Numerical Integration, Haar Projection Numbers and Failure of Unconditional Convergence

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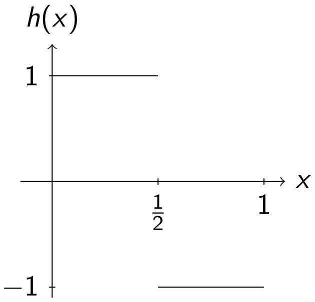
Haar projection numbers

The Haar basis



- The univariate Haar wavelet system

$$h_{j,k}(x) := h(2^j x - k) \quad , \quad j,k \in \mathbb{Z}\,.$$

- The system $\{2^{j/2} h_{j,k}\}_{j,k}$ is an orthonormal basis in $L_2(\mathbb{R})$

An unconditional basis in L_p

- In contrast to the trigonometric system $\{e^{ikx}\}_{k\in\mathbb{Z}}$ the Haar wavelet system is an unconditional basis in L_p for $p \neq 2$

Theorem (Marcinkiewicz '37, Paley '32)

Let $1 < p < \infty$.
(i) *The Haar system $\{h_{j,k}\}_{j,k\in\mathbb{Z}}$ is an unconditional basis in $L_p(\mathbb{R})$.*
(ii) *There exist constants $0 < c < C$ such that*

$$c\|f\|_p \leq \Big\|\Big(\sum_{j\in\mathbb{Z}}\Big|\sum_{k\in\mathbb{Z}} 2^j\langle f, h_{j,k}\rangle \chi_{j,k}\Big|^2\Big)^{1/2}\Big\|_p \leq C\|f\|_p$$

- The statement does neither hold in case $p = 1$ nor $p = \infty$!

100 years of Haar and spline bases

- **1909 Faber**, Über stetige Funktionen
- **1910 Haar**, Zur Theorie der orthogonalen Funktionensysteme
- **1910 Haar**, Über die Orthogonalfunktionen des Herrn Haar
- 1927 Schauder, Zur Theorie stetiger Abbildungen in Funktionalräumen
- 1928 Schauder, Eine Eigenschaft des Haarschen Orthogonalsystems
- **1932 Paley**, A remarkable series of orthogonal functions
- **1937 Marcinkiewicz**, Quelques théoremes sur les séries orthogonales

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- 1973 Triebel, Über die Existenz von Schauderbasen in Sobolev-Räumen
- 1978 Triebel, On Haar bases in Besov spaces
- 1976 Ropela, Spline bases in Besov spaces
- 1995 Bourdaud, Ondelettes et espaces de Besov
- 1997 Kamont, A discrete characterization of Besov spaces
- 1998 DeVore, Konyagin, Temlyakov, Hyperbolic wavelet approximation
- **2010 Triebel** Discrepancy, numerical integration, and hyperbolic cross approximation

More history

	$C([0,1])$	$L_1([0,1])$	$L_2([0,1])$	$L_p([0,1])$ $p \neq 2$
Fourier system	no basis (Bois-Reymond 1873)	no basis (Lebesgue 1909)	orth. basis (Fischer, Riesz 1907)	basis (M. Riesz 1927), **no uncond. basis** (Karlin 1948)
Walsh system	/	no basis (Fine 1949)	orth. basis (Walsh 1923)	basis (Paley 1932), **no uncond. basis** (Karlin 1948)
Haar system	/	basis (Schauder 1928)	**orth. basis (Haar 1910)**	**basis (Schauder 1928), unc. basis (Marcinkiewicz 1937)**
Faber-Schauder system	basis (Faber 1909, 10), Schauder 1927	/	/	/
Franklin system	basis (Franklin 1928)	basis (Cieselski 1963/66), **basis in $\mathbf{H_1}$ (Wojtaszctyk 1982)**	orth. basis (Franklin 1928)	basis (Cieselski 1963), unc. basis (Bochkarev 1974)
General	no unc. basis (Karlin 1948)	no unc. basis (Pelczyński 1960, 61)	many nonequiv. cond. bases (Babenko 1948)	

Smoothness spaces

- **Sobolev regularity**, $1 < p < \infty$, $s \in \mathbb{R}$

$$\|f\|_{W_p^s(\mathbb{R})} := \|\mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\mathcal{F}f]\|_p$$

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- Iterated differences, $m > s$

$$\Delta_h^2 f(x) := f(x+2h) - 2f(x+h) + f(x)$$

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$$\Delta_h^2 f(x) := f(x+2h) - 2f(x+h) + f(x)$$

- **Besov regularity**, $1 \le p, q \le \infty$, $s > 0$,

$$\|f\|_{B^r_{p,q}} := \|f\|_p + \Big(\int_0^1 t^{-rq} \sup_{|h|\le t} \|\Delta_h^m f\|_p^q \frac{dt}{t}\Big)^{1/q}$$

- **Fourier analytical** approach, $s \in \mathbb{R}$

$$\|f\|_{B^r_{p,q}} \asymp \Big(\sum_{j=0}^{\infty} 2^{jrq} \|\Phi_j * f\|_p^q\Big)^{1/q}$$

The Haar basis in $B^s_{p,q}$

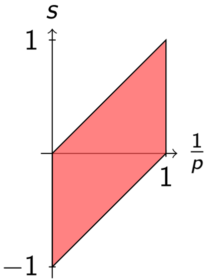
Theorem (Triebel '73, '78)

Let $1 \le p, q < \infty$ *and* $1/p - 1 < s < 1/p$.

(i) *The system* $\{h_{j,k}\}_{j\in\mathbb{N}_{-1},k\in\mathbb{Z}}$ *is an unconditional basis in* $B^s_{p,q}(\mathbb{R})$

(ii) *There exist constants* $0 < c < C$ *such that*

$$c\|f\|_{B^s_{p,q}} \le \Big[\sum_{j=-1}^{\infty} 2^{j(s-1/p)q}\Big(\sum_{k\in\mathbb{Z}} |2^j\langle f, h_{j,k}\rangle|^p\Big)^{q/p}\Big]^{1/q} \le C\|f\|_{B^s_{p,q}}$$



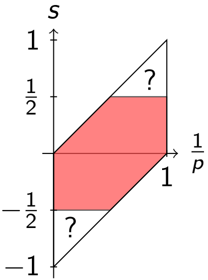
- The condition $1/p - 1 < s < 1/p$ is sharp!
- $h \in B^{1/p}_{p,q} \iff q = \infty$

The Haar basis in $F^s_{p,q}$

Theorem (Triebel '10)

Let $1 < p, q < \infty$ *and* $\max\{-1/p', -1/q'\} < s < \min\{1/p, 1/q\}$.
(i) *The system* $\{h_{j,k}\}_{j\in\mathbb{N}_{-1}, k\in\mathbb{Z}}$ *is an unconditional basis in* $F^s_{p,q}(\mathbb{R})$
(ii) *There exist constants* $0 < c < C$ *such that*

$$c\|f\|_{F^s_{p,q}} \leq \Big\| \Big(\sum_{j\in\mathbb{N}_{-1}} 2^{jsq} \Big| \sum_{k\in\mathbb{Z}} 2^j \langle f, h_{j,k}\rangle \chi_{j,k} \Big|^q \Big)^{1/q} \Big\|_p \leq C\|f\|_{F^s_{p,q}}$$



- shaded region: situation for H^s_p where $q = 2$
- $h \notin F^{1/p}_{p,q}(\mathbb{R})$
- **Problem:** What happens in the question-marked region ?

A negative result

Theorem (Seeger, U. 2015)

Let $1 < p < \infty$*. The Haar system* $\{h_{j,k}\}_{j \in \mathbb{N}_{-1}, k \in \mathbb{Z}}$ *is an unconditional basis in* $W^s_p(\mathbb{R})$ *if and only if*

$$\max\{-1/p', -1/2\} < s < \min\{1/p, 1/2\}\,.$$

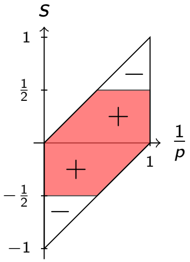


Figure: Haar basis for Sobolev

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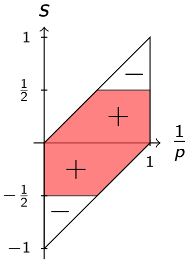

Figure: Haar basis for Sobolev

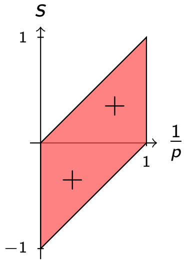

Figure: Haar basis for Besov

Unconditional bases

Definition

Let X be a Banach space. Any biorthogonal system (x_n, x_n^*) with a countable index set A is an unconditional basis in X if

- $\overline{\operatorname{span}}\{x_n\}_n = X$
- There exists a constant $C > 0$ such that for every $x \in X$ and every finite index set $B \subset A$

$$\Big\| \sum_{n \in B} x_n^*(x) x_n \Big\|_X \le C \|x\|_X$$

- **In other words:** the norm of any projection

$$P_B(x) := \sum_{n \in B} x_n^*(x) x_n$$

is bounded by C

Haar projections

- We denote by $HF(h_{j,k})$ the *Haar frequency* 2^j of $h_{j,k}$
- Let $E \subset \{h_{j,k}\}_{j\in\mathbb{N}_{-1},k\in\mathbb{Z}}$ be a finite subset of the Haar system
- The corresponding Haar projection P_E is given by

$$P_E(f) := \sum_{h_{j,k}\in E} 2^j \langle f, h_{j,k}\rangle h_{j,k}$$

- $HF(E) := \{HF(h_{j,k}) \ : \ h_{j,k} \in E\}$

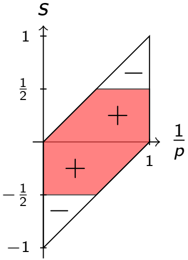

Figure: Haar basis for Sobolev

A quantitative version

- Let $A = \{2^k \; : \; k \geq 0\}$ be an arbitrary set with $\#A \geq 2^N$

Lemma

There are sets E, F with $HF(E), HF(F) \subset A$ such that
(a) **Upper triangle.** *if* $1 < p < 2 < \infty$ *and* $1/2 < s < 1/p$ *then*

$$\|P_E\|_{W^s_p \to W^s_p} \gtrsim 2^{N(s-1/2)}\,,$$

(b) **Lower triangle.** *if* $2 < p < \infty$. *If* $-1/p' < s < -1/2$ *then*

$$\|P_F\|_{W^s_p \to W^s_p} \gtrsim 2^{N(-1/2-s)}\,.$$

- There are also endpoint versions $s = 1/2$, $s = -1/2$
- More general result on Triebel-Lizorkin spaces

Haar projection numbers

- $\mathcal{G}(X, A) = \sup \left\{ \|P_E\|_{X \to X} : \mathrm{HF}(E) \subset A \right\}$

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$$\gamma^*(X; \Lambda) = \sup \left\{ \mathcal{G}(X, A) : \#A \le \Lambda \right\},$$
$$\gamma_*(X; \Lambda) = \inf \left\{ \mathcal{G}(X, A) : \#A \ge \Lambda \right\}.$$

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Theorem (Seeger, U. 2015)

(i) For $1 < p < q < \infty$, $1/q < s < 1/p$,

$$\gamma_*(F^s_{p,q};\Lambda) \approx \gamma^*(F^s_{p,q};\Lambda) \approx \Lambda^{s-\frac{1}{q}}.$$

(ii) For $1 < q < p < \infty$, $-1/p' < s < -1/q'$,

$$\gamma_*(F^s_{p,q};\Lambda) \approx \gamma^*(F^s_{p,q};\Lambda) \approx \Lambda^{-\frac{1}{q'}-s}.$$

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- Endpoint case: γ_* and γ^* differ, but grow in $\log\Lambda$

A probabilistic argument

- $\Lambda \leq \#A + 1$, $2^N \leq \#A < 2^{N+1}$
- $r_j, j = 1, 2, 3, \ldots$, system of Rademacher functions
- For $t_1 \in [0,1]$ define

$$T_{t_1} g(x) = \sum_{2^j \in A} r_j(t_1) \sum_{\mu=0}^{2^j - 1} 2^j \langle h_{j,\mu}, g \rangle h_{j,\mu}(x)$$

- Testfunctions, $t_2 \in [0,1]$

$$f_{N,t_2} = \sum_{2^k \in A} r_k(t_2) 2^{-(k+N)s} \eta_{k,N}$$

- Using **Khinchine's inequality**

$$\Big(\int_0^1 \int_0^1 \big\| T_{t_1} f_{N,t_2} \big\|_{F^s_{p,q}}^q dt_1 dt_2 \Big)^{1/q} \geq c 2^{N(-s-1/q')}$$

Numerical integration

Kink functions

- Computational finance: "**kink functions**", e.g.

$$g(t) = |t - 1/2| \text{ oder } g(t) = |\sin(t)|$$

- Multivariate: "axis parallel kinks" tensorization of univariate kinks

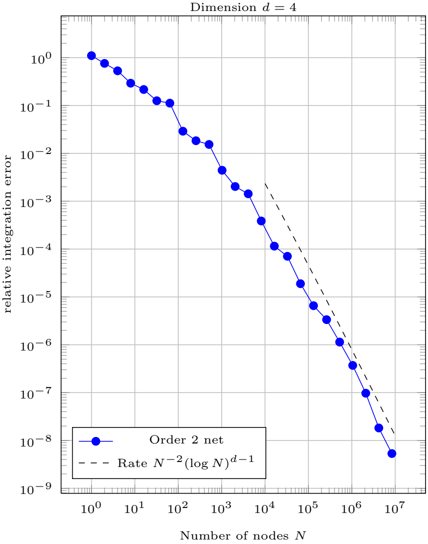
$$f(x_1, ..., x_d) = \sum_{i=1}^{r} g_i^1(x_1) \cdot ... \cdot g_i^d(x_d)$$

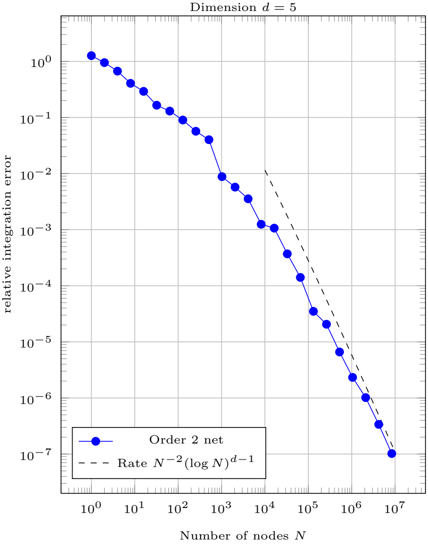
- **Mixed Sobolev-regularity**: $f \in H^{3/2-\varepsilon}_{\text{mix}}$
- Approximation of $I(f) = \int_{[0,1]^d} f(x)\, dx$ with cubature formula

$$Q_N(f) := \sum_{i=1}^{N} \lambda_i f(x^i)$$

- How does the error $|I(f) - Q_N(f)|$ decay?

Integration error kink functions





Mixed Besov regularity

- Plots:

$$|I(f) - Q_N(f)| \sim N^{-2}(\log N)^{d-1}$$

- Where does this rate come from?

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“Sacrifice” integrability and “gain” regularity

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- Minimal worst case error

$$\mathrm{Int}_N(F_d) := \inf_{X_N, \Lambda_N} \sup_{f \in F_d} |I(f) - \Lambda_N(X_N, f)| \,.$$

- **Goal:** $\mathrm{Int}_N(F_d)$ für **mixed Besov-spaces** $F_d := S^r_{p,q}B$ and **mixed Sobolev spaces** S^r_pW

Multivariate functions - mixed Besov regularity

- Iterated differences

$$\Delta_h^2 f(x) := f(x+2h) - 2f(x+h) + f(x)$$

- The Besov norm in case $d = 2$

$$\begin{aligned}
\|f\|_{S^r_{p,q}B} := \|f\|_p &+ \Big(\int_0^1 t^{-rq} \sup_{|h|\le t} \|\Delta^m_{h,1} f\|_p^q \frac{dt}{t}\Big)^{1/q} \\
&+ \Big(\int_0^1 t^{-rq} \sup_{|h|\le t} \|\Delta^m_{h,2} f\|_p^q \frac{dt}{t}\Big)^{1/q} \\
&+ \Big(\int_0^1 \int_0^1 t_1^{-rq} t_2^{-rq} \sup_{\substack{|h_1|\le t_1 \\ |h_2|\le t_2}} \|\Delta^m_{h_1,1} \Delta^m_{h_2,2} f\|_p^q \frac{dt_1}{t_1}\frac{dt_2}{t_2}\Big)^{1/q}
\end{aligned}$$

Optimal QMC

- $R_N(\mathcal{D}_n; f) := I(f) - I_N(\mathcal{D}_n; f)$ ... quasi-Monte Carlo error
- $\mathcal{D}_n$ order-2-digital net with $N = 2^n$ points, **Dick, Pillichshammer**

Theorem (Hinrichs, Markhasin, Oettershagen, U. '15)

Let $1/p < r < 2$. *Then*

$$Int_N(S^r_{p,q}B(\mathbb{T}^d)) \asymp \sup_{f \in S^r_{p,q}B(\mathbb{T}^d)} |R_N(\mathcal{D}_n; f)| \asymp N^{-r}(\log N)^{(d-1)(1-1/q)}.$$

- Restriction on the smoothness $r < 2$ due to the use of the Faber-Schauder system

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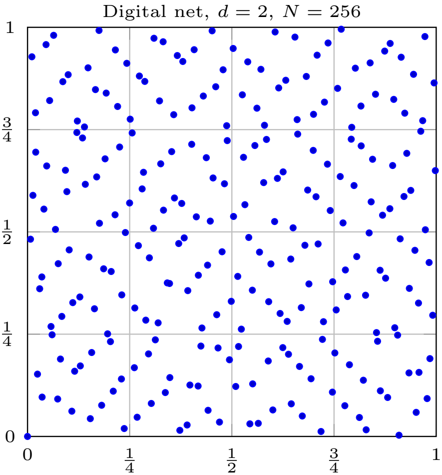
- Restriction on the smoothness $r < 2$ due to the use of the Faber-Schauder system
- Temlyakov 1986: QMC on Korobov lattice, $1/2 < r \leq 1$

$$\mathrm{Int}_N(S^r_{2,\infty}B(\mathbb{T}^d)) \lesssim N^{-r}(\log N)^{(d-1)(r+1/2)}$$

Integration nodes



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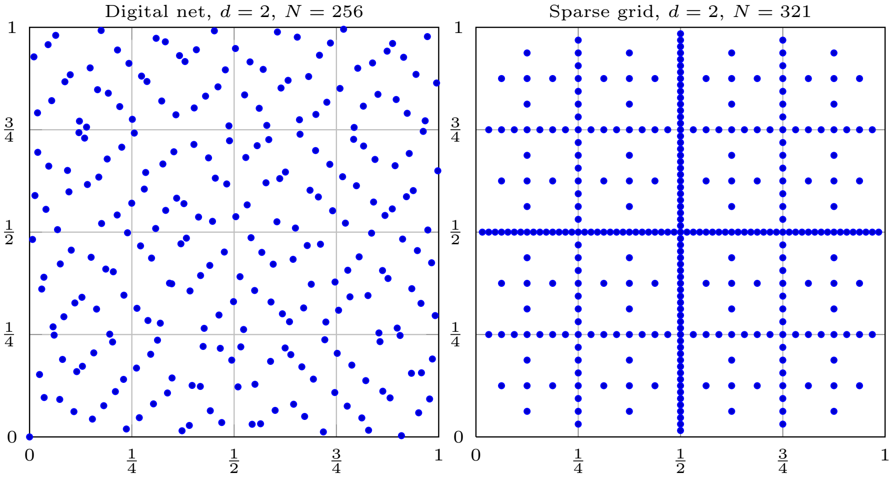
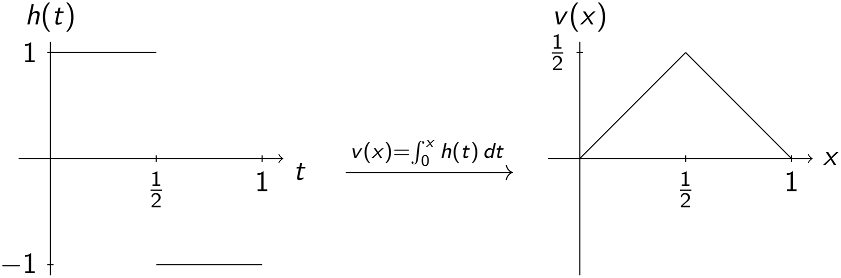


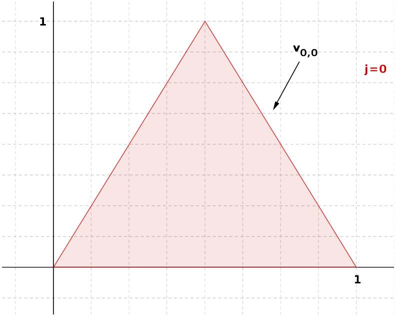
Faber $= \int$ Haar

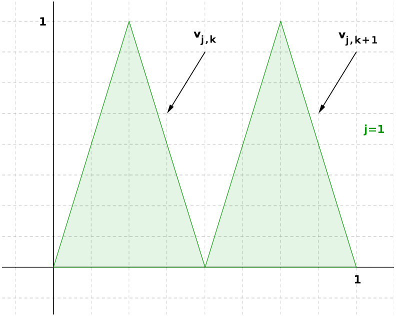


The Faber basis

- Univariate Faber basis $\{x, 1-x, v_{j,k} : j \in \mathbb{N}_0, k \in D_j\}$, decomposition of $f \in C(I)$ into

$$f(x) = f(0)\cdot(1-x) + f(1)\cdot x - \frac{1}{2}\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}(\Delta^2_{2^{-j-1}} f(2^{-j}k) v_{j,k}$$





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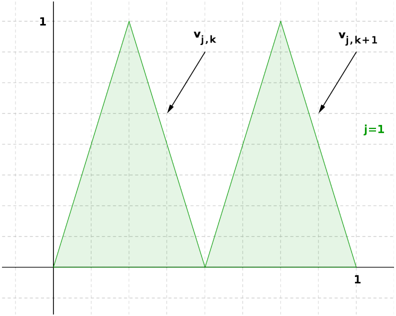

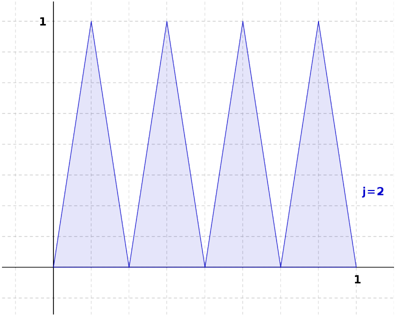

$$
\begin{aligned}
&|R_N(f)| \\
&= \Big| \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} f(x_i) - \int_{\mathbb{T}^2} f(x)\,dx \Big| \\
&= \Big| \sum_{j \in \mathbb{N}^2_{-1}} \sum_{m \in D_j} d^2_{j,m}(f) \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} v_{j,m}(x_i) - \sum_{j \in \mathbb{N}^2_{-1}} \sum_{m \in D_j} d^2_{j,m}(f) \int_{\mathbb{T}^2} v_{j,m}(x)\,dx \Big| \\
&= \Big| \sum_{j \in \mathbb{N}^2_{-1}} \sum_{m \in D_j} d^2_{j,m}(f) c_{j,m} \Big| , \qquad (1)
\end{aligned}
$$

where

$$
c_{j,m} := \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} v_{j,m}(x_i) - \int_{\mathbb{T}^2} v_{j,m}(x)\,dx \quad , \quad j \in \mathbb{N}^2_{-1}, m \in D_j \,.
$$

Integration error

$$
\begin{aligned}
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&= \Big| \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} f(x_i) - \int_{\mathbb{T}^2} f(x)\, dx \Big| \\
&= \Big| \sum_{j \in \mathbb{N}^2_{-1}} \sum_{m \in D_j} d^2_{j,m}(f) \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} v_{j,m}(x_i) - \sum_{j \in \mathbb{N}^2_{-1}} \sum_{m \in D_j} d^2_{j,m}(f) \int_{\mathbb{T}^2} v_{j,m}(x)\, dx \Big| \\
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where

$$
c_{j,m} := \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} v_{j,m}(x_i) - \int_{\mathbb{T}^2} v_{j,m}(x)\, dx \quad , \quad j \in \mathbb{N}^2_{-1}, m \in D_j \, .
$$

$$
\boxed{c_{j,m} = 2^{|j|_1+2} \langle D_{\mathcal{H}_n}, h_{j,m} \rangle} \quad , \quad j \in \mathbb{N}^2_{-1}, m \in D_j
$$

Cubature on sparse grids

- Minimal worst case error

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- Let $N = \sharp \mathcal{S}_m$. Consider **now** the quantity

$$\mathrm{Smol}_N(F_d) := \inf_{\Lambda_N} \sup_{f \in F_d} |I(f) - \Lambda_N(\mathcal{S}_m, f)|$$

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Theorem (Dinh Dũng, U. '13, Temlyakov '90, '15)

Let $d \geq 2$, $1 \leq p, q \leq \infty$, $r > 1/p$. Then

$$Smol_N(S^r_{p,q}B(\mathbb{T}^d)) \asymp N^{-r}(\log N)^{(d-1)(r+1-1/q)}$$

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- Recall: optimal rate

$$\mathrm{Int}_N(S^r_{p,q}B) \asymp N^{-r}(\log N)^{(d-1)(1-1/q)}$$

- However, no restriction on the smoothness!

Frolov's method (1976)

- **Frolov 1976**

- Let $T \in \mathbb{R}^{d\times d}$ be a "suitable" matrix with $\det T = 1$
- $n \in \mathbb{N}$

$$\mathbb{X}_n := n^{-1/d} T(\mathbb{Z}^d) \cap [0,1)^d$$

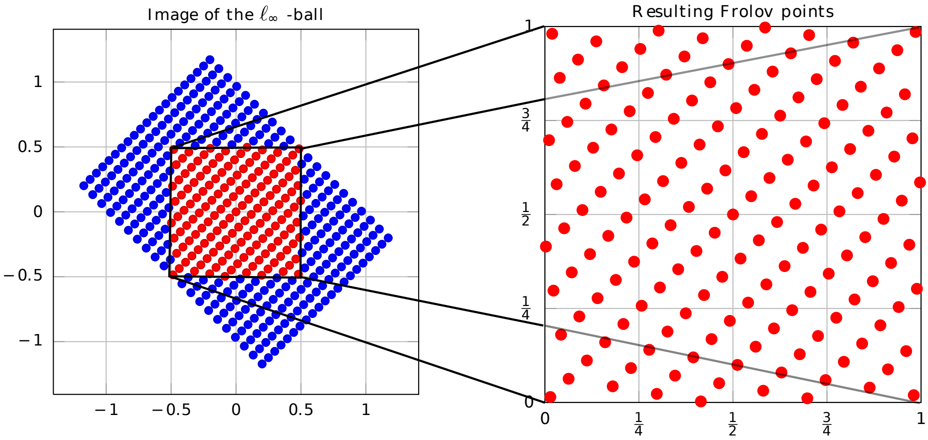
- f continuous and $\operatorname{supp} f \subset [0,1]^d$

$$Q_n(f) = \frac{1}{n} \sum_{x\in\mathbb{X}_n} f(x)$$

- $|\mathbb{X}_n| \asymp n + O((\log n)^{d-1})$
- Poisson summation formula, f continuous, $\operatorname{supp} f \subset [0,1]^d$

$$\det T \sum_{k\in\mathbb{Z}^d} f(Tk) = \sum_{k\in\mathbb{Z}^d} \mathcal{F}f(Bk)$$

Frolov points



Frolov "sees" higher regularity

Theorem (Dubinin' 97, M. Ullrich, T. U. '15)

1. *Let* $1 \le p \le \infty$ *and* $r > 1/p$

$$Int_N(S^r_{p,q}B) \asymp \sup_{f \in S^r_{p,q}B} |R_N(\mathbb{X}_n; f)| \asymp N^{-r}(\log N)^{(d-1)(1-1/q)}$$

2. *Let* $1 < p < \infty$ *und* $r > \max\{1/p, 1/2\}$. *Then*

$$Int_N(S^r_pW) \asymp \sup_{f \in S^r_pW} |R_N(\mathbb{X}_n; f)| \asymp N^{-r}(\log N)^{(d-1)/2}$$

- **Optimal**: asymptotical order (among all cubature formulas)

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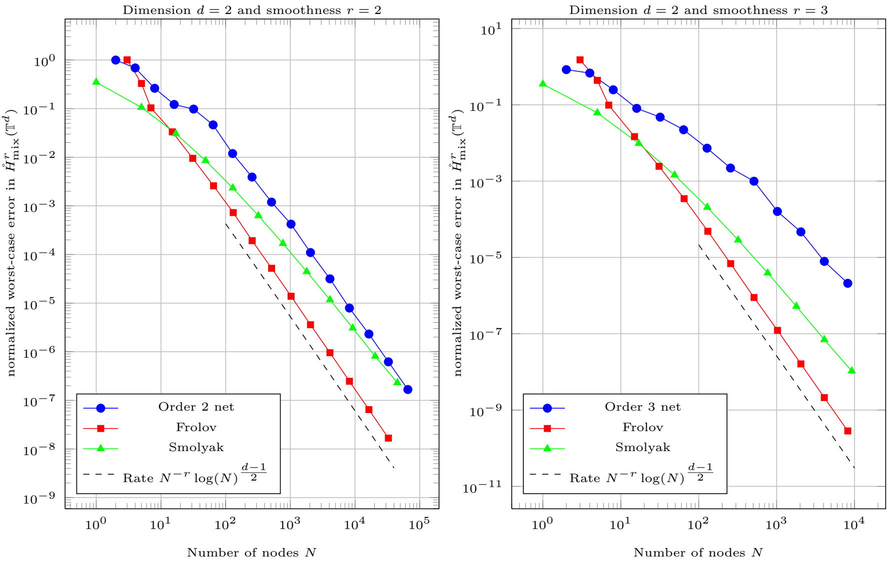
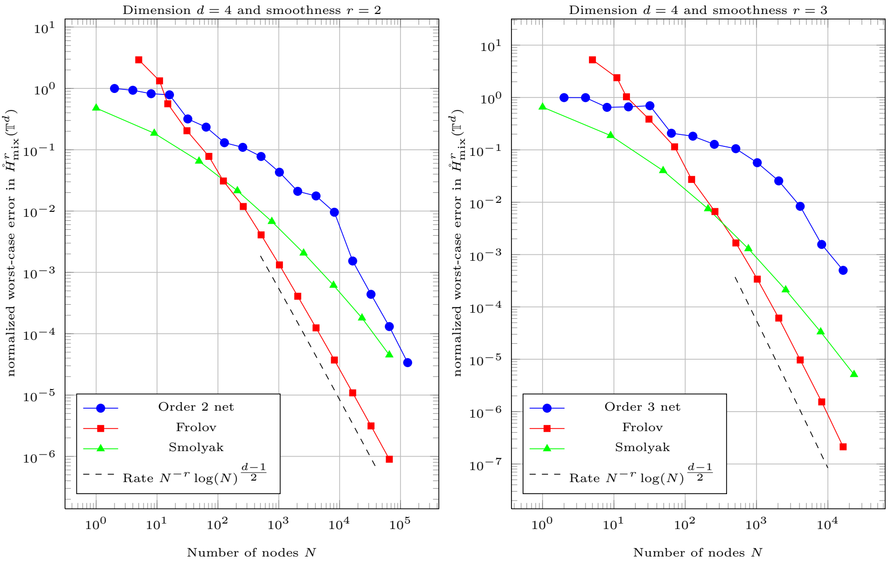
- **Optimal**: asymptotical order (among all cubature formulas)
- **Simple**: rescaling by n
- Method of proof $\Longrightarrow$ results for Triebel-Lizorkin spaces

Numerical results $d = 2$, $r = 2, 3$



Numerical results $d = 4$, $r = 2, 3$



Historical comments

- 1972 Bakhvalov, lower bounds for $\mathrm{Int}_N(S^r_{p,\infty}B)$
- 1976 Frolov, upper bound $\mathrm{Int}_N(S^r_2\mathring{W})$ $1 < p < \infty$
- 1985 Bykowski, upper bound $\mathrm{Int}_N(S^r_2 W)$, $r \in \mathbb{N}$ (change of variable)
- 1990 Temlyakov, upper bound $\mathrm{Int}_N(S^r_p W)$, $r \geq 1$, $2 \leq p < \infty$
- 1992 Dubinin upper bound $\mathrm{Int}_N(S^r_{p,\infty}B)$ for $r > 1/p$, $1 < p \leq \infty$
- 1994 Skriganov, upper bound $\mathrm{Int}_N(S^r_p W)$, $r \in \mathbb{N}$, $1 < p < \infty$
- 1997 Dubinin, upper bound $\mathrm{Int}_N(S^r_{p,q}B)$ for $r > 1/p$, $1 \leq p, q \leq \infty$
- 2013 Hinrichs, Markhasin, upper bound $\mathrm{Int}_N(S^r_{p,q}B)$ and $\mathrm{Int}_N(S^r_{p,q}F)$, $\mathbf{r \leq 1}$, $r > \max\{1/p, 1/q\}$

Small smoothness?

Theorem

1. *Let* $1 < p < \infty$ *und* $r > \max\{1/p, 1/2\}$. *Then*

$$Int_N(S^r_pW) \asymp \sup_{f \in S^r_pW} |R_N(\mathbb{X}_n; f)| \asymp N^{-r}(\log N)^{(d-1)/2}$$

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- **Temlyakov 1991**: The Fibonacci cubature formulas for bivariate periodic functions show an interesting asymptotic behavior in $S^r_p W$ if $2 < p < \infty$ and $1/p < r \leq 1/2$

The tensor Haar basis in spaces with mixed derivative

Theorem (Seeger, U. '15)

Let $1 < p < \infty$. The tensor Haar system $\{h_{j,k}\}_{j \in \mathbb{N}_{-1}^d, k \in \mathbb{Z}^d}$ is an unconditional basis in $S_p^r W(\mathbb{R}^d)$ if and only if

$$\max\left\{\frac{1}{p} - 1, -\frac{1}{2}\right\} < r < \min\left\{\frac{1}{p}, \frac{1}{2}\right\}.$$

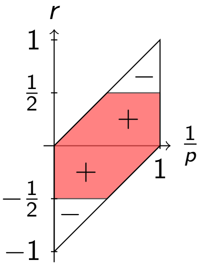

Figure: Haar basis for Sobolev

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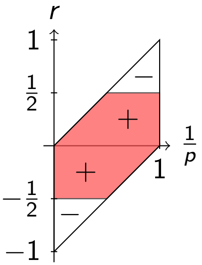

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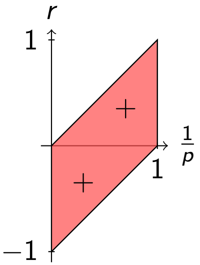

Figure: Haar basis for Besov

The tensor Faber basis in spaces with mixed derivative

Theorem (Byrenheid, Seeger, U. '15)

Let $1 < p < \infty$. The Faber system $\{v_{j,k}\}_{j\in\mathbb{N}_{-1}^d, k\in\mathbb{Z}^d}$ is an unconditional basis in $S^r_pW(\mathbb{R}^d)$ if and only if

$$\max\left\{\frac{1}{2},\frac{1}{p}\right\} < r < \min\left\{1+\frac{1}{p},\frac{3}{2}\right\}.$$

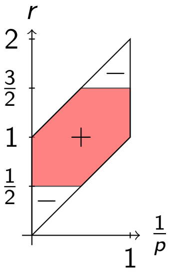

Figure: Faber basis for Sobolev

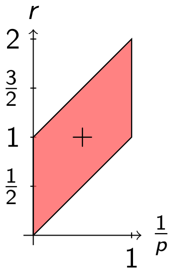

Figure: Faber basis for Besov

Range of small smoothness

Theorem (Byrenheid, Seeger, U. '15)

Let $1 < p < \infty$. The Faber system $\{v_{j,k}\}_{j\in\mathbb{N}^d_{-1},k\in\mathbb{Z}^d}$ is an unconditional basis in $S^r_pW(\mathbb{R}^d)$ if and only if

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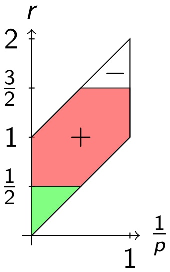

Figure: Small smoothness

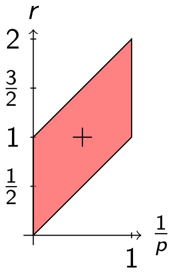

Figure: Faber basis for Besov

Integration for small Sobolev smoothness

Theorem (M. Ullrich., T. U. '15)

Let $1 < p < \infty$ and $r > 1/p$. For the Sobolev spaces S^r_pW it holds in the cases

1. $p > 2$, $1/p < r < 1/2$:

$$\sup_{f \in S^r_pW} |R_N(\mathbb{X}_n; f)| \lesssim N^{-r}(\log N)^{(d-1)(1-r)}$$

2. $p > 2$, $r = 1/2$:

$$\sup_{f \in S^r_pW} |R_N(\mathbb{X}_n; f)| \lesssim N^{-r}(\log N)^{(d-1)/2}\sqrt{\log\log N}$$

- Work in progress: optimality for Frolov's method
- Optimality for general cubature formulas is open!

Thank you for your attention!

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