^{r_0|j|_1 p} \sup_{j\in\mathbb{N}_0^d} \|(2\pi)^{-d/2}\mathcal{F}^{-1}\varphi_j | L_1(\mathbb{R}^d)\|^p \cdot \|f_\ell | L_p(\mathbb{T}^d)\|^p \Big)^{1/p} \\
&\le c_2\, 2^{r_0|\ell|_1}\, \| f_\ell \,| L_p(\mathbb{T}^d)\| .
\end{aligned}
$$

Here we used the standard convolution inequality and the well-known homogeneity property of the Fourier transform. Altogether we obtain

$$
\| f^b - A(m,d,\vec{L}) f^b \,|L_p(\mathbb{T}^d)\| \le c_3 \sum_{\ell\in Q^m_b} 2^{r_0(|\ell|_1-m)}\, |\Lambda^m_\ell|\, \| f_\ell \,|L_p(\mathbb{T}^d)\|
$$

$$
= c_3\, 2^{-r_0 m} \sum_{\ell\in Q^m_b} 2^{(r_0-r)|\ell|_1}\, |\Lambda^m_\ell|\, 2^{r|\ell|_1}\, \| f_\ell \,|L_p(\mathbb{T}^d)\| . \tag{14}
$$

Of course, $|\Lambda^m_\ell|$ denotes the cardinality of the set Λ^m_ℓ. We need to estimate this quantity. Observe

$$
\Lambda^m_\ell \subset \Big[m - \sum_{\substack{n=1\\ n\neq 1}}^{d} (\ell_n + 1 + n_\lambda), \ell_1 + 1 + n_\lambda \Big] \times \cdots \times \Big[m - \sum_{\substack{n=1\\ n\neq d}}^{d} (\ell_n + 1 + n_\lambda), \ell_d + 1 + n_\lambda \Big] .
$$

This implies

$$
|\Lambda_\ell^m| \le \min\Big((|\ell|_1 + d(1+n_\lambda) + 1 - m)^d \ , \ \prod_{n=1}^d (\ell_n + 2 + n_\lambda)\Big) . \tag{15}
$$

Step 3. Estimation (second part). Depending on the size of $|b|_1$ we continue.

Step 3.1. Let $|b|_1 \le 1$. Without loss of generality we may assume that $b_1 = |b|_1$. For given q let q' be such that $(1/q) + (1/q') = 1$. Then we find

$$
\begin{aligned}
& 2^{-mr_0}\Big(\sum_{\ell \in Q_b^m} 2^{q'(r_0-r)|\ell|_1} \, |\Lambda_\ell^m|^{q'}\Big)^{1/q'} \\
& \quad \le \; 2^{-mr_0}\Big(\sum_{\ell \in Q_b^m} 2^{q'(r_0-r)|\bar{\ell}|_1} \, (|\ell|_1 + d(n_\lambda+1) + 1 - m)^{dq'}\Big)^{1/q'} \\
& \quad \le \; 2^{-mr_0}\Big(\sum_{\ell_2,\ldots,\ell_d=0}^{s_m} \sum_{u=0}^{\infty} 2^{q'(r_0-r)(u+m-d(n_\lambda+1)-1)} u^{dq'}\Big)^{1/q'} \\
& \quad \le \; 2^{-mr}\Big(m^{d-1} \sum_{u=0}^{\infty} 2^{q'(r_0-r)(u-d(n_\lambda+1)-1)} u^{dq'}\Big)^{1/q'} \\
& \quad \le \; c_4 \, 2^{-mr} \, m^{(d-1)(1-1/q)} \, . \qquad (16)
\end{aligned}
$$

In this case Hölder's inequality and (16) lead to

$$
\begin{aligned}
& \| \, f^b - A(m,d,\vec{L}) f^b) \, | L_p(\mathbb{T}^d)\| \\
& \le \; c_5 \, 2^{-mr_0}\Big(\sum_{\ell \in Q_b^m} 2^{q'(r_0-r)|\ell|_1} \, |\Lambda_\ell^m|^{q'}\Big)^{1/q'} \Big(\sum_{\ell \in Q_b^m} 2^{r|\ell|_1 q} \, \| \, f_\ell \, | L_p(\mathbb{T}^d)\|^q\Big)^{1/q} \\
& \le \; c_6 \, 2^{-mr} \, m^{(d-1)(1-1/q)} \, \| \, f \, | S^r_{p,q}B(\mathbb{T}^d)\| \, . \qquad (17)
\end{aligned}
$$

Step 3.2. Let $|b|_1 \ge 2$. In this case the estimate becomes easier. We use

$$
2^{-mr_0} \sum_{\ell \in Q_b^m} 2^{(r_0-r)|\ell|_1} \, |\Lambda_\ell^m| \le 2^{-mr_0} \prod_{n=1}^d \Big(\sum_{\ell_n \in I_{b_n}^m} 2^{(r_0-r)\ell_n} (\ell_n + 2 + n_\lambda)\Big) ,
$$

see (15), as well as

$$
\begin{aligned}
& \sum_{\ell_n = s_m+1}^{\infty} 2^{(r_0-r)\ell_n} (\ell_n + 2 + n_\lambda) \\
& \quad = \; 2^{m(r_0-r)} 2^{(r-r_0)(d(n_\lambda+1)-1)} \sum_{u=0}^{\infty} 2^{(r_0-r)u} \, (u + m - d(n_\lambda+1) + 3) \\
& \quad \le \; 2^{m(r_0-r)} \, 2^{(r-r_0)(d(n_\lambda+1)-1)} \sum_{u=0}^{\infty} 2^{(r_0-r)u} \, (u + m + 3) \\
& \quad \le \; c_7 \, 2^{m(r_0-r)} \, m
\end{aligned}
$$

and

$$\sum_{\ell_n=0}^{s_m} 2^{(r_0-r)\ell_n}(\ell_n+2+n_\lambda) \le m \sum_{u=0}^{\infty} 2^{(r_0-r)u} \le c_8\, m.$$

Altogether this leads to

$$\begin{aligned}
&\| f_b - A(m,d,\vec{L}) f_b \,|L_p(\mathbb{T}^d)\| \le c_9 \,\| f\,|S^r_{p,\infty}B(\mathbb{T}^d)\|\, 2^{-mr_0} \sum_{\ell\in Q_b^m} 2^{(r_0-r)|\ell|_1} |\Lambda_\ell^m| \\
&\le \; c_{10}\, \| f\,|S^r_{p,\infty}B(\mathbb{T}^d)\|\, 2^{-mr} m^d 2^{m(r_0-r)(|b|_1-1)} \\
&\le \; c_{11}\, 2^{-mr}\, \| f\,|S^r_{p,\infty}B(\mathbb{T}^d)\|\,, \qquad (18)
\end{aligned}$$

see (14). It remains to sum up over $|b|_1 \le 1$ in (17) and over $2 \le |b|_1 \le d$ in (18), respectively. This completes the proof of Theorem 1 and Theorem 6. ■

4.3 Proof of Theorem 2

Of course, it will be enough to prove the estimate from below. Similar to Temlyakov in [32] we shall use lacunary series (polynomials) as test functions.

Step 1. Preliminaries. Let

$$Lf(t) := \sum_{-N/2\le \ell < N/2} f(t_\ell)\, \Lambda(t-t_\ell)\,, \qquad t_\ell \in J_N\,.$$

If f belongs to the Wiener algebra $\mathcal{A}(\mathbb{T})$, then the Fourier coefficients of Lf are given by the formula

$$c_k(Lf) = N\, c_k(\Lambda) \sum_{\ell\in\mathbb{Z}} c_{k+\ell N}(f)\,, \qquad k \in \mathbb{Z}\,.$$

In particular, if $f(t) = e^{imt}$ one obtains

$$c_0(L(e^{im\cdot})) = N\, c_0(\Lambda) \cdot \begin{cases} 1 & \text{if } \frac{m}{N} \in \mathbb{Z}\,, \\ 0 & \text{otherwise}\,. \end{cases} \qquad (19)$$

The admissible operators L reproduce the function $f(t) = 1$. Hence

$$1 = c_0(Lf) = N\, c_0(\Lambda)\,.$$

Step 2. Test functions. For $m \ge d^2$ we put

$$f_m(x_1,...,x_d) := \sum_{\substack{u_k\ge d\\ |u|_1=m}} e^{iN_{u_1}x_1+...+iN_{u_d}x_d}\,, \qquad (20)$$

where N_j, $j \in \mathbb{N}_0$, is the given sequence of natural numbers appearing in hypothesis (H4). By choosing an appropriate norm in the Besov space it is not difficult to see that

$$\| f_m \,| S^r_{p,q}B(\mathbb{T}^d)\| \asymp 2^{rm}\, m^{(d-1)/q}\,, \tag{21}$$

cf. [30, Lemma 15].

Step 3. Calculation of $c_{(0,\ldots,0)}(A(m,d,\vec{L})f_m)$. Let us first study the number $c_0(\Delta^k_{j_k}(e^{iN_{u_k}\cdot}))$. Putting

$$d_M(N) = \begin{cases} 1 & \text{if } \frac{N}{M} \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}\,,$$

we derive from (19)

$$c_0(\Delta^k_{j_k}(e^{iN_{u_k}\cdot})) = \begin{cases} d_{N_{j_k}}(N_{u_k}) - d_{N_{j_k-1}}(N_{u_k}) & \text{if} \quad j_k \geq 1 \\ d_{N_0}(N_{u_k}) & \text{if} \quad j_k = 0 \end{cases}\,.$$

Now we employ our assumption $N_{j+1}/N_j \in \mathbb{N}$, $j = 0, 1, \ldots$, and obtain

$$c_0(\Delta^k_{j_k}(e^{iN_{u_k}\cdot})) = \begin{cases} -1 & \text{if} \quad j_k = u_k + 1 \\ 1 & \text{if} \quad j_k = 0 \\ 0 & \text{otherwise} \end{cases}\,. \tag{22}$$

This yields

$$\begin{aligned} c_0\big(A(m,d,\vec{L})\, f_m\big) &= \sum_{\substack{u_k \geq d \\ |u|_1 = m}} c_0\big[A(m,d,\vec{L})\big(e^{iN_{u_1}\cdot + \ldots + iN_{u_d}\cdot}\big)\big] \\ &= \sum_{\substack{u_k \geq d \\ |u|_1 = m}} \sum_{j \in T_u} c_0\Big[\Big(\bigotimes_{k=1}^{d} \Delta^k_{j_k}\Big)\big(e^{iN_{u_k}\cdot + \ldots + iN_{u_d}}\big)\Big] \\ &= \sum_{\substack{u_k \geq d \\ |u|_1 = m}} \sum_{j \in T_u} c_0[\Delta^1_{j_1}(e^{iN_{u_1}\cdot})] \cdot \ldots \cdot c_0[\Delta^d_{j_d}(e^{iN_{u_d}\cdot})]\,, \end{aligned} \tag{23}$$

where

$$T_u \;=\; \Big\{(j_1, ..., j_d) \in \mathbb{N}_0^d: \quad |j|_1 \leq m \quad \text{and either} \quad j_k = u_k + 1 \quad \text{or} \quad j_k = 0, \quad k = 1, ..., d\Big\}\,.$$

Clearly, T_u does not contain $(u_1+1, \ldots, u_d+1)$ because of $|u|_1 = m$. Let us decompose the index set T_u into the disjoint subsets $T_u = \bigcup_{\ell=1}^{d} T_u^\ell$, where

$$T_u^\ell = \{(j_1, ..., j_d) \in T_u: \quad \text{exactly } \ell \text{ components of } j \text{ vanish}\}\,, \qquad \ell = 1, ..., d.$$

The set T_u^ℓ contains exactly $\binom{d}{\ell}$ elements for every u and because of (22) we have

$$c_0[\Delta^1_{j_1}(e^{iN_{u_1}\cdot})] \cdot \ldots \cdot c_0[\Delta^d_{j_d}(e^{iN_{u_d}\cdot})] = (-1)^{d-\ell}\,, \qquad j \in T_u^\ell\,.$$

This together with (23) yields

$$\begin{aligned} c_0\big(A(m,d,\vec{L})\, f_m\big) &= \sum_{\substack{u_k\ge d\\ |u|_1=m}} \sum_{\ell=1}^{d} \sum_{j\in T_u^\ell} (-1)^{d-\ell} \\ &= \sum_{\substack{u_k\ge d\\ |u|_1=m}} \sum_{\ell=1}^{d} (-1)^{d-\ell}\binom{d}{\ell} \\ &= \sum_{\substack{u_k\ge d\\ |u|_1=m}} \sum_{\ell=0}^{d-1} (-1)^{\ell}\binom{d}{\ell}\,. \end{aligned}$$

Because of

$$\begin{aligned} \sum_{\ell=0}^{d-1}(-1)^\ell\binom{d}{\ell} &= \binom{d-1}{0} + \sum_{\ell=1}^{d-1}(-1)^\ell\bigg(\binom{d-1}{\ell-1}+\binom{d-1}{\ell}\bigg) \\ &= (-1)^{d-1}\binom{d-1}{d-1} = (-1)^{d-1} \end{aligned}$$

we conclude in view of Remark 4

$$\big|\, c_0\big(f_m - A(m,d,\vec{L})f_m\big)\big| = \big|\, c_0\big(A(m,d,\vec{L})\, f_m\big)\big| = \Big|\sum_{\substack{u_k\ge d\\ |u|_1=m}} (-1)^{d-1}\Big| \asymp m^{d-1}\,.$$

Since we know the behaviour of $\|\, f_m\,|S^r_{p,q}B(\mathbb{T}^d)\|$, see (21), we finally get

$$\begin{aligned} \|\, I - A(m,d,\vec{L})\,|S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)\| &\ge \frac{\|\, f_m - A(m,d,\vec{L})f_m\,|L_p(\mathbb{T}^d)\|}{\|\, f_m\,|S^r_{p,q}B(\mathbb{T}^d)\|} \\ &\ge c\, 2^{-rm}\, m^{(d-1)(1-1/q)} \end{aligned}$$

with some positive constant c independent of $m \in \mathbb{N}$. ■

4.4 Proof of Theorem 3

Again only the estimate from below is of interest. Associated to $L = (L^1, \ldots, L^d)$ is the sequence of grids $\mathcal{G}(m,d,\vec{L})$, $m \in \mathbb{N}_0$, see (3). For simplicity we concentrate on the first component for a moment. Because of (H4) we find

$$\Big|\bigcup_{j=0}^{m} \mathcal{T}_j^1\Big| \le C_2\, 2^{m+1}\,.$$

Consequently, for every $m \in \mathbb{N}_0$ there exists an open interval $\mathcal{I}_m \subset [-\pi, \pi]$, $|\mathcal{I}_m| = \frac{1}{C_2} 2^{-(m+1)}$, such that

$$\mathcal{I}_m \cap \Big(\bigcup_{j=0}^{m} \mathcal{T}_j^1 \Big) = \emptyset \,.$$

Therefore we can find a rectangle $R_m := \mathcal{I}_m \times [-\pi, \pi] \times ... \times [-\pi, \pi]$ satisfying

$$R_m \cap \mathcal{G}(m, d, \vec{L}) = \emptyset \,.$$

The test functions we are going to use are bump functions. Let $\widetilde{B}$ be a $C_0^\infty(\mathbb{R}^d)$ function such that $\operatorname{supp} \widetilde{B} \subset \{x \in \mathbb{R}^d : |x| \leq 1\}$. Its 2π-periodic extension is denoted by B. Obviously, $B \in S^r_{p,q}B(\mathbb{T}^d)$ for all $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $r \geq 0$. Furthermore, if $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_i > 0$, $i = 1, \ldots, d$, is given then $B(\lambda\,\cdot)$ denotes the 2π-periodic extension of $\widetilde{B}(\lambda\cdot)$. Then we have the following, see [30] for details.

Lemma 6 *Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\lambda = (\lambda_1, ..., \lambda_d)$. Let $r > 1/p$. Then there exists a positive constant c such that*

$$\| \, B(\lambda\cdot) \, | S^r_{p,q}B(\mathbb{T}^d)\| \leq c \, \lambda_1^{r-1/p} \cdot \, \ldots \, \cdot \lambda_d^{r-1/p} \, \| \, B \, | S^r_{p,q}B(\mathbb{T}^d)\|$$

holds for all λ, $1 \leq \lambda_i < \infty$, $i = 1, ..., d$.

We choose $\lambda_1 = C_2 \, 2^{m+1}$ and $\lambda_2 = ... = \lambda_d = 1$. If x^m denotes the centre of R_m the function $B(\lambda(\cdot\, - x^m))$ vanishes in $\mathcal{G}(m, d, \vec{L})$. Lemma 6 then implies

$$\frac{\| \, B(\lambda(\cdot \, - x^m)) - A(m, d, \vec{L})(B(\lambda(\cdot \, - x^m))) \, | L_p(\mathbb{T}^d)\|}{\| \, B(\lambda(\cdot \, - x^m)) \, | S^r_{p,1}B(\mathbb{T}^d)\|} \geq c\lambda_1^{-r} \,,$$

where the corresponding constants do not depend on m. ■

4.5 Proof of Corollary 4

The estimate from above follows from Corollary 3. For the estimate from below we shall use some well-known results about Kolmogorov numbers of those embedding operators. Recall, for a Banach space $F \hookrightarrow L_p(\mathbb{T}^d)$ we put

$$d_M(F, L_p(\mathbb{T}^d)) := \inf_{\{u_i\}_{i=1}^M \subset L_p(\mathbb{T}^d)} \; \sup_{\|f|F\| \leq 1} \; \inf_{c_1, \cdots, c_M} \Big\| \, f - \sum_{i=1}^{M} c_i \, u_i \Big| L_p(\mathbb{T}^d) \Big\| \,.$$

Hence $d_M \leq \rho_M$. In case of $F = S^r_{p,\infty}B(\mathbb{T}^d)$ one has the convenient references [36, 11.4.11] and [34, Thm. 3.4.5], but with some additional restrictions what concerns r

and p. For the general case we refer to [14]. Galeev considered a bit different spaces. However, by some standard arguments his estimates carry over to our situation, see e.g. [34, Introduction to Chapt. 3]. For $1 < p < \infty$, $1 \le q \le \infty$ and $r > 0$ this leads to

$$d_M(I, S^r_{p,q}B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \asymp M^{-r}(\log M)^{(d-1)r}\eta(M,d,p,q)\,,$$

where $\eta(M,d,p,q)$ is defined in (12). ■

Remark 16 *For the estimate from below one can also use entropy and approximation numbers. For a definition of these quantities we refer, e.g., to* [12], [36] *or* [39]. *Let $e_n(I,X,Y)$ denote the n-th dyadic entropy number of the embedding operator I which maps the Banach space X into the Banach space Y and let $\lambda_n(I,X,Y)$ denote the n-th approximation number (linear width, see Remark 12) of this embedding. Then trivially $\lambda_M \le \rho_M$ and furthermore $e_n \le c\,\lambda_n$ under certain weak conditions on X and Y which are satisfied in our context, see Theorem 1.3.3 in* [12]. *So, entropy numbers can be used as well for deriving lower bounds of ρ_M. The estimates*

$$e_M(I, S^r_{p,q}B(\mathbb{T}^d), L_p(\mathbb{T}^d)) \ge c \begin{cases} M^{-r}(\log M)^{(d-1)(r+\frac{1}{2}-\frac{1}{q})_+} & \text{if} \quad 1 < p < \infty\,, \\ M^{-r}(\log M)^{(d-1)(r-1/q)_+} & \text{if} \quad p = 1, \infty\,, \end{cases}$$

with some positive constant c (independent of M) are known, at least in a situation very close to ours. For (non-periodic) function spaces on domains it has been proved in [39, Thm. 4.11]. *This can be transferred to the periodic situation. In our case it is enough to construct a bounded linear extension operator from $S^r_{p,q}B((-1,1)^d)$ to $S^r_{p,q}B(\mathbb{T}^d)$ and to apply the multiplicativity of the entropy numbers, see* [12, 1.3.1]. *We omit details and refer to* [6] *where a similar situation is investigated. Under additional restrictions on p and q entropy numbers of the embeddings $I:\ S^r_{p,q}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d))$ are studied in* [4], [11] *and* [33].

4.6 Proof of Theorems 7 and 8

In both cases the estimate from above follows from $S^r_pW(\mathbb{T}^d) \hookrightarrow S^r_{p,2}B(\mathbb{T}^d)$, see (26), in combination with Theorems 1 and 4.

For the estimate from below in Theorem 8 we refer to [30]. To prove the estimate from below in Theorem 7 we employ the same strategy as in the proof of Theorem 2, see Subsection 4.3. The test functions, defined in (20), satisfy

$$\|\, f_m\,|S^r_pW(\mathbb{T}^d)\| \asymp 2^{rm}\, m^{(d-1)/2}\,.$$

see Lemma 10. ■

4.7 Proof of Corollary 5

The estimate from above follows from Corollary 4. For the estimate from below we argue as there. It is enough to use

$$e_M(I:\, S^r_pW(\mathbb{T}^d) \to L_p(\mathbb{T}^d)) \geq c\, M^{-r}(\log M)^{(d-1)r}$$

with some positive constant c (independent of M), see [4] and [39].

5 Appendix - Function Spaces

Let $D(\mathbb{T}^d)$ denote the collection of all infinitely differentiable, complex-valued and in each component 2π-periodic functions equipped with the topology generated by

$$\|\, f\,\|_\alpha := \sup_{x\in\mathbb{T}^d} |D^\alpha f(x)|\, , \qquad \alpha \in \mathbb{N}_0^d\, .$$

The elements of the topological dual $D'(\mathbb{T}^d)$ (equipped with the weak topology) are called periodic distributions. All function spaces considered here in this paper will be continuously embedded into $D'(\mathbb{T}^d)$.

5.1 Nikol'skij-Besov Spaces on the Torus

Let $1 \leq p \leq \infty$ and $r > 0$. Let M be a natural number such that $M - 1 \leq r < M$. A function $f \in L_p(\mathbb{T})$ belongs to the Nikol'skij-Besov space $B^r_{p,\infty}(\mathbb{T})$ if

$$\|\, f\,|B^r_{p,\infty}(\mathbb{T})\| := \|\, f\,|L_p(\mathbb{T})\| + \sup_{t>0}\, t^{-r}\, \omega^M(f,t)_p\, \leq \infty\, .$$

Here

$$\omega^M(f,t)_p := \sup_{|h|<t} \Big\|\, \sum_{j=0}^{M} \binom{M}{j} (-1)^j\, f(x+(M-j)h)\, \Big| L_p(\mathbb{T})\Big\|\, .$$

Our general references for these spaces are [21, 25]. Of certain interest is the following property: we have

$$B^r_{p,\infty}(\mathbb{T}) \hookrightarrow C(\mathbb{T}) \qquad \Longleftrightarrow \qquad r > 1/p\, ,$$

cf. [21, 6.3, 6.10.1] or [25, Rem. 3.5.5/3].

5.2 Besov Spaces on the Torus

For us it is convenient to introduce Besov spaces by making use of a smooth dyadic decomposition of unity. Let $C_0^\infty(\mathbb{R})$ denote the set of all compactly supported, complex-valued and infinite differentiable functions on the real line and Φ the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R})$ satisfying

(i) supp $\varphi_0 \subset \{x : |x| \le 2\}$,

(ii) supp $\varphi_j \subset \{x : 2^{j-1} \le |x| \le 2^{j+1}\}$, $j = 1, 2, ...,$

(iii) $\forall\, \ell \in \mathbb{N}_0$ we have $\sup\limits_{x,j} 2^{j\ell}\, |\varphi_j^{(\ell)}(x)| \le c_\ell < \infty$,

(iv) $\sum\limits_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}$.

Let $1 \le p \le \infty$ and $r > 0$. Then $f \in L_p(\mathbb{T})$ belongs to $B^r_{p,p}(\mathbb{T})$ if

$$\| f \,|B^r_{p,p}(\mathbb{T})\| := \Big(\sum_{j=0}^{\infty} 2^{jrp}\, \| \sum_{k\in\mathbb{Z}} \varphi_j(k)\, c_k(f)\, e^{ikt}\, |L_p(\mathbb{T})\|^p \Big)^{1/p} < \infty \, .$$

Different elements of Φ lead to equivalent norms. For this and other properties, for instance

$$B^r_{p,p}(\mathbb{T}) \hookrightarrow B^r_{p,\infty}(\mathbb{T}) \, , \tag{24}$$

we refer e.g. to [21] and [25, Chapt. 3].

5.3 Sobolev Spaces of Dominating Mixed Smoothness

If r is a natural number and $1 \le p \le \infty$, then the Sobolev space $S^r_pW(\mathbb{T}^d)$ of dominating mixed smoothness of order r is defined as the collection of all $f \in L_p(\mathbb{T}^d)$ such that

$$D^\alpha f \in L_p(\mathbb{T}^2)\, , \qquad \alpha = (\alpha_1, \ldots , \alpha_d)\, , \quad 0 \le \alpha_\ell \le r\, , \quad \ell = 1, \ldots , d\, .$$

Derivatives have to be understood in the weak sense. For general $r > 0$ and $1 < p < \infty$ one may use

$$\sum_{k\in\mathbb{Z}^d} c_k(f)(1 + |k_1|^2)^{r/2} \, \ldots \, (1 + |k_d|^2)^{r/2}\, e^{ikx} \in L_p(\mathbb{T}^d)\, .$$

In case $r \in \mathbb{N}$ this leads to an equivalent characterisation. For $r \in \mathbb{N}$ we endow these classes with the norm

$$\| f \,|S^r_pW(\mathbb{T}^d)\| := \sum_{\substack{\alpha_i \le r \\ i=1,\ldots,d}} \| D^\alpha f \,|L_p(\mathbb{T}^d)\| \, .$$

For $r > 0$, $r \notin \mathbb{N}$, and $1 < p < \infty$ we shall use

$$\| f \,|S^r_pW(\mathbb{T}^d)\| := \Big\| \sum_{k\in\mathbb{Z}^d} c_k(f)(1 + |k_1|^2)^{r/2} \, \ldots \, (1 + |k_d|^2)^{r/2}\, e^{ikx} \Big| L_p(\mathbb{T}^d) \Big\| \, .$$

Sometimes we use the symbol $S^0_pW(\mathbb{T}^d)$ instead of $L_p(\mathbb{T}^d)$.

5.4 Besov Spaces of Dominating Mixed Smoothness

These smooth dyadic decompositions of unity on $\mathbb{R}$ can be used to construct decompositions on $\mathbb{R}^d$ by means of tensor products. Let $\ell = (\ell_1, \dots \ell_d) \in \mathbb{N}_0^d$. Then we put

$$\varphi_\ell(x) := \varphi_{\ell_1}(x_1) \cdot \ \dots \ \cdot \varphi_{\ell_d}(x_d) \,.$$

Hence

$$\sum_{\ell \in \mathbb{N}_0^d} \varphi_\ell(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d \,.$$

As an abbreviation we shall use

$$f_\ell(x) := \sum_{k \in \mathbb{Z}^d} c_k(f)\, \varphi_\ell(k)\, e^{ikx} \,, \qquad x \in \mathbb{T}^d, \quad \ell \in \mathbb{N}_0^d \,, \tag{25}$$

which results in

$$f = \sum_{\ell \in \mathbb{N}_0^d} f_\ell \,,$$

at least in the sense of periodic distributions.
Let $\varphi \in \Phi$, $1 \le p \le \infty$, $1 \le q \le \infty$, and $r > 0$. Then the Besov space $S^r_{p,q}B(\mathbb{T}^d)$ of dominating mixed smoothness is the collection of all functions $f \in L_p(\mathbb{T}^d)$ such that

$$\| \, f \, | S^r_{p,q}B(\mathbb{T}^d) \| := \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{r|\ell|_1 q} \, \| \, f_\ell \, | L_p(\mathbb{T}^d) \|^q \Big)^{1/q} < \infty \,.$$

These classes are Banach spaces independent of the chosen system Φ (in the sense of equivalent norms), cf. [25, Chapt. 2,3] for $d = 2$. Below we shall recall a few facts about these classes. Let $1 < p < \infty$. Then

$$S^r_{p,\min(p,2)}B(\mathbb{T}^d) \hookrightarrow S^r_p W(\mathbb{T}^d) \hookrightarrow S^r_{p,\max(p,2)}B(\mathbb{T}^d) \,, \tag{26}$$

see [34, pp. 20/21] or [25, 2.2.3].

Lemma 7 *Let $1 \le p \le \infty$ and $1 \le q \le \infty$. Let $r > 1/p$. Then*

$$S^r_{p,q}B(\mathbb{T}^d) \hookrightarrow S^{1/p}_{p,1}B(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$$

holds. Furthermore, if $f \in S^{1/p}_{p,1}B(\mathbb{T}^d)$ then

$$\lim_{m \to \infty} \Big\| \, f - \sum_{|\ell|_1 \le m} f_\ell \, \Big| C(\mathbb{T}^d) \Big\| = 0 \,.$$

Proof *Step 1.* The embedding $S^r_{p,q}B(\mathbb{T}^d) \hookrightarrow S^{1/p}_{p,1}B(\mathbb{T}^d)$ is obvious. Using the Nikol'skij inequality, cf. e.g. [25, 3.3.2], we find

$$
\begin{aligned}
\| f \,|L_\infty(\mathbb{T}^d)\| &= \Big\| \sum_{\ell\in\mathbb{N}_0^d} f_\ell \Big| L_\infty(\mathbb{T}^d)\Big\| \\
&\le \sum_{\ell\in\mathbb{N}_0^d} \| f_\ell \,|L_\infty(\mathbb{T}^d)\| \\
&\le c \sum_{\ell\in\mathbb{N}_0^d} 2^{|\ell|_1/p} \| f_\ell \,|L_p(\mathbb{T}^d)\| \\
&= c\, \| f \,|S^{1/p}_{p,1}B(\mathbb{T}^d\| \,. \qquad (27)
\end{aligned}
$$

This proves the boundedness of the elements in $S^{1/p}_{p,1}B(\mathbb{T}^d)$. The continuity follows from the continuity of the pieces f_ℓ, the convergence of $\sum_{|\ell|_1\le m} f_\ell$ in $C(\mathbb{T}^d)$ (employing the same arguments as in (27)) and

$$\lim_{m\to\infty} \sum_{|\ell|_1\le m} f_\ell = f \qquad (\text{convergence in} \quad D'(\mathbb{T}^d))\,.$$

■

Remark 17 *A proof of the nonperiodic counterpart to* $S^{1/p}_{p,1}B(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$ *can be found in* [25, 2.4.1].

5.4.1 Fourier Multipliers

First of all we need some spaces of functions on $\mathbb{R}^d$. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform and its inverse on $L_2(\mathbb{R}^d)$, respectively. Let $\kappa \ge 0$. Then a function $f \in L_2(\mathbb{R}^d)$ belongs to $S^\kappa_2 H(\mathbb{R}^d)$ if

$$\| f \,|S^\kappa_2 H(\mathbb{R}^d)\| := \Big(\int_{\mathbb{R}^d} (1+|\xi_1|^2)^\kappa \,\ldots\, (1+|\xi_d|^2)^\kappa \,|\mathcal{F}(\xi)|^2\, d\xi\Big)^{1/2} < \infty\,.$$

Let $(b^j)_j$ be a sequence in $(0,\infty)^d$ and let $\Lambda = (\Lambda_j)_j$ be a sequence of subsets of $\mathbb{Z}^d$ s.t.

$$\Lambda_j \subset \{\ell\in\mathbb{Z}^d : \quad |\ell_i| \le b^j_i\,, \quad i=1,\ldots,d\}\,, \qquad j\in\mathbb{N}_0^d\,.$$

We say that a sequence $(g_j)_j$ of trigonometric polynomials belongs to $L^\Lambda_p(\mathbb{T}^d,\ell_q)$ if

$$\Big\| (g_j)_j \,|L_p(\mathbb{T}^d,\ell_q)\Big\| := \Big(\int_{\mathbb{T}^d} \Big(\sum_{j\in\mathbb{N}_0^d} |g_j(x)|^q\Big)^{p/q} dx\Big)^{1/p} < \infty$$

and

$$c_k(g_j) = 0 \qquad \text{for all} \quad k\notin\Lambda_j\,, \quad j\in\mathbb{N}_0^d\,.$$

Lemma 8 *Let $1 \le p < \infty$, $1 \le q \le \infty$ and let*

$$\kappa > \frac{1}{\min(p,q)} + \frac{1}{2}$$

If $(M_j)_j$ is a sequence in $S_2^\kappa H(\mathbb{R}^d)$, then there exists a constant c such that

$$\Big\| \sum_{k\in\mathbb{Z}^d} M_j(k)\, c_k(g_j)\, e^{ikx} \Big| L_p(\mathbb{T}^d,\ell_q)\Big\| \le c \sup_{j\in\mathbb{N}_0^d} \|M_j(b^j\,\cdot\,)\,|S_2^\kappa H(\mathbb{R}^d)\|\, \|\, g_j\,|L_p(\mathbb{T}^d,\ell_q)\|$$

holds for all $(g_j)_j \in L_p^\Lambda(\mathbb{T}^d,\ell_q)$. Here c neither depends on $(g_j)_j$ nor on $(M_j)_j$.

Remark 18 *A nonperiodic counterpart to Lemma 8 is proved in* [25, 1.8.3]*. The proof in the periodic situation is similar. Details can be found in [38, 1.3.4].*

5.4.2 Further Littlewood-Paley Characterizations

Similar to isotropic Sobolev spaces also the classes $S_p^rW(\mathbb{T}^d)$ allow a Littlewood-Paley characterisation.

Lemma 9 *Let $1 < p < \infty$ and $r \ge 0$. Let f_ℓ be as in (25). Then*

$$\|\, f\,|S_p^rW(\mathbb{T}^d)\| \asymp \Big\| \Big(\sum_{\ell\in\mathbb{N}_0^d} 2^{2r|\ell|_1}|f_\ell(x)|^2\Big)^{1/2}\Big| L_p(\mathbb{T}^d)\Big\|$$

holds for all $f \in L_p(\mathbb{T}^d)$.

Remark 19 *For $r = 0$ this can be found in Nikol'skij* [21, 1.5.2/(13)]*. For $r > 0$ one has to use a lifting property, we refer to* [25, 2.2.6] *for the nonperiodic counterpart.*

Now we turn to the Lizorkin representation of Besov as well as Sobolev spaces. We need a special covering of $\mathbb{R}^d$. Let

$$\begin{aligned} P_0 &:= [-1,1]\,, \qquad P_j := \{x\in\mathbb{R}^d:\ 2^{j-1} < |x| \le 2^j\}\,, \quad j\in\mathbb{N}\,,\\ \mathcal{P}_j &:= P_{j_1}\times\ldots\times P_{j_d}\,, \qquad j\in\mathbb{N}_0^d\,. \end{aligned}$$

Then

$$\mathbb{R}^d = \bigcup_{j\in\mathbb{N}_0^d} \mathcal{P}_j \qquad \text{and} \qquad \mathcal{P}_j\cap\mathcal{P}_\ell = \emptyset \quad \text{if} \quad j\ne\ell\,.$$

Hence, with

$$\widetilde{f}_\ell(x) := \sum_{k\in\mathcal{P}_\ell} c_k(f)\, e^{ikx}\,, \qquad x\in\mathbb{T}^d,\ \ell\in\mathbb{N}_0^d\,,$$

we find

$$f = \sum_{\ell\in\mathbb{N}_0^d} \widetilde{f}_\ell$$

(convergence in the sense of periodic distributions), compare with (25).

Lemma 10 *Let $1 < p < \infty$, $1 \le q \le \infty$. Then, if $r \ge 0$*

$$\| f \,|S^r_p W(\mathbb{T}^d)\| \asymp \Big\| \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{2r|\ell|_1} \, |\widetilde{f}_\ell(x)|^2 \Big)^{1/2} \Big| L_p(\mathbb{T}^d) \Big\| ,$$

and if $r > 0$

$$\| f \,|S^r_{p,q} B(\mathbb{T}^d)\| \asymp \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{r|\ell|_1 q} \, \| \widetilde{f}_\ell \,| L_p(\mathbb{T}^d)\|^q \Big)^{1/q}$$

holds for all $f \in L_p(\mathbb{T}^d)$.

Remark 20 *The Lemma can be proved by making use of a periodic version of a vector-valued Fourier multiplier theorem of Lizorkin (see* [38, 1.3.5] *and* [25, Thm. 3.4.3/3] *for the non-periodic case), in combination with Lemma 8 and Lemma 9.*

5.4.3 Smooth de la Vallée-Poussin Means

Finally we consider a special decomposition of unity. Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function such that $\psi(t) = 1$ if $|t| \le 1$ and $\operatorname{supp} \psi \subset [-3/2, 3/2]$. Furthermore we assume that ψ is nonincreasing on $[0, \infty)$. Then we put

$$\begin{aligned} \vartheta_0(t) &:= \psi(t)\,, \\ \vartheta_j(t) &:= \psi(2^{-j}t) - \psi(2^{-j+1}t)\,, \qquad j \in \mathbb{N}\,, \\ \Theta_j(x) &:= \vartheta_{j_1}(x_1) \cdot \ldots \cdot \vartheta_{j_d}(x_d)\,, \qquad x \in \mathbb{T}^d, \quad j \in \mathbb{N}_0^d\,. \end{aligned}$$

Clearly, $(\vartheta_j)_j \in \Phi$. It holds

$$\vartheta_j(t) = 1 \qquad \text{if} \qquad \frac{3}{2}\, 2^{j-1} \le |t| \le 2^j$$

as well as

$$\sum_{j \in \mathbb{N}_0^d} \Theta_j(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d\,.$$

In addition we have $\Theta_j \ge 0$ and

$$\sum_{k \in \mathbb{Z}^d} \Theta_j(k) \asymp |\mathcal{P}_j|\,, \qquad j \in \mathbb{N}_0^d\,.$$

For $f \in L_1(\mathbb{T}^d)$ we introduce the decomposition

$$f_\ell^\psi(x) := \sum_{k \in \mathbb{Z}^d} \Theta_\ell(k)\, c_k(f)\, e^{ikx}\,, \qquad x \in \mathbb{T}^d\,, \quad \ell \in \mathbb{N}_0^d\,. \tag{28}$$

As a particular case of the definition we obtain

$$\| f \,|S^r_{p,q} B(\mathbb{T}^d)\| \asymp \Big(\sum_{\ell \in \mathbb{N}_0^d} 2^{r|\ell|_1 q} \, \| f_\ell^\psi \,| L_p(\mathbb{T}^d)\|^q \Big)^{1/q} .$$

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