

Tensor products of Sobolev-Besov spaces and applications to approximation from the hyperbolic cross

*Dedicated to our dear colleague and friend Gérard Bourdaud
on occasion of his 60th birthday*

Winfried Sickel*

*Friedrich-Schiller-University Jena, Mathematical Institute, Ernst-Abbe-Platz 3,
D-07740, Germany*

Tino Ullrich¹

*Hausdorff-Center for Mathematics, Endenicher Allee 60,
D-53115 Bonn, Germany*

Abstract

Besov as well as Sobolev spaces of dominating mixed smoothness are shown to be tensor products of Besov and Sobolev spaces defined on \mathbb{R} . Based on this we derive several useful characterizations from the one-dimensional case to the d -dimensional situation. Finally, consequences for hyperbolic cross approximations, in particular for tensor product splines, are discussed.

Key words:

Tensor products, Besov spaces, fractional Sobolev spaces, Besov spaces of dominating mixed smoothness, Sobolev spaces of dominating mixed smoothness, p -nuclear, projective and injective norm, wavelet decompositions, approximation from hyperbolic crosses, tensor product splines, best n -term approximation.

1991 MSC: 41A25, 41A63, 42B99, 46E35.

* Corresponding author

Email addresses: sickel@minet.uni-jena.de (Winfried Sickel),
tino.ullrich@hcm.uni-bonn.de (Tino Ullrich).

¹ Supported by the German Academic Exchange Service DAAD (D/06/48125)

1 Introduction

The present paper deals with characterizations of iterated tensor products of Sobolev and Besov spaces on \mathbb{R} . Our main results are the identities

$$B_{p,p}^{r_1}(\mathbb{R}) \otimes_{\alpha} \dots \otimes_{\alpha} B_{p,p}^{r_d}(\mathbb{R}) = S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$$

for $0 < p < \infty$, $r_i \in \mathbb{R}$, $i = 1, \dots, d$, and

$$H_p^{r_1}(\mathbb{R}) \otimes_{\alpha} \dots \otimes_{\alpha} H_p^{r_d}(\mathbb{R}) = S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$$

for $1 < p < \infty$ and $r_i \in \mathbb{R}$, $i = 1, \dots, d$. Here α denotes a specific tensor (quasi-)norm depending on p . Based on these formulas we will discuss a new approach to estimates of certain best approximations related to hyperbolic crosses. Here we concentrate on tensor product splines and derive results parallel to the well-known estimates for trigonometric polynomials with frequencies from hyperbolic crosses. In addition we consider best m -term approximation with respect to tensor product wavelet systems oriented on some recent results of Nitsche [22].

Function spaces of dominating mixed smoothness have been introduced by Nikol'skij around 1962. They are systematically studied by Amanov [1], Schmeisser [29] and Vybiral [48]. Connections to hyperbolic cross approximation (in the context of periodic functions) may be found in Temlyakov [37], Sprengel [35] as well as in [34] and [45]. Definitions will be recalled in Appendix A.

There is a well-developed abstract theory for tensor products of Banach spaces, cf. the monograph Defant and Floret [8]. Since we are dealing with function spaces and sequence spaces only we do not need the abstract theory in its full generality. Our approach is based on the treatment of Light and Cheney [20]. Basic concepts of the abstract theory of tensor products of Banach spaces are collected in Appendix B. There we also discuss an extension of the projective norm to tensor products of certain quasi-Banach spaces. This is parallel to the approach suggested by Nitsche [22].

Acknowledgement: We would like to thank Markus Hansen for a critical reading of a preliminary version of this manuscript and for several valuable hints to improve it.

Notation

The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{N}_0$ and \mathbb{Z} denote the real numbers, complex numbers, natural numbers, natural numbers including 0 and the integers. The natural number d is reserved for the dimension of the considered Euclidean space \mathbb{R}^d . The Euclidean distance of $x \in \mathbb{R}^d$ to the origin is given by $|x|_2$, whereas the

ℓ_1^d -norm is denoted by $|x|_1$. We often need further vector-type quantities like indices and parameters. They are denoted by $\bar{\ell}, \bar{k}, \bar{j}$ and \bar{r} with numbered components. As usual we put $\bar{r} + \bar{\ell} = (r_1 + \ell_1, \dots, r_d + \ell_d)$, $\lambda \cdot \bar{\ell} = (\lambda \cdot \ell_1, \dots, \lambda \cdot \ell_d)$, $\lambda \in \mathbb{R}$, and $\bar{k} \cdot \bar{r} = k_1 r_1 + \dots + k_d r_d$.

For a multi-index $\bar{\alpha}$ we define the differential operator $D^{\bar{\alpha}}$ by

$$D^{\bar{\alpha}} = \frac{\partial^{|\bar{\alpha}|_1}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}.$$

Let X and Y be quasi-Banach spaces. Then $\mathcal{L}(X, Y)$ denotes the class of all linear and bounded operators $P : X \rightarrow Y$ equipped with the usual quasi-norm. We also use the notation $a \asymp b$ if there exists a constant $c > 0$ (independent of the context dependent relevant parameters) such that

$$c^{-1} a \leq b \leq c a.$$

Constants will change their value from line to line, sometimes indicated by adding subscripts.

2 Main results

To begin with we describe our results on the representation of certain Besov as well as Sobolev spaces of dominating mixed smoothness as iterated tensor products of Besov and Sobolev spaces defined on \mathbb{R} . As an application we obtain characterizations by means of wavelets under improved conditions. Then we continue with a discussion of some applications to hyperbolic cross approximation.

2.1 Tensor products of Sobolev and Besov spaces

Recall the definitions of the distribution spaces in Appendix A. Basic notions of tensor products can be found in Appendix B.

Theorem 2.1 (Tensor products of Sobolev spaces) *Let $1 < p < \infty$, $d \in \mathbb{N}$, and $r_1, r_2, \dots, r_{d+1} \in \mathbb{R}$. Then*

$$\begin{aligned} H_p^{r_1}(\mathbb{R}) \otimes_{\alpha_p} S_p^{r_2, \dots, r_{d+1}} H(\mathbb{R}^d) &= S_p^{r_1, \dots, r_d} H(\mathbb{R}^d) \otimes_{\alpha_p} H_p^{r_{d+1}}(\mathbb{R}) \\ &= S_p^{r_1, r_2, \dots, r_{d+1}} H(\mathbb{R}^{d+1}). \end{aligned} \quad (2.1)$$

Remark (i) Here and in what follows we identify $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ and $S_p^{r_2, \dots, r_{d+1}} H(\mathbb{R}^d)$ with $H_p^{r_1}(\mathbb{R})$ and $H_p^{r_2}(\mathbb{R})$ if $d = 1$.

- (ii) The norms on the left-hand side and on the right-hand side in (2.1) coincide. That could be of some interest for the investigation of the d -dependence of some operators, in particular for large d .
- (iii) Defining for $m > 2$

$$X_1 \otimes_{\alpha_p} X_2 \otimes_{\alpha_p} \dots \otimes_{\alpha_p} X_m := X_1 \otimes_{\alpha_p} \left(\dots X_{m-2} \otimes_{\alpha_p} (X_{m-1} \otimes_{\alpha_p} X_m) \right)$$

we obtain an interpretation of $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ as an iterated tensor product of the Sobolev spaces of fractional order. In all cases of occurrence of iterated tensor products of quasi-Banach spaces considered in this paper the resulting space will not depend on the order of the tuples which are formed during the process of calculating $X_1 \otimes_{\alpha_p} X_2 \otimes_{\alpha_p} \dots \otimes_{\alpha_p} X_m$, i.e.,

$$(X_1 \otimes_{\alpha_p} X_2) \otimes_{\alpha_p} X_3 = X_1 \otimes_{\alpha_p} (X_2 \otimes_{\alpha_p} X_3).$$

- (iv) In the periodic case with $d = 1$ formula (2.1) is already known, see Sprengel [35].

Theorem 2.2 (Tensor products of Besov spaces)

Let $d \geq 1$ and let $r_1, \dots, r_{d+1} \in \mathbb{R}$.

- (i) Let $1 < p < \infty$. Then the following

$$\begin{aligned} B_{p,p}^{r_1}(\mathbb{R}) \otimes_{\alpha_p} S_{p,p}^{r_2, \dots, r_{d+1}} B(\mathbb{R}^d) &= S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d) \otimes_{\alpha_p} B_{p,p}^{r_{d+1}}(\mathbb{R}) \\ &= S_{p,p}^{r_1, r_2, \dots, r_{d+1}} B(\mathbb{R}^{d+1}) \end{aligned} \quad (2.2)$$

holds true in the sense of equivalent norms.

- (ii) Let $p = \infty$. Then we have

$$\begin{aligned} \mathring{B}_{\infty, \infty}^{r_1}(\mathbb{R}) \otimes_{\lambda} \mathring{S}_{\infty, \infty}^{r_2, \dots, r_{d+1}} B(\mathbb{R}^d) &= \mathring{S}_{\infty, \infty}^{r_1, \dots, r_d} B(\mathbb{R}^d) \otimes_{\lambda} \mathring{B}_{\infty, \infty}^{r_{d+1}}(\mathbb{R}) \\ &= \mathring{S}_{\infty, \infty}^{r_1, r_2, \dots, r_{d+1}} B(\mathbb{R}^{d+1}) \end{aligned}$$

in the sense of equivalent norms.

- (iii) Let $0 < p \leq 1$. Then the following formula

$$\begin{aligned} B_{p,p}^{r_1}(\mathbb{R}) \otimes_{\gamma_p} S_{p,p}^{r_2, \dots, r_{d+1}} B(\mathbb{R}^d) &= S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d) \otimes_{\gamma_p} B_{p,p}^{r_{d+1}}(\mathbb{R}) \\ &= S_{p,p}^{r_1, r_2, \dots, r_{d+1}} B(\mathbb{R}^{d+1}) \end{aligned}$$

holds true in the sense of equivalent quasi-norms.

Remark (a) Similarly as above we identify $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ and $S_{p,p}^{r_2, \dots, r_{d+1}} B(\mathbb{R}^d)$ with $B_{p,p}^{r_1}(\mathbb{R})$ and $B_{p,p}^{r_2}(\mathbb{R})$ if $d = 1$.

(b) The fact (iii) in the remark after Theorem 2.1 applies, mutatis mutandis, for the spaces $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$, $0 < p < \infty$, and $\mathring{S}_{\infty, \infty}^{r_1, \dots, r_d} B(\mathbb{R}^d)$, respectively.

(c) Nitsche [22] has introduced tensor products of Besov spaces with $0 < p < 1$ in connection with best m -term approximation for tensor product wavelets. But in his paper, no relation to spaces of dominating mixed smoothness is given.

(d) In a periodic context (2.2) (with $d = 1$) has been proved earlier in [33].

2.2 Wavelets and spaces of dominating mixed smoothness

Since many years it is well-known that isotropic Besov spaces as well as isotropic Sobolev spaces allow a discretization. That means, there exist isomorphisms onto sequence spaces, see Subsection A.3.1 for details.

The following sequence spaces play a major role in our investigations.

Definition 2.3 Let $0 < p \leq \infty$ and let $r, r_1, \dots, r_d \in \mathbb{R}$.

(i) We put

$$b_p^r := \left\{ (a_{j,k})_{j,k} \subset \mathbb{C} : \|a\|_{b_p^r} := \left(\sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{j(r+\frac{1}{2}-\frac{1}{p})p} |a_{j,k}|^p \right)^{1/p} < \infty \right\}$$

(modification if $p = \infty$).

(ii) Furthermore, let

$$s_p^{r_1, \dots, r_d} b := \left\{ (a_{\bar{j}, \bar{k}})_{\bar{j}, \bar{k}} \subset \mathbb{C} : \|a\|_{s_p^{r_1, \dots, r_d} b} := \left(\sum_{\bar{j} \in \mathbb{N}_0^d} \sum_{\bar{k} \in \mathbb{Z}^d} 2^{(j_1(r_1+\frac{1}{2}-\frac{1}{p})+\dots+j_d(r_d+\frac{1}{2}-\frac{1}{p}))p} |a_{\bar{j}, \bar{k}}|^p \right)^{1/p} < \infty \right\}$$

(modification if $p = \infty$).

(iii) By b_∞^r we denote the closure of the finite sequences with respect to the norm $\|\cdot\|_{b_\infty^r}$.

(iv) By $s_\infty^{r_1, \dots, r_d} b$ we denote the closure of the finite sequences with respect to the norm $\|\cdot\|_{s_\infty^{r_1, \dots, r_d} b}$.

These classes, as indicated by the notation, are related to the Besov spaces. For describing the potential spaces $H_p^r(\mathbb{R})$ and $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ the following more complicated sequence spaces are necessary. To begin with we need a few further notation. Let \mathcal{X} be the characteristic function of the interval $(0, 1)$. We put

$$\mathcal{X}_{j,k}(t) = 2^{j/2} \mathcal{X}(2^j t - k), \quad t \in \mathbb{R}, \quad k \in \mathbb{Z}, \quad j \in \mathbb{N}_0.$$

Let us further define

$$\mathcal{X}_{\bar{j}, \bar{k}} = \mathcal{X}_{j_1, k_1} \otimes \dots \otimes \mathcal{X}_{j_d, k_d}, \quad \bar{j} = (j_1, \dots, j_d) \in \mathbb{N}_0^d, \quad \bar{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d.$$

Definition 2.4 Let $0 < p < \infty$ and $r, r_1, \dots, r_d \in \mathbb{R}$.

(i) We put

$$f_p^r := \left\{ (a_{j,k})_{j,k} \subset \mathbb{C} : \right. \\ \left. \| a |f_p^r \| := \left\| \left(\sum_{j=0}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{jr} |a_{j,k}| \mathcal{X}_{j,k} \right)^2 \right)^{1/2} \Big|_{L_p(\mathbb{R})} \right\| < \infty \right\}.$$

(ii) Furthermore we define

$$s_p^{r_1, \dots, r_d} f := \left\{ (a_{\bar{j}, \bar{k}})_{\bar{j}, \bar{k}} \subset \mathbb{C} : \| a |s_p^{r_1, \dots, r_d} f \| := \right. \\ \left. \left\| \left(\sum_{\bar{j} \in \mathbb{N}_0^d} \left(\sum_{\bar{k} \in \mathbb{Z}^d} 2^{(j_1 r_1 + \dots + j_d r_d)} |a_{\bar{j}, \bar{k}}| \mathcal{X}_{\bar{j}, \bar{k}} \right)^2 \right)^{1/2} \Big|_{L_p(\mathbb{R}^d)} \right\| < \infty \right\}.$$

Our results on tensor products have some nice applications.

Theorem 2.5 (Sequence space isomorphisms)

Let $d > 1$ and $r_1, \dots, r_d \in \mathbb{R}$.

(i) Let $1 < p < \infty$. Let $J_i : H_p^{r_i}(\mathbb{R}) \rightarrow f_p^{r_i}$, $i = 1, \dots, d$, be linear isomorphisms. Then $J_1 \otimes \dots \otimes J_d : S_p^{r_1, \dots, r_d} H(\mathbb{R}^d) \rightarrow s_p^{r_1, \dots, r_d} f$ is a linear isomorphism as well.

(ii) Let $0 < p < \infty$. Let $J_i : B_{p,p}^{r_i}(\mathbb{R}) \rightarrow b_p^{r_i}$, $i = 1, \dots, d$, be linear isomorphisms. Then $J_1 \otimes \dots \otimes J_d : S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d) \rightarrow s_p^{r_1, \dots, r_d} b$ is a linear isomorphism as well.

(iii) Let $p = \infty$. Let $J_i : \mathring{B}_{\infty, \infty}^{r_i}(\mathbb{R}) \rightarrow \mathring{b}_{\infty}^{r_i}$, $i = 1, \dots, d$, be linear isomorphisms. Then $J_1 \otimes \dots \otimes J_d : \mathring{S}_{\infty, \infty}^{r_1, \dots, r_d} B(\mathbb{R}^d) \rightarrow \mathring{s}_{\infty}^{r_1, \dots, r_d} b$ is a linear isomorphism as well.

This has immediate consequences for the discretization of our classes

$S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ and $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$, respectively. We need some further notation. Let $\psi_{j,k}^i : \mathbb{R} \rightarrow \mathbb{C}$, $j \in \mathbb{N}_0$, $k \in \mathbb{Z}$, $i = 1, \dots, d$, be a given system of functions. Then we define Ψ to be the collection of all functions

$$\psi_{\bar{j}, \bar{k}}(x_1, \dots, x_d) := \psi_{j_1, k_1}^1(x_1) \cdot \dots \cdot \psi_{j_d, k_d}^d(x_d), \quad \bar{j} \in \mathbb{N}_0^d, \quad \bar{k} \in \mathbb{Z}^d. \quad (2.3)$$

As usual, $\langle f, g \rangle$ denotes the L_2 scalar product of f and g if both belong to L_2 . However, in many situations we have to interpret $\langle f, g \rangle$ as the application of the tempered distribution f to \bar{g} . Since g will not belong to S in general this question needs some care. For this we refer to Appendix A, Subsection A.3.

Corollary 2.6 Let $d > 1$ and $r_1, \dots, r_d \in \mathbb{R}$.

(i) Let $1 < p < \infty$. Let $(\psi_{j,k}^i)_{j,k}$ be an unconditional basis of $H_p^{r_i}(\mathbb{R})$ such that

$$\| f |H_p^{r_i}(\mathbb{R}) \| \asymp \| \langle f, \psi_{j,k}^i \rangle |f_p^{r_i} \|$$

holds for all $f \in H_p^{r_i}(\mathbb{R})$ and all $i = 1, \dots, d$. Then the system of functions Ψ is an unconditional basis of $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ such that

$$\|f\|_{S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)} \asymp \|\langle f, \psi_{\bar{j}, \bar{k}} \rangle\|_{s_p^{r_1, \dots, r_d} f}$$

holds for all $f \in S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$.

(ii) Let $0 < p < \infty$. Let $(\psi_{j,k}^i)_{j,k}$ be an unconditional basis of $B_{p,p}^{r_i}(\mathbb{R})$ such that

$$\|f\|_{B_{p,p}^{r_i}(\mathbb{R})} \asymp \|\langle f, \psi_{j,k}^i \rangle\|_{b_p^{r_i}}$$

holds for all $f \in B_{p,p}^{r_i}(\mathbb{R})$ and all $i = 1, \dots, d$. Then the system of functions Ψ is an unconditional basis of $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ such that

$$\|f\|_{S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)} \asymp \|\langle f, \psi_{\bar{j}, \bar{k}} \rangle\|_{s_p^{r_1, \dots, r_d} b}$$

holds for all $f \in S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$.

(iii) Let $(\psi_{j,k}^i)_{j,k}$ be an unconditional basis of $\mathring{B}_{\infty, \infty}^{r_i}(\mathbb{R})$ such that

$$\|f\|_{B_{\infty, \infty}^{r_i}(\mathbb{R})} \asymp \|\langle f, \psi_{j,k}^i \rangle\|_{b_{\infty}^{r_i}}$$

holds for all $f \in \mathring{B}_{p,p}^{r_i}(\mathbb{R})$ and all $i = 1, \dots, d$. Then Ψ is an unconditional basis of $\mathring{S}_{\infty, \infty}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ such that

$$\|f\|_{\mathring{S}_{\infty, \infty}^{r_1, \dots, r_d} B(\mathbb{R}^d)} \asymp \|\langle f, \psi_{\bar{j}, \bar{k}} \rangle\|_{s_{\infty}^{r_1, \dots, r_d} b}$$

holds for all $f \in \mathring{S}_{\infty, \infty}^{r_1, \dots, r_d} B(\mathbb{R}^d)$.

Remark Observe that we only need to know the restrictions for the wavelet isomorphism for $d = 1$. No further conditions enter.

2.3 Nonlinear approximation and spaces of dominating mixed smoothness

Let $0 < p < 2$. Let φ be an orthonormal scaling function and ψ be an associated wavelet such that the system $(\psi_{j,k})_{j,k}$ given by

$$\psi_{0,k}(t) := \varphi(t - k) \quad \text{and} \quad \psi_{j+1,k}(t) := 2^{j/2} \psi(2^j t - k),$$

where $t \in \mathbb{R}, k \in \mathbb{Z}$ and $j \in \mathbb{N}_0$, yields an unconditional basis of $B_{p,p}^{1/p-1/2}(\mathbb{R})$ satisfying

$$\|f\|_{B_{p,p}^{\frac{1}{p}-\frac{1}{2}}(\mathbb{R})} \asymp \|\langle f, \psi_{j,k} \rangle\|_{b_p^{\frac{1}{p}-\frac{1}{2}}} = \|\langle f, \psi_{j,k} \rangle\|_{\ell_p}$$

for all $f \in B_{p,p}^{1/p-1/2}(\mathbb{R})$. By Ψ , see (2.3), we denote the corresponding tensor product system. Further, let

$$\Sigma_m := \left\{ \sum_{(\bar{j}, \bar{k}) \in \Lambda} a_{\bar{j}, \bar{k}} \psi_{\bar{j}, \bar{k}} : \Lambda \subset \mathbb{N}_0^d \times \mathbb{Z}^d, \quad |\Lambda| \leq m \right\}, \quad m \in \mathbb{N},$$

(here $|\Lambda|$ denotes the cardinality of the set Λ). The best m -term approximation of $f \in L_2(\mathbb{R}^d)$ by the tensor product wavelet system Ψ with respect to the L_2 -norm is the quantity

$$\sigma_m(f)_2 := \inf \left\{ \|f - g\|_{L_2(\mathbb{R}^d)} : g \in \Sigma_m \right\}.$$

For $s > 0$ and p as above we consider the approximation space $A_p^s(L_2(\mathbb{R}^d))$, see e.g. [9], as the collection of all $f \in L_2(\mathbb{R}^d)$ such that

$$\|f\|_{A_p^s(L_2(\mathbb{R}^d))} := \|f\|_{L_2(\mathbb{R}^d)} + \left(\sum_{m=1}^{\infty} \frac{1}{m} [m^s \sigma_m(f)_2]^p \right)^{1/p} < \infty.$$

Mainly as a corollary of the characterization of $A_p^{\frac{1}{p}-\frac{1}{2}}(L_2(\mathbb{R}^d))$, see e.g. Pietsch [25] or DeVore [9, Thm. 4], and of Corollary 2.6 we obtain the following.

Theorem 2.7 (Nonlinear approximation spaces) *Let $0 < p < 2$. Then we have*

$$A_p^{\frac{1}{p}-\frac{1}{2}}(L_2(\mathbb{R}^d)) = S_{p,p}^{\frac{1}{p}-\frac{1}{2}, \dots, \frac{1}{p}-\frac{1}{2}} B(\mathbb{R}^d)$$

in the sense of equivalent quasi-norms.

Remark With the right-hand side $S_{p,p}^{1/p-1/2, \dots, 1/p-1/2} B(\mathbb{R}^d)$ replaced by the tensor product $B_{p,p}^{1/p-1/2}(\mathbb{R}) \otimes_p \dots \otimes_p B_{p,p}^{1/p-1/2}(\mathbb{R})$ Theorem 2.7 has been proved in Nitsche [22]. Our treatment of tensor products, see Theorem 2.2, allows to identify this approximation space with a Besov space of dominating mixed smoothness. Recall, that these spaces can be described by means of differences and derivatives, cf. [31] and [44].

2.4 Splines and spaces of dominating mixed smoothness

A case of particular importance is given by choosing $J_1 = \dots = J_d$ to be an isomorphism associated to an orthonormal spline wavelet system.

Let $m \in \mathbb{N}$. Let \mathcal{X} be the characteristic function of the interval $(0, 1)$. Then the normalized cardinal B-spline of order $m + 1$ is given by

$$\mathcal{N}_{m+1}(x) := \mathcal{N}_m * \mathcal{X}(x), \quad x \in \mathbb{R}, \quad m \in \mathbb{N},$$

beginning with $\mathcal{N}_1 = \mathcal{X}$. Let \mathcal{F} denote the Fourier transform and \mathcal{F}^{-1} its inverse transform. We normalize these transformations by

$$\mathcal{F}f(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx, \quad \xi \in \mathbb{R}, \quad f \in L_1(\mathbb{R}).$$

By

$$\varphi_m(x) := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \left[\frac{\mathcal{F}\mathcal{N}_m(\xi)}{\left(\sum_{k=-\infty}^{\infty} |\mathcal{F}\mathcal{N}_m(\xi + 2\pi k)|^2 \right)^{1/2}} \right](x), \quad x \in \mathbb{R},$$

we obtain an orthonormal scaling function which is again a spline of order m . Finally, by

$$\psi_m(x) := \sum_{k=-\infty}^{\infty} \langle \varphi_m(t/2), \varphi_m(t-k) \rangle (-1)^k \varphi_m(2x+k+1)$$

we obtain the generator of an orthonormal wavelet system. For $m = 1$ it is easily checked that $-\psi_1(x-1)$ is the Haar wavelet. In general these functions ψ_m have the following properties:

- (i) ψ_m restricted to intervals $[\frac{k}{2}, \frac{k+1}{2}]$, $k \in \mathbb{Z}$, is a polynomial of degree at most $m-1$.
- (ii) $\psi_m \in C^{m-2}(\mathbb{R})$ if $m \geq 2$.
- (iii) $\psi_m^{(m-2)}$ is uniformly Lipschitz continuous on \mathbb{R} if $m \geq 2$.
- (iv) There exist positive numbers τ_m and sequences $(c_k)_k$ and $(d_k)_k$ such that

$$\psi_m(x) = \sum_{k=-\infty}^{\infty} c_k \mathcal{N}_m(2x-k), \quad \mathcal{N}_m(x) = \sum_{k=-\infty}^{\infty} d_k \varphi_m(x-k), \quad x \in \mathbb{R},$$

and

$$\sup_{k \in \mathbb{Z}} (|c_k| + |d_k|) e^{\tau_m |k|} < \infty \quad \text{and} \quad \max_{0 \leq \ell \leq m-2} \sup_{x \in \mathbb{R}} |\psi_m^{(\ell)}(x)| e^{\tau_m |x|} < \infty.$$

- (v) The functions ψ_m satisfy a moment condition of order $m-1$, i.e.

$$\int_{-\infty}^{\infty} x^\ell \psi_m(x) dx = 0, \quad \ell = 0, 1, \dots, m-1.$$

It will be convenient for us to use the following abbreviations:

$$\psi_{0,k}^m(x) := \varphi_m(x-k) \quad \text{and} \quad \psi_{j+1,k}^m(x) := 2^{j/2} \psi_m(2^j x - k), \quad (2.4)$$

where $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and $j \in \mathbb{N}_0$.

Let us now focus on the mapping

$$R_m : f \mapsto (\langle f, \psi_{j,k}^m \rangle)_{j,k}.$$

Proposition 2.8 *Let $m \in \mathbb{N}$.*

- (i) *Let $1 < p < \infty$ and suppose $-m + 1 < r < m - 1$. The mapping R_m generates an isomorphism of $H_p^r(\mathbb{R})$ onto f_p^r .*
- (ii) *Let either $1 \leq p < \infty$ and $-m + 1/p < r < m - 1 + 1/p$ or $0 < p < 1$ and $-m + \frac{1}{p} < r < m$. Then the mapping R_m generates an isomorphism of $B_{p,p}^r(\mathbb{R})$ onto b_p^r .*
- (iii) *Let $-m < r < m - 1$. The mapping R_m generates an isomorphism of $\dot{B}_{\infty,\infty}^r(\mathbb{R})$ onto \dot{b}_{∞}^r .*

Remark (a) The content of this proposition is essentially well-known, see Bourdaud [5], Frazier and Jawerth [12], Meyer [21], Lemarie and Kahane [19] or DeVore [9]. More details will be given later on. However, we wish to mention that the case $m = 1$ has its own history. The characterization of Besov spaces by means of the Haar basis attracted some attention in the literature, see e.g. Ropela [27], Oswald [23] and Triebel [39,40] for earlier papers with this respect.

(b) Wavelet characterizations of Besov spaces with $p < 1$ are investigated e.g. in Bourdaud [5], Cohen [6], Kyriazis and Petrushev [18] and Triebel [42]. Cohen and Triebel concentrate on biorthogonal (orthogonal) wavelets with compact support.

(c) Observe that the interval $-m + 1 < r < m - 1$ is empty if $m = 1$, i.e. for the Haar basis. It still seems to be an open question whether the spaces $H_p^s(\mathbb{R})$, $p \neq 2$, $-1 + 1/p < s < 1/p$, can be characterized by means of the Haar basis. For $p = 2$ this is guaranteed by $H_2^s(\mathbb{R}) = B_{2,2}^s(\mathbb{R})$ in the sense of equivalent norms.

As an immediate conclusion of the preceding proposition and Corollary 2.6 we obtain assertions on the characterization of the spaces $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ and $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$, respectively. Of course, Ψ^m denotes the system Ψ , cf. (2.3), generated by φ_m and ψ_m .

Theorem 2.9 (Spline wavelet isomorphisms)

- Let $d > 1$. (i) *Let $1 < p < \infty$ and $-m + 1 < r_1, \dots, r_d < m - 1$. The spline system Ψ^m is an unconditional basis of $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$. The quantity $\| \langle f, \psi_{j,k}^m \rangle |_{s_p^{r_1, \dots, r_d} f} \|$ represents an equivalent norm in $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$.*
- (ii) *Let either $1 \leq p < \infty$ and $-m + 1/p < r < m - 1 + 1/p$ or $0 < p < 1$ and $-m + \frac{1}{p} < r < m$. The spline system Ψ^m is an unconditional basis of $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$. The quantity $\| \langle f, \psi_{j,k}^m \rangle |_{s_p^{r_1, \dots, r_d} b} \|$ represents an equivalent quasi-norm in $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$.*
- (iii) *Let $-m < r_1, \dots, r_d < m - 1$. The spline system Ψ^m is an unconditional basis of $\dot{S}_{\infty,\infty}^{r_1, \dots, r_d} B(\mathbb{R}^d)$. The quantity $\| \langle f, \psi_{j,k}^m \rangle |_{s_{\infty}^{r_1, \dots, r_d} b} \|$ represents an equivalent quasi-norm in $\dot{S}_{\infty,\infty}^{r_1, \dots, r_d} B(\mathbb{R}^d)$.*

Remark Kamont [17] has proved a similar result for the spaces $S_{p,p}^{r_1, \dots, r_d} B([0, 1]^d)$, $1 < p < \infty$, $r_1 = r_2 = \dots = r_d > 0$, but using a different system of splines, at least if $m > 1$. The case $p = 2$ and $m = 1$ may be found in Oswald [24] as well.

2.5 Approximation from the hyperbolic cross

We concentrate on the case $r_1 = r_2 = \dots = r_d > 0$. Here we make use of the conventions $S_p^r H(\mathbb{R}^d) = S_p^{r, \dots, r} H(\mathbb{R}^d)$ and $S_{p,p}^r B(\mathbb{R}^d) = S_{p,p}^{r, \dots, r} B(\mathbb{R}^d)$. In [10] the following type of approximation related to hyperbolic crosses is considered. Define

$$P_n^m f := \sum_{|\bar{j}|_1 \leq n} \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \psi_{\bar{j}, \bar{k}}^m, \quad n \in \mathbb{N}.$$

We are interested in determining the asymptotic behaviour of the quantities

$$\|I - P_n^m | \mathcal{L}(S_{p,p}^r B(\mathbb{R}^d), L_p(\mathbb{R}^d))\| \quad \text{and} \quad \|I - P_n^m | \mathcal{L}(S_p^r H(\mathbb{R}^d), L_p(\mathbb{R}^d))\|$$

as n tends to infinity. Mainly as a consequence of Theorem 2.9 one obtains the following.

Proposition 2.10 *Let $d > 1$.*

(i) *Let $1 < p < \infty$ and $0 < r < m - 1$. Then*

$$\|I - P_n^m | \mathcal{L}(S_p^r H(\mathbb{R}^d), L_p(\mathbb{R}^d))\| \asymp 2^{-rn}, \quad n \in \mathbb{N},$$

holds.

(ii) *Let $0 < p < \infty$ and $\max(0, \frac{1}{p} - 1) < r < m - 1 + \min(1, 1/p)$. Then we have for $n \in \mathbb{N}$*

$$\|I - P_n^m | \mathcal{L}(S_{p,p}^r B(\mathbb{R}^d), L_p(\mathbb{R}^d))\| \asymp \begin{cases} 2^{-rn} & \text{if } 0 < p \leq 2, \\ n^{(d-1)(\frac{1}{2} - \frac{1}{p})} 2^{-rn} & \text{if } 2 < p < \infty. \end{cases}$$

(iii) *Let $p = \infty$ and $0 < r < m - 1$. Then we have*

$$\|I - P_n^m | \mathcal{L}(S_{\infty, \infty}^r B(\mathbb{R}^d), L_\infty(\mathbb{R}^d))\| \asymp n^{d-1} 2^{-rn}, \quad n \in \mathbb{N}.$$

Next we want to define the error of best approximation of a function $f \in L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, by splines of degree less than m related to the hyperbolic cross. For this purpose it will be convenient to introduce some further notation first. Let

$$V_j^m := \text{span} \left\{ \mathcal{N}_m(2^j \cdot -k) : k \in \mathbb{Z} \right\}, \quad j \in \mathbb{N}_0,$$

and

$$\mathcal{V}_n^m := \text{span} \left\{ \mathcal{N}_m(2^{j_1} \cdot -k_1) \otimes \dots \otimes \mathcal{N}_m(2^{j_d} \cdot -k_d) : \right. \\ \left. \bar{j} \in \mathbb{N}_0^d, \quad |\bar{j}|_1 = n, \quad \bar{k} \in \mathbb{Z}^d \right\},$$

$n \in \mathbb{N}_0$. Sometimes spaces of this type are called *sparse grid ansatz spaces*. Since “span” contains finite sums only we have $\mathcal{V}_n^m \subset L_p(\mathbb{R}^d)$. We put

$$E_n^m(f, L_p(\mathbb{R}^d)) := \inf \left\{ \|f - g\|_{L_p(\mathbb{R}^d)} : g \in \mathcal{V}_n^m \right\}, \quad n \in \mathbb{N}_0.$$

Some comments are necessary. The classes \mathcal{V}_n^m are nested, i.e. $\mathcal{V}_n^m \subset \mathcal{V}_{n+1}^m$, since

$$\mathcal{N}_m(x) = 2^{-m+1} \sum_{k=0}^m \binom{m}{k} \mathcal{N}_m(2x - k), \quad x \in \mathbb{R}.$$

Furthermore, the spaces \mathcal{V}_n^m do not contain our basis functions $\psi_{\bar{j}, \bar{k}}^m$. However, it becomes obvious from Section 2.4/(iv) that $\psi_{\bar{j}, \bar{k}}^m$ belongs to the closure of $\mathcal{V}_n^m, |\bar{j}|_1 = n$, in $L_p(\mathbb{R}^d)$. Alternatively to the quantity $E_n^m(f, L_p(\mathbb{R}^d))$ one could consider the following

$$\tilde{E}_n^m(f, L_p(\mathbb{R}^d)) := \inf \left\{ \|f - g\|_{L_p(\mathbb{R}^d)} : g \in L_p(\mathbb{R}^d), \exists (a_{\bar{j}, \bar{k}})_{\bar{j}, \bar{k}} \text{ s.t.} \right. \\ \left. a_{\bar{j}, \bar{k}} = 0 \text{ if } |\bar{j}|_1 > n \text{ and } g \stackrel{L_p}{=} \sum_{\bar{j} \in \mathbb{N}_0^d} \sum_{\bar{k} \in \mathbb{Z}^d} a_{\bar{j}, \bar{k}} \psi_{\bar{j}, \bar{k}}^m \right\},$$

a concept which is related to the definition of the P_n^m . Fortunately we have

$$E_n^m(f, L_p(\mathbb{R}^d)) = \tilde{E}_n^m(f, L_p(\mathbb{R}^d)).$$

This can be seen by using (iv) in our list of properties of the ψ_m given above. To have a compact formulation we shall use the following quantity:

$$\mathcal{E}_n^m(F)_p := \sup_{\|f\|_F \leq 1} E_n^m(f, L_p(\mathbb{R}^d))$$

where $F \hookrightarrow L_p$ denotes an arbitrary quasi-Banach space.

Theorem 2.11 (Error of best approximation) *Let $d > 1$ and let $m \in \mathbb{N}$.*

(i) *Let $1 < p < \infty$ and $0 < r < m - 1$. Then it holds*

$$\mathcal{E}_n^m(S_p^r H(\mathbb{R}^d))_p \asymp 2^{-rn}, \quad n \in \mathbb{N}.$$

(ii) *Let $1 \leq p < \infty$ and $0 < r < m - 1 + 1/p$. Then it holds*

$$\mathcal{E}_n^m(S_{p,p}^r B(\mathbb{R}^d))_p \asymp \begin{cases} 2^{-rn} & \text{if } 1 \leq p \leq 2, \\ n^{(d-1)(\frac{1}{2}-\frac{1}{p})} 2^{-rn} & \text{if } 2 < p < \infty, \end{cases} \quad (2.5)$$

$n \in \mathbb{N}$.

(iii) Let $0 < p < 1$ and $\frac{1}{p} - 1 < r < m$. Then there exists a constant c such that

$$\mathcal{E}_n^m(S_{p,p}^r B(\mathbb{R}^d))_p \leq c 2^{-rn}$$

holds for all $n \in \mathbb{N}$.

(iv) Let $p = \infty$ and $0 < r < m - 1$. Then there exists a constant c such that

$$\mathcal{E}_n^m(S_{\infty,\infty}^r B(\mathbb{R}^d))_\infty \leq c n^{d-1} 2^{-rn}, \quad n \in \mathbb{N}.$$

Remark (a) Observe $S_{p,p}^r B(\mathbb{R}^d) \hookrightarrow S_p^r H(\mathbb{R}^d)$ if $1 < p \leq 2$ and $S_p^r H(\mathbb{R}^d) \hookrightarrow S_{p,p}^r B(\mathbb{R}^d)$ if $2 \leq p < \infty$, see [46]. All the inclusions are strict except for $p = 2$ where we have coincidence.

(b) DeVore, Konyagin, Temlyakov [10] have also proved the upper bound in (i). But we would like to mention a difference between the results in part (i) and part (ii), respectively. The assumptions in (i) are more restrictive than in part (ii). E.g., if $m = 1$, i.e. for the Haar system, part (i) is not applicable whereas in (ii) the natural restriction $r < 1/p$ shows up.

(c) In case $1 < p < \infty$ we refer to Kamont [17] for corresponding estimates in terms of a certain modulus of smoothness for functions defined on $[0, 1]^d$.

(d) Bazarkhanov [3,4] derived similar estimates by using Meyer wavelets instead of splines.

(e) All estimates stated in Theorem 2.11 have counterparts in the classical periodic context of best approximation by polynomials with frequencies taken from a hyperbolic cross. We refer to Dinh Dung [11], Galeev [13], Romanyuk [26], Temlyakov [37] and [34], [45]. However, the Littlewood-Paley theory for the spline system Ψ^m differs from the Littlewood-Paley theory of the trigonometric system. So, at least partly, our test functions are not the same as used in the quoted literature.

At least for the most interesting case $p = 2$ there is a partial inverse of the inequality (2.5). In fact we have the following equivalence.

Corollary 2.12 (Characterization via approximation)

Let $0 < r < m - 1/2$. A function $f \in L_2(\mathbb{R}^d)$ belongs to $S_{2,2}^r B(\mathbb{R}^d)$ if and only if the sequence $(2^{rn} E_n^m(f, L_2(\mathbb{R}^d)))_n$ belongs to ℓ_2 . Moreover, we have

$$\|f\|_{S_{2,2}^r B(\mathbb{R}^d)} \asymp \|f\|_{L_2(\mathbb{R}^d)} + \left(\sum_{n=0}^{\infty} \left(2^{rn} E_n^m(f, L_2(\mathbb{R}^d)) \right)^2 \right)^{1/2}.$$

Remark For $m = 1$ we refer to Oswald [24]. Let us further mention that there is not much hope to generalize Corollary 2.12 to $p \neq 2$. In a slightly different setting (approximation by entire analytic functions with frequencies in the hyperbolic cross) it has been shown in [30] that the approximation

spaces $A_{p,q}^r(\mathbb{R}^d)$ characterized by a condition

$$\|f\|_{L_p(\mathbb{R}^d)} + \left(\sum_{n=0}^{\infty} \left(2^{rn} \mathcal{E}_n(f, L_p(\mathbb{R}^d)) \right)^q \right)^{1/q} < \infty$$

do not belong to the scales of Besov-Lizorkin-Triebel spaces except $p = q = 2$. Here $\mathcal{E}_n(f, L_p(\mathbb{R}^d))$ has to be understood in a different context of approximation by entire analytic functions with frequencies in the hyperbolic cross. For the approximation spaces related to approximation from hyperbolic crosses we refer to DeVore, Konyagin, Temlyakov [10].

3 Proofs

3.1 Tensor products of Sobolev spaces

The proof follows along the lines of the proof of $L_p(\mathbb{R}^{d+1}) = L_p(\mathbb{R}) \otimes_{\alpha_p} L_p(\mathbb{R}^d)$, cf. [20, Chapt. 1]. We shall prove a more general result first. Let μ be an infinitely differentiable, positive function such that for any multiindex $\bar{\alpha} \in \mathbb{N}_0^d$ there exist a constant $c_{\bar{\alpha}}$ and a natural number $m_{\bar{\alpha}}$ with

$$\sup_{\xi \in \mathbb{R}^d} (1 + |\xi|)^{-m_{\bar{\alpha}}} (|D^{\bar{\alpha}} \mu(\xi)| + |D^{\bar{\alpha}}(1/\mu(\xi))|) \leq c_{\bar{\alpha}}. \quad (3.1)$$

With other words, the functions $\mu, (1/\mu)$ as well as all their derivatives are polynomially bounded. Then, for $1 < p < \infty$, we define

$$H_p(\mu, \mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|\mathcal{F}^{-1}[\mu \mathcal{F}f]\|_{L_p(\mathbb{R}^d)} < \infty \right\}.$$

Distribution spaces of such a type have been discussed by many authors, e.g. by Hörmander [15], Panejach, Voljevich [47] and Bagdasarian [2]. Taking

$$\mu(\xi) := \prod_{i=1}^d (1 + |\xi_i|^2)^{r_i/2}, \quad \xi \in \mathbb{R}^d, \quad r_1, \dots, r_d \in \mathbb{R},$$

then $H_p(\mu, \mathbb{R}^d) = S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$.

Proposition 3.1 *Let $d_1, d_2 \in \mathbb{N}$, $d = d_1 + d_2$, and let $1 < p < \infty$. Let $\mu_1 : \mathbb{R}^{d_1} \rightarrow \mathbb{C}$ and $\mu_2 : \mathbb{R}^{d_2} \rightarrow \mathbb{C}$ be two functions as above. Then*

$$H_p(\mu_1, \mathbb{R}^{d_1}) \otimes_{\alpha_p} H_p(\mu_2, \mathbb{R}^{d_2}) = H_p(\mu_1 \otimes \mu_2, \mathbb{R}^d).$$

The proof of this proposition requires some preparation.

The set of all finite rectangles in \mathbb{R}^d with sides parallel to the axes will be

denoted by $\mathcal{R}(\mathbb{R}^d)$. The corresponding set of characteristic functions is denoted by $\mathcal{R}^*(\mathbb{R}^d)$. Since $I_\mu : f \rightarrow \mathcal{F}^{-1}\mu\mathcal{F}f$ is an isometry of $H_p(\mu, \mathbb{R}^d)$ onto $L_p(\mathbb{R}^d)$ the inverse mapping sends dense subsets of $L_p(\mathbb{R}^d)$ into dense subsets in $H_p(\mu, \mathbb{R}^d)$. The inverse mapping is given by $I_{(1/\mu)}$ (here we need (3.1)). The outcome of this simple observation is the following.

Lemma 3.2 *Let $1 < p < \infty$. Then the image of the set*

$$K(\mathbb{R}^d) := \left\{ \sum_{j=1}^n \eta_j \chi_j : n \in \mathbb{N}, \eta_j \in \mathbb{C}, \chi_j \in \mathcal{R}^*(\mathbb{R}^d), \right. \\ \left. |\text{supp } \chi_j \cap \text{supp } \chi_i| = 0, i \neq j \right\}$$

under the mapping $I_{(1/\mu)}$ is dense in $H_p(\mu, \mathbb{R}^d)$.

After these preparations we are in the position to prove Proposition 3.1.

Proof of Proposition 3.1. *Step 1.* To begin with we shall verify the continuous embedding

$$H_p(\mu_1, \mathbb{R}^{d_1}) \otimes_{\alpha_p} H_p(\mu_2, \mathbb{R}^{d_2}) \hookrightarrow H_p(\mu_1 \otimes \mu_2, \mathbb{R}^d) \quad (3.2)$$

with $d = d_1 + d_2$.

Let $h = \sum_{i=1}^n f_i \otimes g_i$, where $f_i \in H_p(\mu_1, \mathbb{R}^{d_1})$, $g_i \in H_p(\mu_2, \mathbb{R}^{d_2})$ and $i = 1, \dots, n$. The linearity of $I_{\mu_1 \otimes \mu_2}$ as well as of I_{μ_2} yields

$$\begin{aligned} \|h\|_{H_p(\mu_1 \otimes \mu_2, \mathbb{R}^d)}^p &= \left\| I_{\mu_1 \otimes \mu_2} \left(\sum_{i=1}^n f_i \otimes g_i \right) \right\|_{L_p(\mathbb{R}^d)}^p \\ &= \int_{\mathbb{R}^{d_1}} \left\| \sum_{i=1}^n I_{\mu_1} f_i(x) \cdot I_{\mu_2} g_i \right\|_{L_p(\mathbb{R}^{d_2})}^p dx \\ &= \int_{\mathbb{R}^{d_1}} \left\| I_{\mu_2} \left(\sum_{i=1}^n I_{\mu_1} f_i(x) \cdot g_i \right) \right\|_{L_p(\mathbb{R}^{d_2})}^p dx \\ &= \int_{\mathbb{R}^{d_1}} \left\| \sum_{i=1}^n I_{\mu_1} f_i(x) \cdot g_i \right\|_{H_p(\mu_2, \mathbb{R}^{d_2})}^p dx. \end{aligned}$$

Now we make use of the fact

$$\|g\|_X = \sup_{\substack{\psi \in X' \\ \|\psi\|_{X'} \leq 1}} |\psi(g)|$$

for normed spaces X . With $X = H_p(\mu_2, \mathbb{R}^{d_2})$ and Hölder's inequality ($1/p + 1/p' = 1$) we obtain

$$\begin{aligned}
\|h|_{H_p(\mu_1 \otimes \mu_2, \mathbb{R}^d)}\|^p &= \int_{\mathbb{R}^{d_1}} \sup_{\|\psi|_{X'}\| \leq 1} \left| \sum_{i=1}^n I_{\mu_1} f_i(x) \cdot \psi(g_i) \right|^p dx \\
&\leq \int_{\mathbb{R}^{d_1}} \left(\sum_{i=1}^n |I_{\mu_1} f_i(x)|^p \right) \cdot \sup_{\|\psi|_{X'}\| \leq 1} \left(\sum_{i=1}^n |\psi(g_i)|^{p'} \right)^{p/p'} dx \\
&= \sup_{\|\psi|_{X'}\| \leq 1} \left(\sum_{i=1}^n |\psi(g_i)|^{p'} \right)^{p/p'} \sum_{i=1}^n \int_{\mathbb{R}} |I_{\mu_1} f_i(x)|^p dx \\
&= \sup_{\|\psi|_{X'}\| \leq 1} \left(\sum_{i=1}^n |\psi(g_i)|^{p'} \right)^{p/p'} \sum_{i=1}^n \|I_{\mu_1} f_i|_{L_p(\mathbb{R})}\|^p.
\end{aligned}$$

Taking the infimum over all representations of h we end up with

$$\|h|_{H_p(\mu_1 \otimes \mu_2, \mathbb{R}^d)}\| \leq \alpha_p(h, H_p(\mu_1, \mathbb{R}^{d_1}), H_p(\mu_2, \mathbb{R}^{d_2})),$$

which implies (3.2) (see also (B.4) and (B.3)).

Step 2. We proceed with a proof of

$$H_p(\mu_1 \otimes \mu_2, \mathbb{R}^d) \hookrightarrow H_p(\mu_1, \mathbb{R}^{d_1}) \otimes_{\alpha_p} H_p(\mu_2, \mathbb{R}^{d_2}). \quad (3.3)$$

First we consider a function $h \in I_{(1/(\mu_1 \otimes \mu_2))} K(\mathbb{R}^d)$, i.e.

$$I_{\mu_1 \otimes \mu_2} h = \sum_{j=1}^n \eta_j \mathcal{X}_j = \sum_{j=1}^n f_j \otimes g_j,$$

where f_j and g_j are multiples of characteristic functions of finite rectangles. It is not difficult to see that we may choose these functions f_j, g_j in such a way that $\|g_j|_{L_p(\mathbb{R}^{d_2})}\| = 1$, $j = 1, \dots, n$, and $\text{supp } g_j \cap \text{supp } g_i = \emptyset$ if $i \neq j$. Thanks to this special choice of the functions g_j we find

$$\begin{aligned}
&\alpha_p(h, H_p(\mu_1, \mathbb{R}^{d_1}), H_p(\mu_2, \mathbb{R}^{d_2})) \\
&\leq \left(\sum_{i=1}^n \|I_{1/\mu_1} f_i|_{H_p(\mu_1, \mathbb{R}^{d_1})}\|^p \right)^{1/p} \cdot \sup_{\|\eta\|_p \leq 1} \left\| \sum_{i=1}^n \eta_i I_{1/\mu_2} g_i \Big|_{H_p(\mu_2, \mathbb{R}^{d_2})} \right\| \\
&= \left(\sum_{i=1}^n \|f_i|_{L_p(\mathbb{R}^{d_1})}\|^p \right)^{1/p} \cdot \sup_{\|\eta\|_p \leq 1} \left\| \sum_{i=1}^n \eta_i g_i \Big|_{L_p(\mathbb{R}^{d_2})} \right\| \\
&= \left(\sum_{i=1}^n \|f_i|_{L_p(\mathbb{R}^{d_1})}\|^p \right)^{1/p} \cdot \sup_{\|\eta\|_p \leq 1} \left(\sum_{i=1}^n |\eta_i|^p \|g_i|_{L_p(\mathbb{R}^{d_2})}\|^p \right)^{1/p} \\
&= \left(\sum_{i=1}^n \|f_i|_{L_p(\mathbb{R})}\|^p \right)^{1/p}.
\end{aligned}$$

On the other hand we conclude

$$\begin{aligned}
\|h|H_p(\mu_1 \otimes \mu_2, \mathbb{R}^d)\|^p &= \left\| \sum_{j=1}^n f_j \otimes g_j \Big|_{L_p(\mathbb{R}^d)} \right\|^p \\
&= \sum_{j=1}^n \int_{\mathbb{R}^{d_1}} |f_j(x)|^p \|g_j\|_{L_p(\mathbb{R}^{d_2})}^p dx \\
&= \sum_{i=1}^n \|f_i\|_{L_p(\mathbb{R})}^p.
\end{aligned}$$

Hence, the special choice of the functions g_j implies

$$\|h|H_p(\mu_1 \otimes \mu_2, \mathbb{R}^d)\| = \left(\sum_{j=1}^n \|f_j\|_{L_p(\mathbb{R})}^p \right)^{1/p}.$$

With (3.4) we obtain

$$\alpha_p(h, H_p(\mu_1, \mathbb{R}^{d_1}), H_p(\mu_2, \mathbb{R}^{d_2})) \leq \|h|H_p(\mu_1 \otimes \mu_2, \mathbb{R}^d)\|$$

for all such functions $h \in I_{(1/(\mu_1 \otimes \mu_2))}K(\mathbb{R}^d)$. A density argument, see Lemma 3.2, completes the proof of (3.3). \square

Proof of Theorem 2.1. Choosing either

$$\mu_1(\eta) := (1 + |\eta|^2)^{r_1/2} \quad \text{and} \quad \mu_2(\xi) := \prod_{i=1}^d (1 + |\xi_i|^2)^{r_{i+1}/2},$$

$\eta \in \mathbb{R}, \xi \in \mathbb{R}^d$, or

$$\mu_1(\xi) := \prod_{i=1}^d (1 + |\xi_i|^2)^{r_i/2} \quad \text{and} \quad \mu_2(\eta) := (1 + |\eta|^2)^{r_{d+1}/2}$$

then the theorem follows immediately from Proposition 3.1. \square

3.2 Tensor products of weighted sequence spaces

Let I be a countable index set. Let $w = (w(j))_{j \in I}$ be a sequence of positive real numbers. Let $0 < p < \infty$. Then $\ell_p(w, I)$ consists of all sequences $a = (a_j)_{j \in I}$ of complex numbers such that

$$\|a\|_{\ell_p(w, I)} := \left(\sum_{j \in I} |a_j w(j)|^p \right)^{1/p} < \infty.$$

Clearly, $\ell_p(w_1 \otimes w_2, \mathbb{N}^2)$ means the collection of all sequences $(a_{j,k})_{j,k \in \mathbb{N}}$ such that

$$\|a\|_{\ell_p(w_1 \otimes w_2, \mathbb{N}^2)} := \left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_{j,k} w_1(j) w_2(k)|^p \right)^{1/p} < \infty.$$

Furthermore, $c_0(w, I)$ denotes the closure of the set of finite sequences with respect to the norm

$$\|a\|_{c_0(w, I)} := \|a\|_{\ell_\infty(w, I)} = \sup_{j \in I} |a_j w(j)|.$$

Proposition 3.3 *Let $1 < p < \infty$. Then*

$$\ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N}) = \ell_p(w_1 \otimes w_2, \mathbb{N}^2). \quad (3.4)$$

The norms on the left-hand side and on the right-hand side coincide.

Remark Formula (3.4) represents a special case of the more general formula

$$L_p(\mu_1) \otimes_{\alpha_p} L_p(\mu_2) = L_p(\mu_1 \otimes \mu_2).$$

valid for arbitrary measures μ_1 and μ_2 , see Defant and Floret [8, 7.2, p. 79, 186].

For the convenience of the reader we shall give an elementary proof of Proposition 3.3. The only nontrivial fact we will use within this proof is $(\ell_p)' = \ell_{p'}$ where $1/p + 1/p' = 1$.

Proof of Proposition 3.3 . Step 1. We shall prove

$$\ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N}) \hookrightarrow \ell_p(w_1 \otimes w_2, \mathbb{N}^2).$$

Let $h \in \ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N})$ be given by

$$h = (h_{k,\ell})_{k,\ell}, \quad h_{k,\ell} = \sum_{i=1}^n a_k^i b_\ell^i, \quad k, \ell \in \mathbb{N},$$

where $(a_k^i)_k \in \ell_p(w_1, \mathbb{N})$, $(b_\ell^i)_\ell \in \ell_p(w_2, \mathbb{N})$, $i = 1, \dots, n$. Then, using $(\ell_p)' = \ell_{p'}$ where $1/p + 1/p' = 1$ and Hölder's inequality, we obtain

$$\begin{aligned}
& \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} w_1^p(k) w_2^p(\ell) \left| \sum_{i=1}^n a_k^i b_\ell^i \right|^p = \sum_{k=1}^{\infty} w_1^p(k) \left\| \left(\sum_{i=1}^n w_2(\ell) a_k^i b_\ell^i \right)_\ell \right\|_{\ell_p}^p \\
&= \sum_{k=1}^{\infty} w_1^p(k) \sup_{\|\delta\|_{\ell_{p'}} \leq 1} \left| \sum_{i=1}^n \sum_{\ell=1}^{\infty} w_2(\ell) \delta_\ell a_k^i b_\ell^i \right|^p \\
&\leq \sum_{k=1}^{\infty} w_1^p(k) \left(\sum_{i=1}^n |a_k^i|^p \right) \sup_{\|\delta\|_{\ell_{p'}} \leq 1} \left(\sum_{i=1}^n \left| \sum_{\ell=1}^{\infty} w_2(\ell) \delta_\ell b_\ell^i \right|^{p'} \right)^{p/p'} \\
&= \left(\sum_{i=1}^n \|a^i\|_{\ell_p(w_1, \mathbb{N})}^p \right) \sup_{\|\delta\|_{\ell_{p'}} \leq 1} \sup_{\|\lambda\|_{\ell_p} \leq 1} \left| \sum_{i=1}^n \lambda_i \sum_{\ell=1}^{\infty} w_2(\ell) \delta_\ell b_\ell^i \right|^p \\
&= \left(\sum_{i=1}^n \|a^i\|_{\ell_p(w_1, \mathbb{N})}^p \right) \sup_{\|\lambda\|_{\ell_p} \leq 1} \left\| \sum_{i=1}^n \lambda_i b^i \right\|_{\ell_p(w_2, \mathbb{N})}^p.
\end{aligned}$$

Since this is true for all representations of h we conclude

$$\|h\|_{\ell_p(w_1 \otimes w_2, \mathbb{N}^2)} \leq \|h\|_{\ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N})}.$$

Step 2. Now we deal with

$$\ell_p(w_1 \otimes w_2, \mathbb{N}^2) \hookrightarrow \ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N}).$$

Therefore, let $h = (h_{k,\ell})_{k,\ell} \in \ell_p(w_1 \otimes w_2, \mathbb{N}^2)$ such that only finitely many components are not vanishing. Let $h_{k,\ell} = 0$ if either $k > M$ or if $\ell > N$. Then

$$h = \sum_{k=1}^M \sum_{\ell=1}^N h_{k,\ell} (e_k \otimes e_\ell) = \sum_{\ell=1}^N \left(\sum_{k=1}^M h_{k,\ell} w_2(\ell) e_k \right) \otimes (1/w_2(\ell)) e_\ell, \quad (3.5)$$

where e_k denotes the elements of the canonical basis. Let $a_\ell := \sum_{k=1}^M h_{k,\ell} e_k$. It follows

$$\begin{aligned}
& \|h\|_{\ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N})}^p \\
&\leq \left(w_2(\ell)^p \sum_{\ell=1}^N \|a_\ell\|_{\ell_p(w_1, \mathbb{N})}^p \right) \sup_{\|\lambda\|_{\ell_p} \leq 1} \left\| \sum_{\ell=1}^N \lambda_\ell \frac{e_\ell}{w_2(\ell)} \right\|_{\ell_p(w_2, \mathbb{N})}^p \\
&= \sum_{\ell=1}^N \sum_{k=1}^M w_2^p(\ell) |h_{k,\ell} w_1(k)|^p.
\end{aligned}$$

Hence

$$\|h\|_{\ell_p(w_1, \mathbb{N}) \otimes_{\alpha_p} \ell_p(w_2, \mathbb{N})} \leq \|h\|_{\ell_p(w_1 \otimes w_2, \mathbb{N}^2)}.$$

A density argument completes the proof. \square

Proposition 3.4 *Let $0 < p \leq 1$. Then*

$$\ell_p(w_1, \mathbb{N}) \otimes_{\gamma_p} \ell_p(w_2, \mathbb{N}) = \ell_p(w_1 \otimes w_2, \mathbb{N}^2). \quad (3.6)$$

The quasi-norms on the left-hand side and on the right-hand side coincide.

Remark (i) Formula (3.6) with $p = 1$ is well-known. We refer to [20, Cor. 1.16].
(ii) In the framework of a more general concept of tensor products of quasi-Banach spaces, Nitsche [22] has proved a similar result for the unweighted case.

Proof of Proposition 3.4. *Step 1.* We shall prove

$$\ell_p(w_1, \mathbb{N}) \otimes_{\gamma_p} \ell_p(w_2, \mathbb{N}) \hookrightarrow \ell_p(w_1 \otimes w_2, \mathbb{N}^2).$$

Let $h \in \ell_p(w_1, \mathbb{N}) \otimes \ell_p(w_2, \mathbb{N})$ be given by

$$h = (h_{k,\ell})_{k,\ell}, \quad h_{k,\ell} = \sum_{i=1}^n a_k^i b_\ell^i, \quad k, \ell \in \mathbb{N},$$

where $(a_k^i)_k \in \ell_p(w_1, \mathbb{N})$, $(b_\ell^i)_\ell \in \ell_p(w_2, \mathbb{N})$, $i = 1, \dots, n$. Then the elementary inequality $(\sum_i |c_i|)^p \leq \sum_i |c_i|^p$ yields

$$\begin{aligned} \|h\|_{\ell_p(w_1 \otimes w_2, \mathbb{N}^2)}^p &= \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} w_1(k)^p w_2(\ell)^p \left| \sum_{i=1}^n a_k^i b_\ell^i \right|^p \\ &\leq \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} w_1(k)^p w_2(\ell)^p \sum_{i=1}^n |a_k^i|^p \cdot |b_\ell^i|^p \\ &= \sum_{i=1}^n \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} w_1(k)^p w_2(\ell)^p |a_k^i|^p \cdot |b_\ell^i|^p \\ &= \sum_{i=1}^n \|a^i\|_{\ell_p(w_1, \mathbb{N})}^p \cdot \|b^i\|_{\ell_p(w_2, \mathbb{N})}^p. \end{aligned}$$

Since this is true for all representations of h we conclude

$$\|h\|_{\ell_p(w_1 \otimes w_2, \mathbb{N}^2)} \leq \gamma_p(h, \ell_p(w_1, \mathbb{N}), \ell_p(w_2, \mathbb{N})).$$

Step 2. It remains to prove

$$\ell_p(w_1 \otimes w_2, \mathbb{N}^2) \hookrightarrow \ell_p(w_1, \mathbb{N}) \otimes_{\gamma_p} \ell_p(w_2, \mathbb{N}).$$

We follow the arguments from Step 2 in the proof of the previous Proposition. By the same density argument it will be enough to deal with finite sequences. Therefore, let

$$h = \sum_{k=1}^M \sum_{\ell=1}^N h_{k,\ell} (e_k \otimes e_\ell) = \sum_{\ell=1}^N \left(\sum_{k=1}^M h_{k,\ell} w_2(\ell) e_k \right) \otimes (1/w_2(\ell)) e_\ell.$$

Then we obtain

$$\begin{aligned}
& \| h |_{\ell_p(w_1, \mathbb{N})} \otimes_{\gamma_p} \ell_p(w_2, \mathbb{N}) \| ^p \\
& \leq \sum_{\ell=1}^N \left\| \sum_{k=1}^M h_{k,\ell} w_2(\ell) e_k \right|_{\ell_p(w_1, \mathbb{N})} \|^p \| e_\ell / w_2(\ell) |_{\ell_p(w_2, \mathbb{N})} \|^p \\
& = \sum_{\ell=1}^N \sum_{k=1}^M |w_2(\ell)|^p |w_1(k)|^p |h_{k,\ell}|^p \\
& = \| h |_{\ell_p(w_1 \otimes w_2, \mathbb{N}^2)} \| ^p .
\end{aligned}$$

This proves the claim. \square

Proposition 3.5 *Let $p = \infty$. Then*

$$c_0(w_1, \mathbb{N}) \otimes_\lambda c_0(w_2, \mathbb{N}) = c_0(w_1 \otimes w_2, \mathbb{N}^2). \quad (3.7)$$

The norms on the left-hand side and on the right-hand side coincide.

Proof. Let $h \in c_0(w_1, \mathbb{N}) \otimes c_0(w_2, \mathbb{N})$ be given by

$$h = (h_{k,\ell})_{k,\ell}, \quad h_{k,\ell} = \sum_{i=1}^n a_k^i b_\ell^i, \quad k, \ell \in \mathbb{N},$$

where $a^i := (a_k^i)_k \in c_0(w_1, \mathbb{N})$, $b^i := (b_\ell^i)_\ell \in c_0(w_2, \mathbb{N})$ and $i = 1, \dots, n$. Here we suppose $a_k^i = b_\ell^i = 0$ if $k, \ell > n$. Let $X = c_0(w_1, \mathbb{N})$. Obviously,

$$\begin{aligned}
\| h |_{c_0(w_1 \otimes w_2, \mathbb{N}^2)} \| &= \sup_{k,\ell \in \mathbb{N}} w_1(k) w_2(\ell) \left| \sum_{i=1}^n a_k^i b_\ell^i \right| \\
&= \sup_{\ell=1,\dots,n} w_2(\ell) \left\| \left(\sum_{i=1}^n a_k^i b_\ell^i \right)_k \right|_{c_0(w_1, \mathbb{N})} \right\| \\
&= \sup_{\ell=1,\dots,n} w_2(\ell) \sup_{\|\psi\|_{X'} \leq 1} \left| \psi \left(\left(\sum_{i=1}^n a_k^i b_\ell^i \right)_k \right) \right| \\
&= \sup_{\|\psi\|_{X'} \leq 1} \sup_{\ell=1,\dots,n} w_2(\ell) \left| \sum_{i=1}^n \psi(a^i) b_\ell^i \right| \\
&= \lambda(h, c_0(w_1, \mathbb{N}), c_0(w_2, \mathbb{N})).
\end{aligned}$$

Vice versa, if

$$h = \sum_{k=1}^M \sum_{\ell=1}^N h_{k,\ell} (e_k \otimes e_\ell) = \sum_{\ell=1}^N \left(\sum_{k=1}^M h_{k,\ell} w_2(\ell) e_k \right) \otimes (1/w_2(\ell)) e_\ell.$$

(we put $h_{k,\ell} = 0$ if either $\ell > N$ or $k > M$) and using the abbreviations

$$a^\ell := \sum_{k=1}^M h_{k,\ell} w_2(\ell) e_k \quad \text{and} \quad b^\ell := (1/w_2(\ell)) e_\ell,$$

we obtain

$$\begin{aligned}
\|h|_{c_0(w_1, \mathbb{N})} \otimes_\lambda c_0(w_2, \mathbb{N})\| &= \sup_{\|\psi|_{X'}\| \leq 1} \left\| \sum_{\ell=1}^N \psi(a^\ell) b^\ell \Big|_{c_0(w_2, \mathbb{N})} \right\| \\
&= \sup_{\|\psi|_{X'}\| \leq 1} \sup_{\ell=1, \dots, N} |\psi(a^\ell)| \\
&= \sup_{\ell \in \mathbb{N}} \left\| \sum_{k=1}^M h_{k, \ell} w_2(\ell) e_k \Big|_{c_0(w_1, \mathbb{N})} \right\| \\
&= \sup_{\ell \in \mathbb{N}} w_2(\ell) \sup_{k \in \mathbb{N}} w_1(k) |h_{k, \ell}| \\
&= \|h|_{c_0(w_1 \otimes w_2, \mathbb{N}^2)}\|.
\end{aligned}$$

A density argument completes the proof of (3.7). \square

We formulate a few consequences of Propositions 3.3, 3.4, 3.5 for more specialized sequence spaces, see Definition 2.3. Switching in these propositions from \mathbb{N} to $\mathbb{N}_0 \times \mathbb{Z}$ and from \mathbb{N} to $\mathbb{N}_0^d \times \mathbb{Z}^d$ (simply by renumbering) and selecting the appropriate weights we obtain the following.

Corollary 3.6 *Let $r_1, \dots, r_d, r_{d+1} \in \mathbb{R}$.*

(i) *Let $1 < p < \infty$. Then*

$$b_p^{r_1} \otimes_{\alpha_p} s_p^{r_2, \dots, r_{d+1}} b = s_p^{r_1, \dots, r_d} b \otimes_{\alpha_p} b_p^{r_{d+1}} = s_p^{r_1, r_2, \dots, r_{d+1}} b.$$

(ii) *Let $0 < p \leq 1$. Then*

$$b_p^{r_1} \otimes_{\gamma_p} s_p^{r_2, \dots, r_{d+1}} b = s_p^{r_1, \dots, r_d} b \otimes_{\gamma_p} b_p^{r_{d+1}} = s_p^{r_1, r_2, \dots, r_{d+1}} b.$$

(iii) *Let $p = \infty$. Then*

$$\mathring{b}_\infty^{r_1} \otimes_\lambda \mathring{s}_\infty^{r_2, \dots, r_{d+1}} b = \mathring{s}_\infty^{r_1, \dots, r_d} b \otimes_\lambda \mathring{b}_\infty^{r_{d+1}} = \mathring{s}_\infty^{r_1, r_2, \dots, r_{d+1}} b.$$

In (i)-(iii) all quasi-norms coincide.

Now we investigate the counterpart of Corollary 3.6 for the spaces f_p^r .

Lemma 3.7 *Let $1 < p < \infty$ and $r_1, \dots, r_{d+1} \in \mathbb{R}$. Then*

$$f_p^{r_1} \otimes_{\alpha_p} s_p^{r_2, \dots, r_{d+1}} f = s_p^{r_1, \dots, r_d} f \otimes_{\alpha_p} f_p^{r_{d+1}} = s_p^{r_1, r_2, \dots, r_{d+1}} f$$

in the sense of equivalent norms.

Proof. We argue similar as in the proof of Theorem 2.2 below. We combine Propositions A.8, A.9 with the observation that the isomorphism J_d used in

Proposition A.9 is the tensor product of the isomorphisms J used in Proposition A.8. This is a simple consequence of the fact that J_d^{-1} and $J^{-1} \otimes \dots \otimes J^{-1}$ coincide on the space $f_p^{r_1} \otimes \dots \otimes f_p^{r_d}$. Hence, they coincide on $f_p^{r_1} \otimes_{\alpha_p} \dots \otimes_{\alpha_p} f_p^{r_d}$. From Proposition A.9 we know

$$J_d : S_p^{r_1, \dots, r_d} H(\mathbb{R}^d) \rightarrow s_p^{r_1, \dots, r_d} f$$

and by general properties of tensor product operators and Proposition A.8 we conclude

$$J_d : H_p^{r_1}(\mathbb{R}) \otimes_{\alpha_p} \dots \otimes_{\alpha_p} H_p^{r_d}(\mathbb{R}) \rightarrow f_p^{r_1} \otimes_{\alpha_p} \dots \otimes_{\alpha_p} f_p^{r_d}.$$

The notation has to be understood as the iterated tensor product, see the remark after Theorem 2.1. Since $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d) = H_p^{r_1}(\mathbb{R}) \otimes_{\alpha_p} \dots \otimes_{\alpha_p} H_p^{r_d}(\mathbb{R})$, see Theorem 2.1, we have done. \square

3.3 Tensor products of Besov spaces

The heart of the matter consists in the assertions on tensor products of weighted sequence spaces of the previous paragraph.

Proof of Theorem 2.2 As in proof of Lemma 3.7 we first combine Propositions A.7, A.9 with the observation that the isomorphism J_d used in Proposition A.9 is the tensor product of the isomorphisms J used in Proposition A.7. This follows from the coincidence of J_d^{-1} and $J^{-1} \otimes \dots \otimes J^{-1}$ on the space $b_p^{r_1} \otimes \dots \otimes b_p^{r_d}$. Hence, they coincide on $b_p^{r_1} \otimes_{\delta_p} \dots \otimes_{\delta_p} b_p^{r_d} = s_p^{r_1, \dots, r_d} b$ (Corollary 3.6(i),(ii)), where $\delta_p := \alpha_p$ if $1 < p < \infty$ and $\delta_p := \gamma_p$ if $0 < p \leq 1$. Therefore, $J^{-1} \otimes \dots \otimes J^{-1}$ yields an isomorphism of $b_p^{r_1} \otimes_{\delta_p} \dots \otimes_{\delta_p} b_p^{r_d}$ onto $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$. On the other hand we know that $J \otimes \dots \otimes J$ is an isomorphism of $B_{p,p}^{r_1}(\mathbb{R}) \otimes_{\delta_p} \dots \otimes_{\delta_p} B_{p,p}^{r_d}(\mathbb{R})$ onto $b_p^{r_1} \otimes_{\delta_p} \dots \otimes_{\delta_p} b_p^{r_d}$ using Lemma B.1, Lemma B.6 and Proposition A.7. But this means that $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ and $B_{p,p}^{r_1}(\mathbb{R}) \otimes_{\delta_p} \dots \otimes_{\delta_p} B_{p,p}^{r_d}(\mathbb{R})$ coincide (up to equivalent quasi-norms). This proves Theorem 2.2(i),(iii). To prove (ii) we use a similar argument, now in connection with Corollary 3.6(iii) and Subsection 3.6. \square

3.4 Sequence space isomorphisms

Proof of Theorem 2.5. It is enough to combine Theorem 2.1, Theorem 2.2 with Lemma B.1 and Lemma B.6. \square

3.5 Nonlinear approximation spaces

Proof of Theorem 2.7. By means of [9, Thm. 4] we know that $f \in A_p^{\frac{1}{p}-\frac{1}{2}}(L_2(\mathbb{R}^d))$ if and only if $(\langle f, \psi_{\bar{j}, \bar{k}} \rangle)_{\bar{j}, \bar{k}} \in \ell_p$ and

$$\|f|A_p^{\frac{1}{p}-\frac{1}{2}}(L_2(\mathbb{R}^d))\| \asymp \|(\langle f, \psi_{\bar{j}, \bar{k}} \rangle)_{\bar{j}, \bar{k}}| \ell_p\|$$

In view of Corollary 2.6(ii) we see

$$\|(\langle f, \psi_{\bar{j}, \bar{k}} \rangle)_{\bar{j}, \bar{k}}| \ell_p\| \asymp \|(\langle f, \psi_{\bar{j}, \bar{k}} \rangle)_{\bar{j}, \bar{k}}| S_p^{\frac{1}{p}-\frac{1}{2}, \dots, \frac{1}{p}-\frac{1}{2}} b\| \asymp \|f|S_{p,p}^{\frac{1}{p}-\frac{1}{2}, \dots, \frac{1}{p}-\frac{1}{2}} B(\mathbb{R}^d)\|.$$

The proof is complete. \square

3.6 Spline wavelet isomorphisms

Proof of Proposition 2.8. *Step 1.* The assertion in part (i) is covered by Theorems 3.5 and 3.7 in Frazier and Jawerth [12] (see also the remarks on the top of page 132 concerning the inhomogeneous counterparts), since the functions $\psi_{j,k}^m$ are smooth molecules.

Step 2. For part (ii) we refer to Bourdaud [5], for $p \geq 1$ also to Lemarie and Kahane [19, Part II, Chapt. 6, Thm. 5] and DeVore [9]. Only the case $p = \infty$ requires an additional comment. The references cover $R_m \in \mathcal{L}(B_{\infty, \infty}^r(\mathbb{R}), b_{\infty}^r)$ as well as

$$\left\| \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} a_{j,k} \psi_{j,k}^m \Big| B_{\infty, \infty}^r(\mathbb{R}) \right\| \leq c \| (a_{j,k})_{j,k} | b_{\infty}^r \|$$

with c independent of $(a_{j,k})_{j,k}$ and $-m < r < m - 1$. Furthermore, observe that the space \mathring{b}_{∞}^r consists of all sequences $a = (a_{j,k})_{j,k} \in b_{\infty}^r$ such that

$$\lim_{j+|k| \rightarrow \infty} 2^{j(r+1/2)} |a_{j,k}| = 0. \quad (3.8)$$

Obviously, the image of $f \in C_0^{\infty}(\mathbb{R})$ under the mapping R_m belongs to b_{∞}^s for any $s < m - 1$. This and the fact that

$$\lim_{|k| \rightarrow \infty} \langle f, \psi_{j,k}^m \rangle = 0 \quad , \quad j \in \mathbb{N}_0 \quad ,$$

imply (3.8) with $a_{j,k} := \langle f, \psi_{j,k}^m \rangle$. Therefore, we have $R_m(C_0^{\infty}(\mathbb{R})) \subset \mathring{b}_{\infty}^r$ with $r < m - 1$. Vice versa, the functions φ_m, ψ_m belong to $C^{m-2}(\mathbb{R})$ and the derivatives $\varphi_m^{(m-2)}, \psi_m^{(m-2)}$ are uniformly Lipschitz continuous and rapidly decaying. Let $\varrho, \omega \in C_0^{\infty}(\mathbb{R})$ be functions such that $\text{supp } \varrho, \text{supp } \omega$ are compact, $\varrho(x) = 1$ if $|x| \leq 1$, and $\int \omega(x) dx = 1$. Then it is not difficult to see that

$$\varrho(\lambda \cdot) \left(\frac{1}{\varepsilon} \omega\left(\frac{\cdot}{\varepsilon}\right) * \varphi \right) \xrightarrow{\lambda, \varepsilon \rightarrow 0} \varphi \quad \text{as well as} \quad \varrho(\lambda \cdot) \left(\frac{1}{\varepsilon} \omega\left(\frac{\cdot}{\varepsilon}\right) * \psi \right) \xrightarrow{\lambda, \varepsilon \rightarrow 0} \psi$$

holds with respect to $\|\cdot\|_{B_{\infty,\infty}^r(\mathbb{R})}$ for $r < m - 1$. Assume now $\{a_{j,k}\}_{j,k} \in \mathring{b}_{\infty}^r$ and define

$$f_N := \sum_{j=0}^N \sum_{|k| < N} a_{j,k} \psi_{j,k}^m \in \mathring{B}_{\infty,\infty}^r(\mathbb{R}), \quad N \in \mathbb{N}.$$

Finally, observe for $N < M$

$$\begin{aligned} \left\| f_N - f_M \right\|_{B_{\infty,\infty}^r(\mathbb{R})} &\leq c \left(\sup_{j \geq N+1} 2^{j(r+1/2)} \sup_{k \in \mathbb{Z}} |a_{j,k}| \right. \\ &\quad \left. + \sup_{j=0,\dots,N} 2^{j(r+1/2)} \sup_{|k| \geq N} |a_{j,k}| \right) \xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

by the given characterization of \mathring{b}_{∞}^r . This proves the norm convergence of the sequence of partial sums $\{f_N\}_N$ in $\mathring{B}_{\infty,\infty}^r(\mathbb{R})$. The limit is called f and by standard arguments we have $\langle f, \psi_{j,k}^m \rangle = a_{j,k}$ for all j, k . Hence $R_m(\mathring{B}_{\infty,\infty}^r(\mathbb{R})) = \mathring{b}_{\infty}^r$. \square

Proof of Theorem 2.9. Theorem 2.9 follows from Proposition 2.8 combined with Corollary 2.6. \square

3.7 Error of best approximation

To begin with we shall prove Proposition 2.10. In case $1 < p < \infty$ our main tool will be Littlewood-Paley theory in a form developed in [10].

Lemma 3.8 *Let $1 < p < \infty$. Let $m \in \mathbb{N}$. Then*

$$\begin{aligned} \left\| \sum_{\bar{j} \in \mathbb{N}_0^d} \sum_{\bar{k} \in \mathbb{Z}^d} a_{\bar{j},\bar{k}} \psi_{\bar{j},\bar{k}}^m \right\|_{L_p(\mathbb{R}^d)} &\asymp \left\| \left(\sum_{\bar{j} \in \mathbb{N}_0^d} \sum_{\bar{k} \in \mathbb{Z}^d} |a_{\bar{j},\bar{k}}|^2 \chi_{\bar{j},\bar{k}}^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \\ &\asymp \left\| \left(\sum_{\bar{j} \in \mathbb{N}_0^d} \sum_{\bar{k} \in \mathbb{Z}^d} |a_{\bar{j},\bar{k}}|^2 (\psi_{\bar{j},\bar{k}}^m)^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \end{aligned}$$

holds for all finite sequences $(a_{\bar{j},\bar{k}})_{\bar{j},\bar{k}}$.

Proof. Let $m = 1$. Then ψ_1 is essentially the Haar wavelet. For the Haar system the claim is explicitly stated and proved in [10] (see the comments before Lemma 2.2).

Let $m \geq 2$. Then ψ_m is uniformly Lipschitz continuous, exponentially decaying and satisfies a moment condition. Hence, Theorem 4.4 in [10] is applicable.

Since $\Psi^m = (\psi_{\bar{j}, \bar{k}}^m)_{\bar{j}, \bar{k}}$ is an orthonormal system we may apply Corollary 4.5 in [10] as well. \square

Proof of Proposition 2.10. *Step 1.* Proof of (i). Employing Lemma 3.8 we obtain

$$\begin{aligned} \|f - P_n^m f\|_{L_p(\mathbb{R}^d)} &= \left\| \sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \psi_{\bar{j}, \bar{k}}^m \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c \left\| \left(\sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle|^2 \mathcal{X}_{\bar{j}, \bar{k}}^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c 2^{-nr} \| \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \|_{s_p^r f}. \end{aligned}$$

This proves the estimate from above in (i). Now we turn to the estimate from below. Our test functions are

$$g_n := 2^{-(n+1)(r+\frac{1}{2}-\frac{1}{p})} \psi_{\bar{j}, \bar{0}} \quad \text{for some } |\bar{j}|_1 = n+1.$$

Because of $\|g_n\|_{S_p^r H(\mathbb{R}^d)} \asymp 1$, $\|g_n\|_{L_p(\mathbb{R}^d)} \asymp 2^{-(n+1)r}$, see Theorem 2.9, and $P_n^m g_n = 0$ we obtain

$$2^{-nr} \leq c \|I - P_n^m\|_{\mathcal{L}(S_p^r H(\mathbb{R}^d), L_p(\mathbb{R}^d))}$$

for some positive constant c independent of n . This completes the proof of (i). *Step 2.* Let $1 < p \leq 2$. We shall use the elementary inequality

$$\|(f_\ell)_\ell\|_{L_p(\ell_2)} \leq \|(f_\ell)_\ell\|_{\ell_{\min(p,2)}(L_p)}$$

together with Lemma 3.8. Then

$$\begin{aligned} \|f - P_n^m f\|_{L_p(\mathbb{R}^d)} &= \left\| \sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \psi_{\bar{j}, \bar{k}}^m \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c \left\| \left(\sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle|^2 \mathcal{X}_{\bar{j}, \bar{k}}^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \\ &\leq c \left(\sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle|^p \| \mathcal{X}_{\bar{j}, \bar{k}} \|_{L_p(\mathbb{R}^d)}^p \right)^{1/p} \\ &= c \left(\sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle|^p 2^{|\bar{j}|_1(\frac{1}{2}-\frac{1}{p})p} \right)^{1/p} \\ &\leq c 2^{-nr} \| \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \|_{s_p^r b}. \end{aligned}$$

This proves the estimate from above in part (ii) under the given restrictions.

Step 3. Let $0 < p \leq 1$. Using the elementary inequality $(\sum_j |c_j|)^p \leq \sum_j |c_j|^p$ we find

$$\begin{aligned}
\left\| \sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \psi_{\bar{j}, \bar{k}}^m \Big| L_p(\mathbb{R}^d) \right\|^p &\leq \sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle|^p \|\psi_{\bar{j}, \bar{k}}^m\|_{L_p(\mathbb{R}^d)}^p \\
&\asymp \sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} 2^{|\bar{j}|_1(\frac{1}{2} - \frac{1}{p})p} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle|^p \\
&\leq 2^{-nrp} \|(\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle)_{\bar{j}, \bar{k}}\|_{s_p^r b}^p.
\end{aligned}$$

Hence, the sequence $(\sum_{|\bar{j}|_1 \leq n} \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \psi_{\bar{j}, \bar{k}}^m)_n$ converges in $L_p(\mathbb{R}^d)$. But the same sequence converges in $L_q(\mathbb{R}^d)$, where

$$\frac{1}{q} < \frac{1}{p} - r \quad \text{and} \quad 1 < q < \infty,$$

This follows from $S_{p,p}^r B(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d)$, see e.g. [31, 2.4.1] or [14], and Step 2. Hence, also with respect to the L_p -norm the limit of this sequence is $f \in S_{p,p}^r B(\mathbb{R}^d)$ itself. This implies the estimate from above in part (i) with $0 < p \leq 1$.

Step 4. Let $2 < p < \infty$. We argue as in Step 1. In addition we shall apply Hölder's inequality with $1/2 = 1/p + 1/u$. With $f_{\bar{j}} = (\sum_{\bar{k} \in \mathbb{Z}^d} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle|^2 \mathcal{X}_{\bar{j}, \bar{k}}^2)^{1/2}$ it follows

$$\begin{aligned}
\|f - P_n^m f\|_{L_p(\mathbb{R}^d)} &= \left\| \sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \psi_{\bar{j}, \bar{k}}^m \Big| L_p(\mathbb{R}^d) \right\| \\
&\leq c_1 \left\| \left(\sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle|^2 \mathcal{X}_{\bar{j}, \bar{k}}^2 \right)^{1/2} \Big| L_p(\mathbb{R}^d) \right\| \\
&= c_1 \left\| \left(\sum_{|\bar{j}|_1 > n} f_{\bar{j}}^2 \right)^{1/2} \Big| L_p(\mathbb{R}^d) \right\| \\
&\leq c_1 \left(\sum_{|\bar{j}|_1 > n} \|f_{\bar{j}}\|_{L_p(\mathbb{R}^d)}^2 \right)^{1/2} \\
&\leq c_1 \left(\sum_{|\bar{j}|_1 > n} 2^{|\bar{j}|_1 r p} \|f_{\bar{j}}\|_{L_p(\mathbb{R}^d)}^p \right)^{1/p} \left(\sum_{|\bar{j}|_1 > n} 2^{-|\bar{j}|_1 r u} \right)^{1/u} \\
&\leq c_2 n^{(d-1)/u} 2^{-nr} \left(\sum_{|\bar{j}|_1 > n} 2^{|\bar{j}|_1 r p} \|f_{\bar{j}}\|_{L_p(\mathbb{R}^d)}^p \right)^{1/p}.
\end{aligned}$$

Since

$$\|f_{\bar{j}}\|_{L_p(\mathbb{R}^d)}^p = 2^{|\bar{j}|_1(\frac{1}{2} - \frac{1}{p})p} \sum_{\bar{k} \in \mathbb{Z}^d} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle|^p$$

the estimate from above in (ii) follows.

Step 5. It remains to deal with $p = \infty$. We find

$$\begin{aligned}
\|f - P_n^m f\|_{L_\infty(\mathbb{R}^d)} &= \left\| \sum_{|\bar{j}|_1 > n} \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \psi_{\bar{j}, \bar{k}}^m \right\|_{L_\infty(\mathbb{R}^d)} \\
&\leq \sum_{|\bar{j}|_1 > n} \left\| \sum_{\bar{k} \in \mathbb{Z}^d} \langle f, \psi_{\bar{j}, \bar{k}}^m \rangle \psi_{\bar{j}, \bar{k}}^m \right\|_{L_\infty(\mathbb{R}^d)} \\
&\leq c_1 \sum_{|\bar{j}|_1 > n} 2^{|\bar{j}|_1/2} \sup_{\bar{k} \in \mathbb{Z}^d} |\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle| \\
&\leq c_1 \left(\sum_{|\bar{j}|_1 > n} 2^{-r|\bar{j}|_1} \right) \|(\langle f, \psi_{\bar{j}, \bar{k}}^m \rangle)_{\bar{j}, \bar{k}}\|_{s_\infty^r} b,
\end{aligned}$$

where we used again Section 2.4/(iv) in the third line. Since $\sum_{|\bar{j}|_1 > n} 2^{-r|\bar{j}|_1} \leq c_2 n^{d-1} 2^{-nr}$ the estimate in part (iii) follows.

Step 6. Estimate from below in (ii).

Substep 6.1. Let $0 < p \leq 2$. We shall use the same test functions as in Step 1. Because of $\|g_n\|_{S_{p,p}^r B(\mathbb{R}^d)} \asymp 1$, $\|g_n\|_{L_p(\mathbb{R}^d)} \asymp 2^{-(n+1)r}$ if $|\bar{j}|_1 = n+1$, see Theorem 2.9, and $P_n^m g_n = 0$ we obtain

$$2^{-nr} \leq c \|I - P_n^m\|_{\mathcal{L}(S_{p,p}^r B(\mathbb{R}^d), L_p(\mathbb{R}^d))}$$

for some positive constant c independent of n .

Substep 6.2. Let $2 < p < \infty$. We put

$$I_{\bar{j}} := \left\{ k \in \mathbb{Z}^d : 0 \leq k_\ell < 2^{j_\ell}, \ell = 1, \dots, d \right\}$$

and

$$f_n := n^{-(d-1)/p} 2^{-(n+1)(r+\frac{1}{2})} \sum_{|\bar{j}|_1 = n+1} \sum_{\bar{k} \in I_{\bar{j}}} \psi_{\bar{j}, \bar{k}}^m, \quad n \in \mathbb{N}.$$

Then Theorem 2.9 implies

$$\|f_n\|_{S_{p,p}^r B(\mathbb{R}^d)} \asymp n^{-(d-1)/p} \left(\sum_{|\bar{j}|_1 = n+1} 2^{-|\bar{j}|_1} |I_{\bar{j}}| \right)^{1/p} \asymp 1.$$

Furthermore, by means of Lemma 3.8,

$$\begin{aligned}
\|f_n\|_{L_p(\mathbb{R}^d)} &\asymp n^{-(d-1)/p} 2^{-(n+1)(r+\frac{1}{2})} \left\| \left(\sum_{|\bar{j}|_1 = n+1} \sum_{\bar{k} \in I_{\bar{j}}} \chi_{\bar{j}, \bar{k}}^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}^d)} \\
&\asymp n^{-(d-1)/p} 2^{-(n+1)r} \left\| \left(\sum_{|\bar{j}|_1 = n+1} 1 \right)^{1/2} \right\|_{L_p([0, 1]^d)} \\
&\asymp n^{(d-1)(\frac{1}{2} - \frac{1}{p})} 2^{-(n+1)r}.
\end{aligned}$$

This completes the proof of (ii).

Step 7. Estimate from below in (iii).

Substep 7.1. Preparations. We claim that there is at least one integer k such

that $\psi_m(k) \neq 0$, $m > 1$.

We need a few more facts about wavelets. Let $R_m(\xi) := \sum_{k=-\infty}^{\infty} |\mathcal{F}\mathcal{N}_m(\xi + 2\pi k)|^2$ be the autocorrelation function of \mathcal{N}_m . These functions R_m are trigonometric polynomials (apply Poisson's Summation formula) bounded away from zero. Our scaling functions φ_m are given by (see Section 2.4)

$$\mathcal{F}\varphi_m(\xi) = \frac{\mathcal{F}\mathcal{N}_m(\xi)}{\sqrt{2\pi R_m(\xi)}}, \quad \xi \in \mathbb{R}, \quad (3.9)$$

where

$$\mathcal{F}\mathcal{N}_m(\xi) = \frac{1}{\sqrt{2\pi}} e^{-im\xi/2} \left(\frac{\sin(\xi/2)}{\xi/2} \right)^m, \quad \xi \in \mathbb{R}, \quad (3.10)$$

and satisfy a refinement equation given by

$$\mathcal{F}\varphi_m(2\xi) = M_m(\xi) \mathcal{F}\varphi_m(\xi) \quad \text{with} \quad M_m(\xi) := \frac{\sqrt{R_m(\xi)}}{\sqrt{R_m(2\xi)}} \cos^m(\xi/2) e^{-im\xi/2}.$$

Obviously, the functions M_m are 2π periodic C^∞ functions satisfying $M_m(0) = 1$ and $M_m(\pi) = 0$. The Fourier transform of ψ_m is then given by

$$\mathcal{F}\psi_m(2\xi) = e^{i\xi} \overline{M_m(\pi + \xi)} \mathcal{F}\varphi_m(\xi), \quad \xi \in \mathbb{R}.$$

Let us further introduce the 2π -periodic functions

$$\Theta_m(\xi) := \sum_{\ell=-\infty}^{\infty} \mathcal{F}\mathcal{N}_m(\xi + 2\pi\ell), \quad \Phi_m(\xi) := \sum_{\ell=-\infty}^{\infty} \mathcal{F}\varphi_m(\xi + 2\pi\ell)$$

and

$$\Psi_m(\xi) := \sum_{\ell=-\infty}^{\infty} \mathcal{F}\psi_m(\xi + 2\pi\ell), \quad \xi \in \mathbb{R}.$$

The decay of $\mathcal{F}\mathcal{N}_m$, $m > 1$, guarantees absolute convergence of these three series (taking into account the boundedness of R_m and M_m as well) and continuity of the limit functions. The Fourier series of $\Psi_m(\xi)$ is given by $(1/\sqrt{2\pi}) \sum_{k=-\infty}^{\infty} \psi_m(k) e^{ik\xi}$, see Poisson's summation formula in [36, Cor. 7.2.6]. From the decay properties of ψ_m , see Section 2.4, it follows that this series also converges uniformly and hence $\Psi_m(\xi) = (1/\sqrt{2\pi}) \sum_{k=-\infty}^{\infty} \psi_m(k) e^{ik\xi}$ for all ξ . Observe $\Phi_m(\xi) = \Theta_m(\xi)/\sqrt{2\pi R_m(\xi)}$, see (3.9). Let us calculate the value of Ψ_m at some particular points. We obtain

$$\begin{aligned}
\Psi_m(\xi) &= e^{i\xi/2} \sum_{k=-\infty}^{\infty} (-1)^k \overline{M_m((k+1)\pi + \xi/2)} \mathcal{F}\varphi_m(\pi k + \xi/2) \\
&= e^{i\xi/2} \left(\sum_{\ell=-\infty}^{\infty} \overline{M_m((2\ell+1)\pi + \xi/2)} \mathcal{F}\varphi_m(2\pi\ell + \xi/2) \right. \\
&\quad \left. - \sum_{\ell=-\infty}^{\infty} \overline{M_m((2\ell+2)\pi + \xi/2)} \mathcal{F}\varphi_m(\pi(2\ell+1) + \xi/2) \right) \\
&= e^{i\xi/2} \left(\overline{M_m(\pi + \xi/2)} \sum_{\ell=-\infty}^{\infty} \mathcal{F}\varphi_m(2\pi\ell + \xi/2) \right. \\
&\quad \left. - \overline{M_m(\xi/2)} \sum_{\ell=-\infty}^{\infty} \mathcal{F}\varphi_m(\pi(2\ell+1) + \xi/2) \right).
\end{aligned}$$

Using the functions defined above we arrive at

$$\Psi_m(\xi) = e^{i\xi/2} \left(\overline{M_m(\pi + \xi/2)} \Phi_m(\xi/2) - \overline{M_m(\xi/2)} \Phi_m(\pi + \xi/2) \right) \quad (3.11)$$

An easy calculation using equation (3.10) gives us

$$\Theta_m(\xi) = \frac{1}{\sqrt{2\pi}} 2^m e^{-im\xi/2} \sin^m(\xi/2) \sum_{\ell=-\infty}^{\infty} \frac{1}{(\xi + 2\pi\ell)^m}, \quad \xi \in \mathbb{R}.$$

We introduce a further abbreviation for the Eisenstein series

$$\varepsilon_m(\xi) := \sum_{\ell=-\infty}^{\infty} \frac{1}{(\xi + 2\pi\ell)^m}, \quad \xi \in \mathbb{R}.$$

This yields in particular

$$\begin{aligned}
\Theta_m(\pi) &= \frac{1}{\sqrt{2\pi}} 2^m (-i)^m \varepsilon_m(\pi) \\
\Theta_m(\pi/2) &= \frac{1}{\sqrt{2\pi}} 2^m e^{-im\pi/4} \sin^m(\pi/4) \varepsilon_m(\pi/2) \\
\Theta_m(3\pi/2) &= \frac{1}{\sqrt{2\pi}} 2^m (-i)^m e^{-im\pi/4} \sin^m(\pi/4) \varepsilon_m(3\pi/2).
\end{aligned} \quad (3.12)$$

If m is even then we have $\Theta_m(\pi) \neq 0$. This and (3.11) yield

$$\Psi_m(0) = -\Phi_m(\pi) = -\frac{\Theta_m(\pi)}{\sqrt{2\pi} R_m(\pi)} \neq 0. \quad (3.13)$$

In case $m > 1$ is odd we claim $\Psi_m(\pi) \neq 0$. We argue by contradiction. Assume

$$0 = \Psi_m(\pi) = i \left(\overline{M_m(3\pi/2)} \Phi_m(\pi/2) - \overline{M_m(\pi/2)} \Phi_m(3\pi/2) \right),$$

see (3.11), then it follows

$$\frac{\overline{M_m(3\pi/2)}}{\sqrt{R_m(\pi/2)}} \frac{\Theta_m(\pi/2)}{\sqrt{R_m(\pi/2)}} = \frac{\overline{M_m(\pi/2)}}{\sqrt{R_m(3\pi/2)}} \frac{\Theta_m(3\pi/2)}{\sqrt{R_m(3\pi/2)}}.$$

Using our formula for M_m this turns out to be equivalent to

$$R_m(3\pi/2) (-i)^m \Theta_m(\pi/2) = R_m(\pi/2) \Theta_m(3\pi/2).$$

With the help of (3.12) this identity can be transformed into

$$R_m(3\pi/2) \varepsilon_m(\pi/2) = R_m(\pi/2) \varepsilon_m(3\pi/2).$$

Next we use $R_m(\xi) = (1/\sqrt{2\pi}) e^{im\xi} \Theta_{2m}(\xi)$ which gives

$$\varepsilon_{2m}(3\pi/2) \varepsilon_m(\pi/2) = \varepsilon_{2m}(\pi/2) \varepsilon_m(3\pi/2), \quad (3.14)$$

(see (3.12)). Therefore, (3.14) can not be true since $\varepsilon_{2m}(\xi) > 0$ for all ξ , $\varepsilon_m(\pi/2) > 0$ and $\varepsilon_m(3\pi/2) < 0$. This and (3.13) imply that for all $m > 1$ the continuous function Ψ_m is not identically zero and consequently

$$0 < \|\Psi_m|_{L_2(-\pi, \pi)}\|^2 = 2\pi \sum_{k=-\infty}^{\infty} |\psi_m(k)|^2.$$

Substep 7.2. The shifts $\psi_m(\cdot - k)$, $k \in \mathbb{Z}$, are, of course, also wavelets and satisfy the same list of properties as ψ_m itself. The spaces \mathcal{V}_n^m and the operators P_n^m remain unchanged if we replace the generator ψ by $\psi(\cdot - k)$ for some $k \in \mathbb{Z}$. In what follows we assume that ψ denotes a shift of ψ_m (ignoring m) such that $\psi(0) \neq 0$. Consequently there exists a positive constant c such that for all $n \in \mathbb{N}$

$$cn^{d-1} 2^{n/2} \leq \left\| \sum'_{|\bar{j}|_1=n+1} \psi_{\bar{j}, \bar{0}} \Big| L_\infty(\mathbb{R}^d) \right\|$$

holds. Here \sum' indicates that we sum only over those vectors \bar{j} satisfying $j_\ell > 0$ for all ℓ . Furthermore

$$\left\| \sum'_{|\bar{j}|_1=n+1} \psi_{\bar{j}, \bar{0}} \Big| S_{\infty, \infty}^r B(\mathbb{R}^d) \right\| \asymp 2^{n(r+1/2)}, \quad n \in \mathbb{N}.$$

Hence

$$n^{d-1} 2^{-nr} \leq c \|I - P_n^m |_{\mathcal{L}(S_{\infty, \infty}^r B(\mathbb{R}^d), L_\infty(\mathbb{R}^d))}\|$$

for some positive constant c independent of n . \square

Proof of Theorem 2.11. The Littlewood-Paley argument in Lemma 3.8 yields the uniform boundedness of $\|P_n^m |_{\mathcal{L}(L_p(\mathbb{R}^d), L_p(\mathbb{R}^d))}\|$, $n \in \mathbb{N}_0$ if $1 < p < \infty$. Since P_n^m is a projection a standard argument leads to

$$\begin{aligned} E_n^m(f, L_p(\mathbb{R}^d)) &\leq \|f - P_n^m f\|_{L_p(\mathbb{R}^d)} \\ &\leq (1 + \|P_n^m | \mathcal{L}(L_p(\mathbb{R}^d), L_p(\mathbb{R}^d))\|) E_n^m(f, L_p(\mathbb{R}^d)). \end{aligned}$$

Hence $\mathcal{E}_n^m(F)_p \asymp \|I - P_n^m | \mathcal{L}(F, L_p(\mathbb{R}^d))\|$ for any space F . This proves the theorem if $1 < p < \infty$. If $p = 1$ we use the test functions $g_n = 2^{-(n+1)(r-1/2)} \psi_{\bar{j},0}$ for a certain $|\bar{j}|_1 = n + 1$ (see Step 6 in the proof of Proposition 2.10). This gives $\|g_n | S_{1,1}^r B(\mathbb{R}^d)\| \asymp 1$, $\|g_n | L_2(\mathbb{R}^d)\| = 2^{-(n+1)(r-1/2)}$ and $\|g_n | L_\infty(\mathbb{R}^d)\| \leq c 2^{-(n+1)(r-1)}$. Let $h \in \mathcal{V}_n^m$. Then, by an argument we learned from [37, p. 252], we obtain

$$\begin{aligned} 2^{-2(n+1)(r-1/2)} &= \|g_n | L_2(\mathbb{R}^d)\|^2 = \langle g_n, g_n \rangle = \langle g_n, g_n - h \rangle \\ &\leq \|g_n - h\|_{L_1(\mathbb{R}^d)} \|g_n\|_{L_\infty(\mathbb{R}^d)} \\ &\leq c 2^{-(n+1)(r-1)} \|g_n - h\|_{L_1(\mathbb{R}^d)}, \end{aligned}$$

which implies immediately $\mathcal{E}_n^m(S_{1,1}^r B(\mathbb{R}^d))_1 \geq c 2^{-rn}$ for some c independent of n . Finally, for all values of p the quantity $\|I - P_n^m | \mathcal{L}(S_{p,p}^r B(\mathbb{R}^d), L_p(\mathbb{R}^d))\|$ yields an upper bound for $\mathcal{E}_n^m(S_{p,p}^r B(\mathbb{R}^d))_p$ since $E_n^m(f, L_p(\mathbb{R}^d)) = \tilde{E}_n^m(f, L_p(\mathbb{R}^d))$. \square

3.8 Characterization via approximation

Proof of Corollary 2.12. For $p = 2$ we have the identity

$$E_n^m(f, L_2(\mathbb{R}^d)) = \|f - P_n^m f\|_{L_2(\mathbb{R}^d)}$$

To continue we shall use some standard arguments. Observe

$$\|f | S_{2,2}^r B(\mathbb{R}^d)\| \asymp \|P_0^m f\|_{L_2(\mathbb{R}^d)} + \left(\sum_{n=0}^{\infty} 2^{2nr} \|P_{n+1}^m f - P_n^m f\|_{L_2(\mathbb{R}^d)}^2 \right)^{1/2}.$$

From this equivalence the inequality

$$\|f | S_{2,2}^r B(\mathbb{R}^d)\| \leq c \left(\|f\|_{L_2(\mathbb{R}^d)} + \left(\sum_{n=0}^{\infty} 2^{2nr} E_n^m(f, L_2(\mathbb{R}^d))^2 \right)^{1/2} \right)$$

follows by triangle inequality and the uniform boundedness of $\|P_n^m | \mathcal{L}(L_2(\mathbb{R}^d))\|$. Vice versa, using

$$\lim_{n \rightarrow \infty} P_n^m f = f \quad \text{for all } f \in L_2(\mathbb{R}^d),$$

we derive

$$\begin{aligned}
E_n^m(f, L_2(\mathbb{R}^d))^2 &= \left\| \sum_{\ell=n}^{\infty} P_{\ell+1}^m f - P_{\ell}^m f \right\|_{L_2(\mathbb{R}^d)}^2 \\
&= \sum_{\ell=n}^{\infty} \left\| P_{\ell+1}^m f - P_{\ell}^m f \right\|_{L_2(\mathbb{R}^d)}^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\| (2^{nr} E_n^m(f, L_2(\mathbb{R}^d)))_n \|_{\ell_2}^2 &= \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} 2^{2nr} \left\| P_{\ell+n+1}^m f - P_{\ell+n}^m f \right\|_{L_2(\mathbb{R}^d)}^2 \\
&\leq c \sum_{\ell=0}^{\infty} 2^{-2\ell r} \| f \|_{S_{2,2}^r B(\mathbb{R}^d)}^2.
\end{aligned}$$

Since $r > 0$ the claim follows. \square

A Appendix - Distribution spaces

As usual, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on \mathbb{R}^d . Its locally convex topology is generated by the (semi-)norms

$$\| \varphi \|_{k,\ell} = \sup_{x \in \mathbb{R}^d} (1 + |x|)^k \sum_{|\bar{\alpha}|_1 \leq \ell} |D^{\bar{\alpha}} \varphi(x)|, \quad k, \ell \in \mathbb{N}_0.$$

In other words, a sequence $\{\varphi_j\}_j \subset \mathcal{S}(\mathbb{R}^d)$ converges to $\varphi \in \mathcal{S}(\mathbb{R}^d)$ in $\mathcal{S}(\mathbb{R}^d)$ if and only if $\| \varphi - \varphi_j \|_{k,\ell} \xrightarrow{j \rightarrow \infty} 0$ holds for all pairs $(k, \ell) \in \mathbb{N}_0^2$. Then we shall write $\varphi_j \xrightarrow{\mathcal{S}} \varphi$. The space $\mathcal{S}'(\mathbb{R}^d)$ denotes the topological dual of $\mathcal{S}(\mathbb{R}^d)$. We equip $\mathcal{S}'(\mathbb{R}^d)$ with the weak topology. The Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$ will be denoted by \mathcal{F} and its inverse transform by \mathcal{F}^{-1} .

A.1 Classes of distributions on \mathbb{R}

Here we recall the definition and a few properties of Besov and Sobolev spaces defined on \mathbb{R} . We shall use the Fourier analytic approach, see e.g. [41]. Let $\varphi \in C_0^\infty(\mathbb{R})$ be a function such that $\varphi(t) = 1$ in an open set containing the origin. Then by means of

$$\varphi_0(t) = \varphi(t), \quad \varphi_j(t) = \varphi(2^{-j}t) - \varphi(2^{-j+1}t), \quad t \in \mathbb{R}, \quad j \in \mathbb{N},$$

we get a smooth dyadic decomposition of unity. First we deal with Besov spaces.

Definition A.1 Let $0 < p \leq \infty$ and $s \in \mathbb{R}$. The Besov space $B_{p,p}^s(\mathbb{R})$ is then the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{B_{p,p}^s(\mathbb{R})} := \left(\sum_{j=0}^{\infty} 2^{jsp} \|\mathcal{F}^{-1}[\varphi_j \mathcal{F}f](\cdot)\|_{L_p(\mathbb{R})}^p \right)^{1/p}$$

is finite (modification if $p = \infty$). By $\mathring{B}_{p,p}^s(\mathbb{R})$ we denote the closure of $C_0^\infty(\mathbb{R})$ with respect to the quasi-norm $\|\cdot\|_{B_{p,p}^s(\mathbb{R})}$.

In a similar way one could introduce Sobolev spaces of fractional order. However, here we prefer the interpretation as potential spaces.

Definition A.2 Let $1 < p < \infty$ and $s \in \mathbb{R}$. The fractional Sobolev space $H_p^s(\mathbb{R})$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{H_p^s(\mathbb{R})} := \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{s/2} \mathcal{F}f(\xi)](\cdot)\|_{L_p(\mathbb{R})}$$

is finite.

Remark (i) These quasi-Banach spaces $B_{p,p}^s(\mathbb{R})$ and $H_p^s(\mathbb{R})$ can be characterized in various ways, e.g. by differences and derivatives, whenever $s > \max(0, 1/p - 1)$. We refer to [41] for details.

(ii) The spaces $B_{p,p}^s(\mathbb{R})$ and $H_p^s(\mathbb{R})$ do not coincide as sets except the case $p = 2$. For $p = 2$ we have $B_{2,2}^s(\mathbb{R}) = H_2^s(\mathbb{R})$ in the sense of equivalent norms.

Let us recall the definition $\sigma_p := \max(0, \frac{1}{p} - 1)$.

Lemma A.3 Let $s \in \mathbb{R}$.

(i) Let $0 < p < \infty$ and $1/p + 1/p' = 1$, where we put $p' = \infty$ if $p \leq 1$. Then $\mathcal{S}(\mathbb{R})$ is dense in $B_{p,p}^s(\mathbb{R})$ and the dual space of $B_{p,p}^s(\mathbb{R})$ can be identified with $B_{p',p'}^{-s+\sigma_p}(\mathbb{R})$.

(ii) The dual space of $\mathring{B}_{\infty,\infty}^s(\mathbb{R})$ can be identified with $B_{1,1}^{-s}(\mathbb{R})$.

(iii) Let $1 < p < \infty$, $1/p + 1/p' = 1$ and $s \in \mathbb{R}$. Then $\mathcal{S}(\mathbb{R})$ is dense in $H_p^s(\mathbb{R})$ and the dual space of $H_p^s(\mathbb{R})$ can be identified with $H_{p'}^{-s}(\mathbb{R})$.

Remark We refer to [41, Thm. 2.3.3] for the density assertions and to [41, Thm. 2.11.2, 2.11.3] and the references given there for the assertions concerning duality.

A.2 Spaces of dominating mixed smoothness on \mathbb{R}^d

Detailed treatments of Besov as well as Sobolev spaces of dominating mixed smoothness are given at various places, we refer to the monographs [1,31], the survey [29] as well as to the booklet [48].

If φ_j , $j \in \mathbb{N}_0$, is a smooth dyadic decomposition of unity as introduced in Subsection A.1, then by means of

$$\varphi_{\bar{j}} := \varphi_{j_1} \otimes \dots \otimes \varphi_{j_d}, \quad \bar{j} = (j_1, \dots, j_d) \in \mathbb{N}_0^d,$$

we obtain a smooth decomposition of unity on \mathbb{R}^d .

Definition A.4 Let $0 < p \leq \infty$ and $r_1, \dots, r_d \in \mathbb{R}$. Then the Besov space $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)} := \left(\sum_{\bar{j} \in \mathbb{N}_0^d} 2^{(j_1 r_1 + \dots + j_d r_d)p} \|\mathcal{F}^{-1}[\varphi_{\bar{j}} \mathcal{F}f](\cdot)\|_{L_p(\mathbb{R}^d)}^p \right)^{1/p}$$

is finite (modification if $p = \infty$). By $\mathring{S}_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ we denote the closure of $S_0^\infty(\mathbb{R})$ with respect to the quasi-norm $\|\cdot\|_{S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)}$.

Again we introduce Sobolev type spaces as potential spaces.

Definition A.5 Let $1 < p < \infty$ and $r_1, \dots, r_d \in \mathbb{R}$. The fractional Sobolev space with dominating mixed smoothness $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ is then the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)} := \left\| \mathcal{F}^{-1} \left[\prod_{i=1}^d (1 + |\xi_i|^2)^{r_i/2} \mathcal{F}f(\xi) \right] (\cdot) \right\|_{L_p(\mathbb{R}^d)}$$

is finite.

Remark (i) These classes $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ as well as $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ are quasi-Banach spaces. If $\min(r_1, \dots, r_d) > \max(0, (1/p) - 1)$ then they can be characterized by differences, we refer to [31] and [44] for details.

(ii) Again the spaces $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ and $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ do not coincide as sets except the case $p = 2$. For $p = 2$ it holds $S_{2,2}^{r_1, \dots, r_d} B(\mathbb{R}^d) = S_2^{r_1, \dots, r_d} H(\mathbb{R}^d)$ in the sense of equivalent norms.

Also here we need to know about density of $\mathcal{S}(\mathbb{R}^d)$ and duality. Recall $\sigma_p := \max(0, 1/p - 1)$.

Lemma A.6 Let $r \in \mathbb{R}$.

(i) Let $0 < p < \infty$ and $1/p + 1/p' = 1$, where we put $p' = \infty$ if $p \leq 1$. Then $\mathcal{S}(\mathbb{R}^d)$ is dense in $S_{p,p}^r B(\mathbb{R}^d)$ and the dual space of $S_{p,p}^r B(\mathbb{R}^d)$ can be identified with $S_{p',p'}^{-r+\sigma_p}(\mathbb{R}^d)$.

(ii) The dual space of $\mathring{S}_{\infty,\infty}^r B(\mathbb{R})$ can be identified with $S_{1,1}^{-r} B(\mathbb{R}^d)$.

(iii) Let $1 < p < \infty$, $1/p + 1/p' = 1$ and $r \in \mathbb{R}$. Then $\mathcal{S}(\mathbb{R})$ is dense in $S_p^r H(\mathbb{R}^d)$ and the dual space of $S_p^r H(\mathbb{R}^d)$ can be identified with $S_{p'}^{-r} H(\mathbb{R}^d)$.

Remark For the proof we refer to Vybiral [48, p. 42].

A.3 Discretization of Besov and Sobolev spaces

There are different ways to discretize Besov or Sobolev spaces. Most convenient for us will be the use of Daubechies wavelets. In connection with the characterization of function spaces we refer to [5–7,21,42,48,49].

A.3.1 Wavelet bases of Besov and Sobolev spaces on the real line

Let φ be an orthonormal compactly supported scaling function belonging to $C^N(\mathbb{R})$. Let ψ be an associated compactly supported orthonormal wavelet. Then this function satisfies a moment condition of order N , see [7, Thm. 5.5.1] or [49, Prop. 3.1]. We shall use the same abbreviations as done in (2.4).

Proposition A.7 *Let $0 < p < \infty$ and suppose*

$$\max\left(r, \max\left(0, \frac{1}{p} - 1\right) - r\right) < N. \quad (\text{A.1})$$

(i) *The mapping J defined by*

$$f \mapsto (\langle f, \psi_{j,k} \rangle)_{j,k}$$

generates an isomorphism of $B_{p,p}^r(\mathbb{R})$ onto b_p^r .

(ii) *In case $p = \infty$ the mapping $J : B_{\infty,\infty}^r(\mathbb{R}) \rightarrow b_\infty^r$ is continuous. Furthermore, if $a = (a_{j,k})_{j,k} \in b_\infty^r$ then the tempered distribution f given by $f := \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} a_{j,k} \psi_{j,k}$ belongs to $B_{\infty,\infty}^r(\mathbb{R})$ and there exists a constant c such that*

$$\left\| \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} a_{j,k} \psi_{j,k} \right\|_{B_{\infty,\infty}^r(\mathbb{R})} \leq c \|a\|_{b_\infty^r}$$

holds for all such sequences a .

Remark (i) Proofs in case $p \geq 1$ may be found in many places, see e.g. Meyer [21] and Wojtaszczyk [49]. For $p < 1$ we refer to Bourdaud [5], Cohen [6] ($r > \max(0, 1/p - 1)$), Kyriazis and Petrushev [18] and to Triebel [42] (general case). See also the survey DeVore [9] and the references given there.

(ii) Since ψ does not belong to $\mathcal{S}(\mathbb{R}^d)$ the interpretation of the symbol $\langle f, \psi_{j,k} \rangle$ needs some care. However, because of the compact support of φ and ψ we have

$$\varphi, \psi \in B_{p,\infty}^N(\mathbb{R}) \quad \text{for all } 0 < p \leq \infty.$$

Now our restriction (A.1) allows an interpretation of $\langle f, \psi_{j,k} \rangle$ by Lemma A.3.

Proposition A.8 *Let $1 < p < \infty$ and suppose $|r| < N$. The mapping J defined by*

$$f \mapsto (\langle f, \psi_{j,k} \rangle)_{j,k}$$

generates an isomorphism of $H_p^r(\mathbb{R})$ onto f_p^r .

Remark This result, even in a more general form, can be found in Triebel [42]. We also refer to Frazier and Jawerth [12] and Kyriazis and Petrushev [18], where corresponding estimates for more general systems (not only orthonormal) are treated.

A.3.2 Wavelet bases of Besov and Sobolev spaces on \mathbb{R}^d

An extension to spaces of dominating mixed smoothness has been given in Vybiral [48]. Let φ and ψ be as in the preceding subsection. Defining

$$\psi_{\bar{j},\bar{k}}(x_1, \dots, x_d) := \psi_{j_1, k_1}(x_1) \cdot \dots \cdot \psi_{j_d, k_d}(x_d)$$

we end up with the following characterization of $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$.

Proposition A.9 *Let $d > 1$ and $r_1, \dots, r_d \in \mathbb{R}$.*

(i) *Let $1 < p < \infty$. Let $N = N(r_1, \dots, r_d, p)$ be sufficiently large. The mapping J_d defined by*

$$f \mapsto (\langle f, \psi_{\bar{j},\bar{k}} \rangle)_{\bar{j},\bar{k}}$$

generates an isomorphism of $S_p^{r_1, \dots, r_d} H(\mathbb{R}^d)$ onto $s_p^{r_1, \dots, r_d} f$.

(ii) *Let $0 < p < \infty$. Let $N = N(r_1, \dots, r_d, p)$ be sufficiently large. The mapping J_d defined by*

$$f \mapsto (\langle f, \psi_{\bar{j},\bar{k}} \rangle)_{\bar{j},\bar{k}}$$

generates an isomorphism of $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ onto $s_p^{r_1, \dots, r_d} b$.

(iii) *Let $p = \infty$. Let $N = N(r_1, \dots, r_d)$ be sufficiently large. The mapping $J_d : S_{\infty, \infty}^{r_1, \dots, r_d} B(\mathbb{R}^d) \rightarrow s_{\infty}^{r_1, \dots, r_d} b$ is continuous. Furthermore, if $a = (a_{\bar{j},\bar{k}})_{\bar{j},\bar{k}} \in s_{\infty}^{r_1, \dots, r_d} b$ then the tempered distribution $f := \sum_{\bar{j} \in \mathbb{N}_0^d} \sum_{\bar{k} \in \mathbb{Z}^d} a_{\bar{j},\bar{k}} \psi_{\bar{j},\bar{k}}$ belongs to $S_{\infty, \infty}^{r_1, \dots, r_d} B(\mathbb{R}^d)$ and there exists a constant c such that*

$$\|f\|_{S_{\infty, \infty}^{r_1, \dots, r_d} B(\mathbb{R}^d)} \leq c \|a\|_{s_{\infty}^{r_1, \dots, r_d} b}$$

holds for all such sequences a .

Remark (i) Of some importance will be the fact that the mapping J_d is the tensor product of the isomorphism discussed in Propositions A.7, A.8.

(ii) Again the interpretation of $\langle f, \psi_{\bar{j},\bar{k}} \rangle$ needs some care. In view of Lemma A.6 we can argue as in the remark after Proposition A.7.

(iii) In the periodic setting, i.e. on the d -dimensional torus \mathbb{T}^d , Schmeisser [28] constructed unconditional Schauder bases of $S_{p,p}^{r_1, \dots, r_d} B(\mathbb{T}^d)$ consisting of trigonometric polynomials.

B Appendix - Tensor products

We shall deal with tensor products in several different situations.

B.1 Tensor products of Banach spaces

We follow [20], but see also [8].

Let X and Y be Banach spaces. X' denotes the dual of X . Consider the set of all formal expressions $\sum_{i=1}^n f_i \otimes g_i$, $n \in \mathbb{N}$, $f_i \in X'$ and $g_i \in Y$. We introduce an equivalence relation by means of

$$\sum_{i=1}^n f_i \otimes g_i \sim \sum_{j=1}^m u_j \otimes v_j$$

if both expressions generate the same operator $A : X' \rightarrow Y$, i.e.

$$\sum_{i=1}^n \varphi(f_i) g_i = \sum_{j=1}^m \varphi(u_j) v_j \quad \text{for all } \varphi \in X'. \quad (\text{B.1})$$

Then the algebraic tensor product $X \otimes Y$ of X and Y is defined to be the set of all such equivalence classes. One can equip this set with several different norms. We are interested in so-called uniform norms only. Let X_1, X_2, Y_1, Y_2 be Banach spaces. For $T_i \in \mathcal{L}(X_i, Y_i)$, $i = 1, 2$, we define their tensor product by

$$(T_1 \otimes T_2)h := \sum_{i=1}^n (T_1 f_i) \otimes (T_2 g_i), \quad h = \sum_{i=1}^n f_i \otimes g_i \in X_1 \otimes X_2. \quad (\text{B.2})$$

We call a norm $\alpha(\cdot, X, Y)$ on $X \otimes Y$ a uniform tensor norm if it satisfies

$$\alpha((T_1 \otimes T_2)h, Y_1, Y_2) \leq \|T_1|_{\mathcal{L}(X_1, Y_1)}\| \cdot \|T_2|_{\mathcal{L}(X_2, Y_2)}\| \alpha(h, X_1, X_2).$$

for all $h = \sum_{j=1}^n f_j \otimes g_j \in X_1 \otimes X_2$ and all $T_1 \in \mathcal{L}(X_1, Y_1)$, $T_2 \in \mathcal{L}(X_2, Y_2)$. The completion of $X \otimes Y$ with respect to the tensor norm α will be denoted by $X \otimes_\alpha Y$. If α is uniform then $T_1 \otimes T_2$ has a unique extension to $X_1 \otimes_\alpha X_2$ which we again denote by $T_1 \otimes T_2$. Simple, but important, is the next property we need.

Lemma B.1 *Let X_1, X_2, Y_1, Y_2 be Banach spaces and let $\alpha(\cdot, X, Y)$ be a uniform tensor norm. Further we suppose that $T_1 \in \mathcal{L}(X_1, Y_1)$ and $T_2 \in \mathcal{L}(X_2, Y_2)$ are linear isomorphisms. Then the operator $T_1 \otimes T_2$ is a linear isomorphism from $X_1 \otimes_\alpha X_2$ onto $Y_1 \otimes_\alpha Y_2$.*

Proof. Obviously, $T_1 \otimes T_2 \in \mathcal{L}(X_1 \otimes_\alpha X_2, Y_1 \otimes_\alpha Y_2)$ and $T_1^{-1} \otimes T_2^{-1} \in \mathcal{L}(Y_1 \otimes_\alpha Y_2, X_1 \otimes_\alpha X_2)$. So it remains to show that

$$(T_1 \otimes_\alpha T_2)^{-1} = T_1^{-1} \otimes_\alpha T_2^{-1}$$

in the algebraical sense, which is a simple consequence of

$$(T_1 \otimes T_2) \circ (T_1^{-1} \otimes T_2^{-1}) = I \quad \text{on } Y_1 \otimes Y_2 \quad ,$$

$$(T_1^{-1} \otimes T_2^{-1}) \circ (T_1 \otimes T_2) = I \quad \text{on } X_1 \otimes X_2$$

and a limit argument. Here I denotes the identity on the corresponding space. \square

Next we recall three well-known constructions of tensor norms, namely the injective, the projective and the p -nuclear norm.

Definition B.2 *Let X and Y be Banach spaces.*

(i) *Let $h \in X \otimes Y$ be given by*

$$h = \sum_{j=1}^n f_j \otimes g_j, \quad f_j \in X, \quad g_j \in Y.$$

Then the injective tensor norm $\lambda(\cdot, X, Y)$ is defined as

$$\lambda(h, X, Y) = \sup \left\{ \left\| \sum_{j=1}^n \psi(f_j) \cdot g_j \right\|_Y : \psi \in X', \|\psi|_{X'}\| \leq 1 \right\}.$$

(ii) *The projective tensor norm $\gamma(\cdot, X, Y)$ is defined by*

$$\gamma(h, X, Y) = \inf \left\{ \sum_{j=1}^n \|f_j|_X\| \|g_j|_Y\| : f_j \in X, g_j \in Y, h = \sum_{j=1}^n f_j \otimes g_j \right\}.$$

(iii) *Let $1 \leq p \leq \infty$ and let $1/p + 1/p' = 1$. Then the p -nuclear tensor norm $\alpha_p(\cdot, X, Y)$ is given by*

$$\alpha_p(h, X, Y) := \inf \left\{ \left(\sum_{i=1}^n \|f_i|_X\|^p \right)^{1/p} \cdot \sup \left\{ \left(\sum_{i=1}^n |\psi(g_i)|^{p'} \right)^{1/p'} : \psi \in Y', \|\psi|_{Y'}\| \leq 1 \right\} \right\},$$

where the infimum is taken over all representations of h (as in (ii)).

Remark (i) All three expressions define norms, we refer to [20, Chapt. 1]. In particular, λ is independent of the representation of h .

(ii) In Definition B.2(iii) one can replace

$$\sup \left\{ \left(\sum_{i=1}^n |\psi(g_i)|^{p'} \right)^{1/p'} : \psi \in Y', \|\psi|_{Y'}\| \leq 1 \right\} \quad (\text{B.3})$$

by

$$\sup \left\{ \left\| \sum_{i=1}^n \lambda_i g_i \Big| Y \right\| : \left(\sum_{i=1}^n |\lambda_i|^p \right)^{1/p} \leq 1 \right\}, \quad (\text{B.4})$$

see [20, Lem. 1.44].

B.2 Tensor products of distributions

As usual, we put $\mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$. It is equipped with a topology: a sequence $\{\varphi_j\}_j \subset \mathcal{D}(\mathbb{R}^d)$ converges to a $\varphi \in \mathcal{D}(\mathbb{R}^d)$ if $\text{supp } \varphi_j \subset K$, $j = 1, 2, \dots$, where $K \subset \mathbb{R}^d$ is a compact subset and $\{D^{\bar{\alpha}} \varphi_j\}_j$ converges uniformly to $D^{\bar{\alpha}} \varphi$ for every multi-index $\bar{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. The topological dual of $\mathcal{D}(\mathbb{R}^d)$ is denoted by $\mathcal{D}'(\mathbb{R}^d)$. Tensor products of distributions is a well-developed subject, mainly in the framework of $\mathcal{D}'(\mathbb{R}^d)$, we refer e.g. to [32, Chapt. IV], [38, III.13] as well as [16, Chapt. X]. We need the following.

Lemma B.3 *Let $T \in \mathcal{S}'(\mathbb{R}^{d_1})$ and $S \in \mathcal{S}'(\mathbb{R}^{d_2})$. Then there exists a unique distribution $U \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$, called the tensor product of T and S and denoted by $T \otimes^D S$, such that for all functions $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$ and $\psi \in \mathcal{S}(\mathbb{R}^{d_2})$*

$$U(\varphi(x) \cdot \psi(y)) = T(\varphi(x))\mathcal{S}(\psi(y))$$

holds true. Furthermore, U is given explicitly by the formula

$$U(\rho(x, y)) = S_y(T_x(\rho(x, y))) = T_x(S_y(\rho(x, y))), \quad \rho \in \mathcal{S}(R^{d_1+d_2}).$$

Proof. In [16, Chapt. X] it is proved that

$$(T \otimes^D S)(\rho(x, y)) = S_y(T_x(\rho(x, y))) = T_x(S_y(\rho(x, y))) \quad , \quad \rho \in \mathcal{S}(R^{d_1+d_2}),$$

defines a tempered distribution belonging to $\mathcal{S}'(\mathbb{R}^{d_1+d_2})$. The uniqueness of this distribution can be proved following the lines of the proof for the \mathcal{D}' -counterpart of Lemma B.3 (see e.g. [38, III.13]) making use of the facts: $\mathcal{D}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d)$ (topological embedding); $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$; and the set

$$\left\{ \rho = \sum_{j=1}^N \varphi_1^j \otimes \dots \otimes \varphi_d^j : \varphi_k^j \in \mathcal{D}(\mathbb{R}) , N \in \mathbb{N}, j = 1, \dots, N, k = 1, \dots, d \right\}$$

is dense in $\mathcal{D}(\mathbb{R}^d)$. The proof is complete. \square

By means of the linearity of distributions this implies the following.

Proposition B.4 *Let $T_i \in \mathcal{S}'(\mathbb{R}^{d_1})$ and $S_i \in \mathcal{S}'(\mathbb{R}^{d_2})$, $i = 1, \dots, n$. Then there exists a unique distribution $U \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$ such that for all functions $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$ and $\psi \in \mathcal{S}(\mathbb{R}^{d_2})$*

$$U(\varphi(x) \cdot \psi(y)) = \sum_{i=1}^n T_i(\varphi(x)) S_i(\psi(y))$$

holds true. Furthermore, U is given explicitly by

$$U = \sum_{i=1}^n T_i \otimes^D S_i.$$

B.3 Tensor products of spaces of distributions

Tensor products of functions and sequences have been the original source for the introduction of the abstract tensor product. However, since we are dealing with quasi-Banach spaces of distributions and sequences, we prefer to make a few comments to the coincidence of these tensor products. First we recall the notion of a reasonable cross-norm. A norm α on $X \otimes Y$ is called a cross-norm, if $\alpha(f \otimes g, X, Y) = \|f\|_X \|g\|_Y$ for all $f \in X$ and all $g \in Y$. Further, a cross-norm α is called reasonable if, for all $\varphi \in X'$ and all $\psi \in Y'$, the linear form $\varphi \otimes \psi$ is bounded on $X \otimes_\alpha Y$ and has norm $\|\varphi\|_{X'} \|\psi\|_{Y'}$.

Let X and Y be spaces of tempered distributions such that $\mathcal{S}(\mathbb{R}^{d_1}) \hookrightarrow X \hookrightarrow \mathcal{S}'(\mathbb{R}^{d_1})$ and $\mathcal{S}(\mathbb{R}^{d_2}) \hookrightarrow Y \hookrightarrow \mathcal{S}'(\mathbb{R}^{d_2})$, respectively. Similar to Subsection B.1 we can introduce the following set

$$X \otimes^D Y := \left\{ h = \sum_{j=1}^n f_j \otimes^D g_j : f_j \in X, g_j \in Y, n \in \mathbb{N}, j = 1, \dots, n \right\}.$$

Because of $X \hookrightarrow \mathcal{S}'$ we have $\mathcal{S}(\mathbb{R}^{d_1}) \subset X'$ in the sense that a fixed function $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$ defines a continuous linear functional on X via

$$f \mapsto f(\varphi) \quad , \quad f \in X, \quad (\text{B.5})$$

(analogously $\mathcal{S}(\mathbb{R}^{d_2}) \subset Y'$). Let us additionally assume a dense embedding

$$\overline{\mathcal{S}(\mathbb{R}^{d_1})}^{\|\cdot\|_{X'}} = X'$$

with the above interpretation. Under these assumptions the set $X \otimes^D Y$ equipped and completed with a reasonable cross-norm α is isomorphic to $X \otimes_\alpha Y$ (see Subsection B.1). Indeed, let us suppose

$$\sum_{i=1}^n f_i \otimes g_i = \sum_{j=1}^m u_j \otimes g_j \quad (\text{B.6})$$

in the sense of Subsection B.1. This implies

$$\sum_{i=1}^n \eta(f_i) \cdot \beta(g_i) = \sum_{j=1}^m \eta(u_j) \cdot \beta(g_j)$$

for all $\eta \in X'$ and $\beta \in Y'$ and therefore (in the sense of (B.5))

$$\sum_{i=1}^n f_i(\varphi) \cdot g_i(\psi) = \sum_{j=1}^m u_j(\varphi) \cdot g_j(\psi)$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$ and $\psi \in \mathcal{S}(\mathbb{R}^{d_2})$. Then Proposition B.4 implies

$$\sum_{i=1}^n f_i \otimes^D g_i = \sum_{j=1}^m u_j \otimes^D g_j. \quad (\text{B.7})$$

Vice versa, if we assume (B.7), then arguing backwards, we find that

$$\sum_{i=1}^n f_i(\varphi) \cdot g_i = \sum_{j=1}^m u_j(\varphi) \cdot g_j$$

holds in Y for every $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$. Finally, because of the dense inclusion $\mathcal{S}(\mathbb{R}^{d_1}) \subset X'$, we conclude the equality (B.6) (in the sense of Subsection B.1) by a limit argument. Hence, in the case of Banach spaces X and Y of distributions with the properties above we have the coincidence of both approaches. We therefore write $X \otimes_\alpha Y$ for both constructions.

Remark (i) All the spaces under consideration in Section 2 have the properties required in the previous consideration, see Lemma A.3 and Lemma A.6.
(ii) The injective, the projective and the p -nuclear norm are reasonable cross-norms, cf. [20, Lem. 1.6, 1.8, 1.46].

B.4 Tensor products of sequence spaces

Let I and J denote countable index-sets. Then $F(I)$ is the class (\mathbb{C} -vector space) of all functions $f : I \mapsto \mathbb{C}$. Similar as above we consider subspaces $X \subset F(I)$ and $Y \subset F(J)$ and define the tensor product $f \otimes g \in F(I \times J)$ of $f \in X$ and $g \in Y$ by

$$(f \otimes g)(i, j) := f(i) g(j), \quad i \in I, \quad j \in J.$$

B.5 Tensor products of certain quasi-Banach spaces of distributions

We want to generalize the concept of the projective tensor-norm γ (see Definition B.2 (ii)) in order to include also special quasi-Banach spaces either of

type $X \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$ or of type $\ell_p(w)$.

Let us mention that there is no hope for a general abstract theory of tensor product for quasi-Banach spaces. At least one of the reasons consists in the somehow “poor” dual spaces of some quasi-Banach spaces. This has to be compared with (B.1). So we concentrate on those situations where the tensor product has a meaning from the very beginning, i.e. for distributions and sequences of complex numbers.

Definition B.5 *Let $0 < p < 1$.*

(i) *Let X and Y be quasi-Banach spaces such that $X \hookrightarrow \mathcal{S}'(\mathbb{R}^{d_1})$ and $Y \hookrightarrow \mathcal{S}'(\mathbb{R}^{d_2})$. Then we define the projective tensor p -norm γ_p by*

$$\gamma_p(h, X, Y) := \inf \left\{ \left(\sum_{j=1}^n \|f_j|X\|^p \|g_j|Y\|^p \right)^{1/p} : f_j \in X, g_j \in Y, h = \sum_{j=1}^n f_j \otimes^D g_j \right\}.$$

(ii) *Let X and Y be quasi-Banach spaces such that $X = \ell_{q_1}(w_1)$ and $Y = \ell_{q_2}(w_2)$ for some $q_1, q_2 \in (0, \infty]$. Then the projective tensor p -norm is defined as*

$$\gamma_p(h, X, Y) := \inf \left\{ \left(\sum_{j=1}^n \|f_j|X\|^p \|g_j|Y\|^p \right)^{1/p} : f_j \in X, g_j \in Y, h = \sum_{j=1}^n f_j \otimes^s g_j \right\}.$$

Remark (i) γ_p defines a uniform quasi-norm (p -norm) on $X \otimes Y$. The inequality

$$\gamma_p(h_1 + h_2, X, Y)^p \leq \gamma_p(h_1, X, Y)^p + \gamma_p(h_2, X, Y)^p$$

as well as the uniformness are obvious.

(ii) Different attempts to introduce tensor products of quasi-Banach spaces have been undertaken by Turpin [43] and Nitsche [22]. In particular the approach of Nitsche applies to so-called placid q -Banach spaces. Let us mention that $\ell_q, B_{q,q}^r(\mathbb{R})$ as well as $S_{q,q}^{r_1, \dots, r_d}(\mathbb{R}^d)$ are placid q -quasi-Banach spaces if $0 < q < 1$.

We can carry over the definition of the tensor product of operators to the present case, see (B.2). One has to check that this definition does not depend on the chosen representation of $h \in X \otimes Y$. But this can be done as in case of Banach spaces, cf. [20, p. 19]. Now we are in position to formulate a supplement to Lemma B.1.

Lemma B.6 *Let X_1, X_2, Y_1, Y_2 be quasi-Banach spaces such that the pairs (X_1, X_2) and (Y_1, Y_2) are admissible in Definition B.5. Let $0 < p \leq 1$. Further we suppose that $T_1 \in \mathcal{L}(X_1, Y_1)$ and $T_2 \in \mathcal{L}(X_2, Y_2)$ are linear isomorphisms. Then the operator $T_1 \otimes T_2$ is a linear isomorphism from $X_1 \otimes_{\gamma_p} X_2$ onto $Y_1 \otimes_{\gamma_p} Y_2$.*

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