Sparse approximation with respect to the Faber-Schauder System

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Motivation: quantum mechanics

- Hamilton operator of the elektronic Schrödinger equation

$$H = -\frac{1}{2}\sum_{i=1}^{N}\Delta_i - \sum_{i=1}^{N}\sum_{\nu=1}^{K}\frac{Z_\nu}{|x^i - a_\nu|} + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}\frac{1}{|x^i - x^j|}$$

- Operates on functions $f(x^1, ..., x^N)$, $x^i \in \mathbb{R}^3$
- Δ_i...Laplace operator wrt. $x^i = (x^i_1, x^i_2, x^i_3) \in \mathbb{R}^3$
- $\{x^i\}_{i=1}^N$ Positions of N electrons
- $\{a_\nu\}_{\nu=1}^K$ Positions of the (fixed) nuclei
- $\{Z_\nu\}_{\nu=1}^K$ Charges of the nuclei
- Invariant states: **Eigenfunctions** of H

$$\boxed{Hf = \lambda f}$$

Mixed Sobolev regularity

- $f(x^1, ..., x^N)$
- Number of electrons N large $\Longrightarrow$ increasing complexity!

$$d = 3N$$

- **Yserentant '04:** Some eigenfunctions of H possess mixed weak derivatives in L_2 with increasing order (in the number of electrons N)
- product weights on the "frequency space"

$$\|f\|_{mix}^2 = \int\limits_{\mathbb{R}^{3N}} \Big(1 + \sum_{i=1}^{N} |\omega^i|^2\Big) \prod_{i \in I} \Big(1 + |\omega^i|^2\Big) |\hat{f}(\omega^1, ..., \omega^N)|^2 d\omega < \infty$$

Approximation of the eigenfunctions

- **Motivation:** Galerkin methods
- Discretization of the problem $\Longrightarrow$ finite elements (hat functions)

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- **Motivation:** Galerkin methods
- Discretization of the problem $\Longrightarrow$ finite elements (hat functions)
- Look for an "optimal" m-dimensional subspace in the sense of the **approximation/Kolmogorov numbers**
- **F**...function space, X...error norm

$$a_m(\mathbf{F}, X) := \inf_{rank\ A \leq m} \sup_{f \in \mathbf{F}} \|f - Af\|_X$$

- Fourier partial sum

$$Af = S_m f := \sum_{|k| \leq m} \hat{f}(k) e^{ik \cdot x}$$

- $1 = a_0 \geq a_1 \geq ... \geq a_m$
- See talks by Sickel, Kühn, Mayer,...

Adaptive approximation

- Kolmogorov numbers: **worst-case error** with respect to a class $\mathbf{F}$

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- In practice: **One single instance** of the problem...
- $\Longrightarrow$ **Adaptive approximation**
- Requires **nonlinear methods**

A univariate example

- $f(x) = |sin(x)|$

$$\|f - S_m(f)\|_\infty \asymp m^{-1}$$

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- Trigonometric system: $\{e^{ikx} \ : \ k \in \mathbb{Z}\}$
- **Majorov** (1970s): $|\Lambda_m| = m$

$$\Big\|f - \sum_{k\in\Lambda_m} \lambda_k e^{ikx}\Big\|_\infty \asymp m^{-3/2}$$

(Improves on a result by **Ismagilov**: $m^{-6/5+\varepsilon}$)

- Constructive approach!

Sparse approximation

- Best m-term approximation wrt. a dictionary $\mathcal{D} = \{\varphi_j\}_j$
- X quasi-Banach space, $f \in X$

$$\sigma_m(f,\mathcal{D})_X = \inf\Big\{\Big\|f - \sum_{j\in\Lambda}\lambda_j\varphi_j\Big\|_X \ :\ |\Lambda| \leq m, \lambda_j \in \mathbb{C}, \varphi_j \in \mathcal{D}\Big\}$$

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- $f \in \mathbf{F}$
- Best m-term widths

$$\sigma_m(\mathbf{F}, \mathcal{D})_X := \sup_{f \in \mathbf{F}} \sigma_m(f, \mathcal{D})_X$$

- $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m$

Largest coefficients?

- $0 < p < 1$

$$\|x\|_p := \Big(\sum_{j=1}^{\infty} |x_i|^p\Big)^{1/p} \leq 1$$

- $0 < p < q$, $\mathcal{D} = \{e_j\}_j$

$$\sigma_m(x, \mathcal{D})_{\ell_2} \lesssim m^{-(1/p-1/2)}$$

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- Note, the trigonometric system is not an unconditional basis in L_p, $p \neq 2$
- **Temlyakov, Konyagin**: $\mathcal{D}$ is greedy

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- **Pietsch/ DeVore, Popov** 1980s

$$\mathcal{A}_p^{\alpha}(\ell_2) = \ell_p \quad , \quad \alpha = 1/p - 1/2\,.$$

Functions with bounded mixed derivative

- r **smoothness**, $1 < p < \infty$ **integrability**
- r integer, $d = 2$

$$\|f\|_{S^r_pW} := \|f\|_p + \left\|\frac{\partial^r f}{\partial x_1^r}\right\|_p + \left\|\frac{\partial^r f}{\partial x_2^r}\right\|_p + \left\|\frac{\partial^{2r} f}{\partial x_1^r \partial x_2^r}\right\|_p$$

- $f : \mathbb{T}^d \to \mathbb{C}$, multivariate, periodic in every component
- $\hat{f}(k)$, Fourier coefficient of the "frequency" $k \in \mathbb{Z}^d$

$$\|f\|_{S^r_pW} := \Big\| \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \prod_{i=1}^{d} (1 + |k_i|^2)^{r/2} e^{i2\pi kx} \Big\|_p$$

- $S^r_{2,2}B = H^r_{mix} = S^r_2 W$

Tensor products

- $f_1, \ldots, f_d \in W^r_p(\mathbb{R})$
- Consequence of the definition

$$f_1 \otimes \ldots \otimes f_d \in S^r_p W(\mathbb{R}^d)$$

- p-nuclear tensor norm α_p (Grothendieck)

$$S^r_p W(\mathbb{R}^d) = W^r_p(\mathbb{R}) \otimes_{\alpha_p} \cdots \otimes_{\alpha_p} W^r_p(\mathbb{R})$$

- Sobolev embedding: $1 < p < q < \infty$

$$S^r_p W(\mathbb{R}^d) \hookrightarrow S^{r-1/p+1/q}_q W(\mathbb{R}^d)$$

- **Compact embedding into $C(\mathbb{T}^d)$:** $r > 1/p$

$$S^r_p W(\mathbb{T}^d) \hookrightarrow C(\mathbb{T}^d)$$

Littlewood-Paley decomposition

- $f = \sum_j \delta_j f(x)$
- $1 < p < \infty$, Littlewood-Paley

$$\|f\|_p \asymp \Big\| \Big(\sum_{j \in \mathbb{Z}^d} |\delta_{(j_1,\ldots,j_d)} f(x)|^2 \Big)^{1/2} \Big\|_p$$

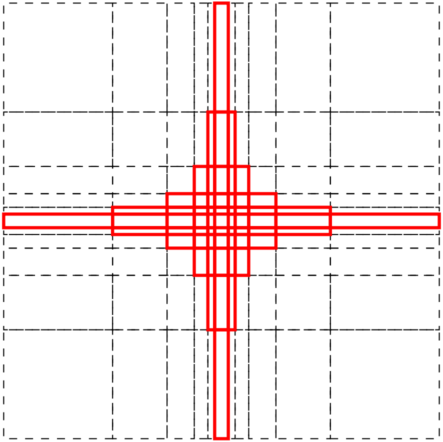
- Now for $S^r_p W$

$$\|f\|_{S^r_p W} \asymp \Big\| \Big(\sum_{j \in \mathbb{Z}^d} 2^{2(r_1 j_1 + \ldots + r_d j_d)} |\delta_{(j_1,\ldots,j_d)} f(x)|^2 \Big)^{1/2} \Big\|_p$$

- Recall: classical definition

$$\|f\|_{S^r_p W} := \Big\| \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \prod_{i=1}^{d} (1 + |k_i|^2)^{r/2} \; e^{i 2\pi k x} \Big\|_p$$

Step hyperbolic cross

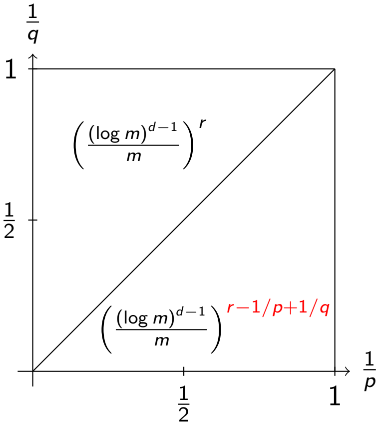


- Take the dyadic blocks from the (step-)hyperbolic cross $\mathcal{H}_n$

$$P_{\mathcal{H}_n} f := \sum_{j_1+\ldots+j_d \le n} \delta_{j_1,\ldots,j_d} f$$

- $f - P_{\mathcal{H}_n} f = \sum_{j_1+\ldots+j_d > n} \delta_{j_1,\ldots,j_d} f$
- L_p-**Error:** $\lesssim 2^{-r \cdot n}$
- **Cost:** $m \asymp 2^n n^{d-1}$
- **Rate:** $m^{-r} (\log m)^{(d-1)r}$

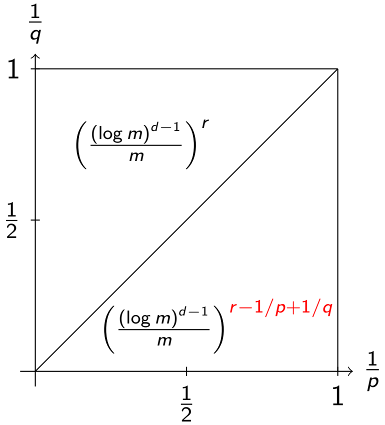
Hyperbolic cross and sparse trig. approximation



Best approximation from the hyperbolic cross

Hyperbolic cross and sparse trig. approximation

Temlyakov (recently)



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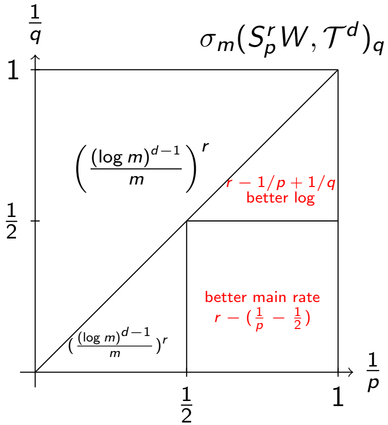

Sparse trigonometric approximation
"large" smoothness

Wavelet dictionaries

- $(\psi_{j,k})_{j,k}$ univariate wavelet basis $j \in \mathbb{N}_0, k \in \mathbb{Z}$
- Hyperbolic wavelets, **tensorization over all scales**

$$\psi_{\bar{j},\bar{k}}(x_1,\ldots,x_d) := \psi_{j_1,k_1}(x_1)\cdot\ldots\cdot\psi_{j_d,k_d}(x_d) \quad , \quad \bar{j} \in \mathbb{N}_0^d, \bar{k} \in \mathbb{Z}^d$$

- In contrast to $\mathcal{T}^d = \{e^{ik\cdot x}\}_{k\in\mathbb{Z}^d}$ the wavelet dictionary $\Psi^d = \{\psi_{\bar{j},\bar{k}}\}_{\bar{j},\bar{k}}$ is an unconditional basis in L_p for all $1 < p < \infty$

$$\|f\|_p \asymp \Big\| \Big(\sum_{\bar{j}\in\mathbb{N}_0^d} \Big| \sum_{\bar{k}\in\mathbb{Z}^d} 2^{|\bar{j}|_1} \langle f, \psi_{\bar{j},\bar{k}} \rangle \chi_{\bar{j},\bar{k}} \Big|^2 \Big)^{1/2} \Big\|_p$$

- **Disadvantage:** the basis is not "democratic" for $d \geq 2$

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- **Temlyakov, Konyagin**:

$$\boxed{\text{greedy} = \text{unconditional} + \text{democratic}}$$

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- Instead of greedy: levelwise thresholding!

Best m-term widths

- $(\psi_{j,k})_{j,k}$ wavelet with sufficient smoothness and vanishing moments

Theorem (Hansen, Sickel '09)

Let $1 < p, q < \infty$ *and* $r > (1/p - 1/q)_+$. *Then*

$$\sigma_m(S^r_pW, \Psi^d)_q \asymp m^{-r}(\log m)^{(d-1)r} \quad , \quad m \in \mathbb{N}$$

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$$\sigma_m(S^r_pW, \Psi^d)_q \asymp m^{-r}(\log m)^{(d-1)r} \quad , \quad m \in \mathbb{N}$$

- Non-compact embedding: $S^{1/p-1/q}_pW \hookrightarrow L_q$

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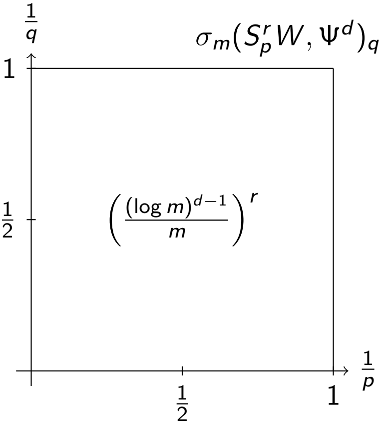
Sparse hyperbolic wavelet approximation

Temlyakov,Hansen,Sickel,Dũng,...



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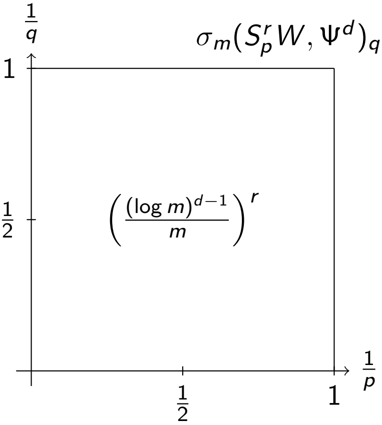
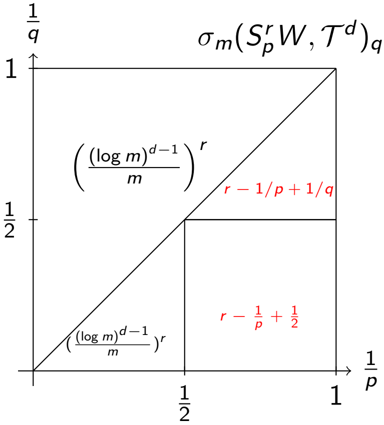


Sparse wavelet approximation

Temlyakov

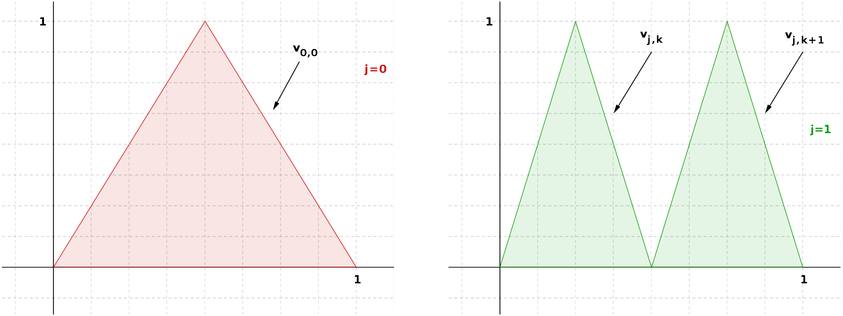


Sparse trigonometric approximation

The Faber-Schauder basis

- **Faber 1908:** Univariate hat functions: decomposition of $f \in C(\mathbb{T})$

$$f(x) = f(0) - \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} d_{j,k}(f) v_{j,k}$$



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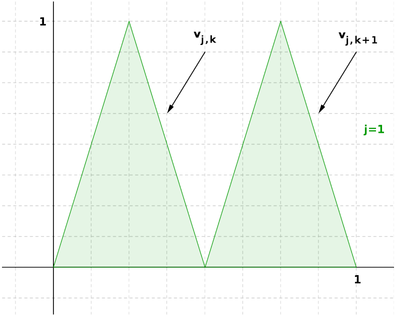
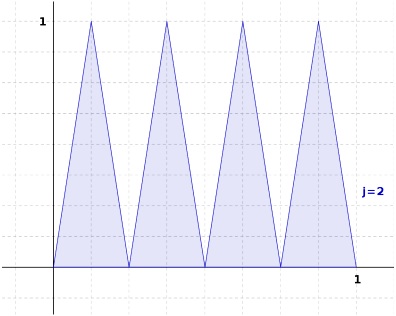
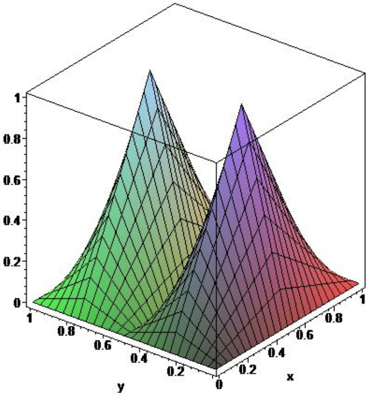
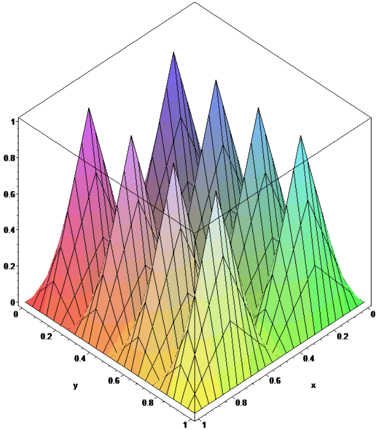




Tensorized Faber-Schauder basis





Sparse Faber-Schauder approximation

- Let $\mathcal{F}^d := (v_{\bar{j},\bar{k}})_{\bar{j},\bar{k}}$ be the Faber-Schauder dictionary
- **Problem:** No unconditional basis in any $L_q(\mathbb{T}^d)$
- Solution: $f = \sum_j \sum_k d^2_{\bar{j},\bar{k}}(f) v_{\bar{j},\bar{k}}$

$$\|f\|_q \leq \sum_{\bar{j}} \Big\| \sum_{\bar{k}} d^2_{\bar{j},\bar{k}}(f) v_{\bar{j},\bar{k}} \Big\|_q$$

right-hand side is related to $s^0_{q,1}b$ sequence space

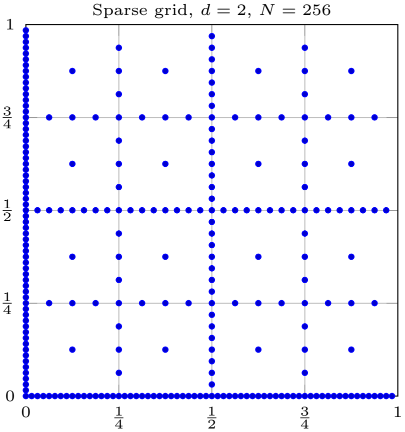
- Those target spaces have been considered by **Hansen, Sickel '09**

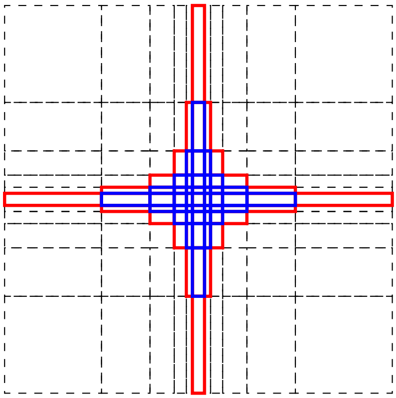
Theorem (Byrenheid, U.)

Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $2 > r > \max\{1/p, 1/2\}$ then

$$\sigma_m(S^r_pW, \mathcal{F}^d)_q \lesssim m^{-r}(\log m)^{(d-1)(r+1/2)}$$

Which coefficients (points) are used?





Linear approximation, sampling recovery

- **Sampling nodes** $\xi = \{\xi^1, ..., \xi^m\}$
- Linear recovery operators of type

$$R_{\xi,\Psi} f := \sum_{i=1}^{m} f(\xi^i)\psi^i(\cdot)$$

- Sampling numbers

$$\varrho_m(\mathbf{F}, X) := \inf_{\substack{\xi^1,...,\xi^m \\ \psi^1,...,\psi^m}} \sup_{f \in \mathbf{F}} \|f - R_{\xi,\Psi} f\|_X$$

- **Temlyakov '93, Ismagilov '74**: $r > 1/2$

$$a_m(H^r_{mix}, L_\infty) \asymp \varrho_m(H^r_{mix}, L_2) \asymp m^{-(r-1/2)}(\log m)^{(d-1)r}$$

Unconditional tensor Faber basis

Theorem (Byrenheid, U. '16, Triebel '10)

Let $1 < p < \infty$. *The tensorized Faber-Schauder system* $\{v_{\bar{j},\bar{k}}\}_{\bar{j}\in\mathbb{N}_{-1},\bar{k}\in\mathbb{Z}}$ *is an uncond. basis in* $S^r_pW(\mathbb{T}^d)$ *if*

$\max\left\{1/2, 1/p\right\} < r < \min\left\{3/2, 1+1/p\right\}$. *Then*

$$\|f\|_{S^r_pW} \asymp \Big\|\Big(\sum_{\bar{j}\in\mathbb{N}_0^d} 2^{|\bar{j}|_1 r2}\Big|\sum_{\bar{k}\in D_{\bar{j}}} d^2_{\bar{j},\bar{k}}(f)\chi_{\bar{j},\bar{k}}(\cdot)\Big|^2\Big)^{1/2}\Big\|_p$$

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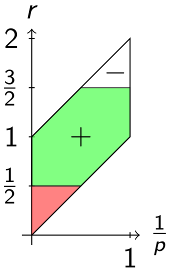

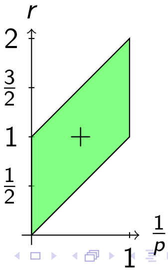

Comments on the proof

- Conjecture by **Triebel '10**
- Extending the univariate proof technique to d dimensions requires a non-trivial **direction-adapted** use of complex interpolation by **Mendez, Mitrea**
- Vector-smoothness $r = (r_1, ..., r_d)$

Small smoothness effect?

Theorem (Byrenheid, U. '16)

Let $2 < p < \infty$, $1 \leq q \leq \infty$ *and* $1/p < r \leq 1/2$. *Then*

$$\sigma_m(S^r_pW, \mathcal{F}^d)_q \lesssim m^{-r}(\log m)^{(d-1)(1+\varepsilon)}$$

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Recall: large smoothness...

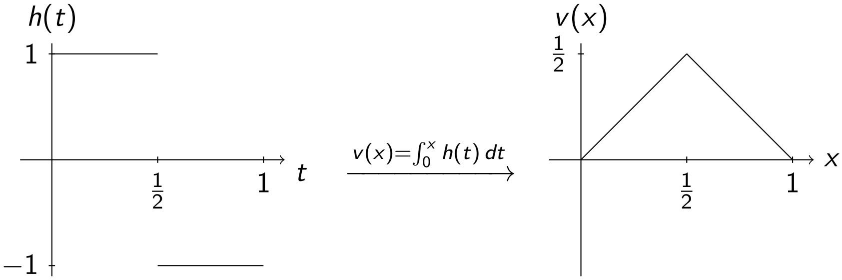
Theorem

Let $1 < p < \infty$, $1 < q \leq \infty$ *and* $2 > r > \max\{1/p, 1/2\}$ *then*

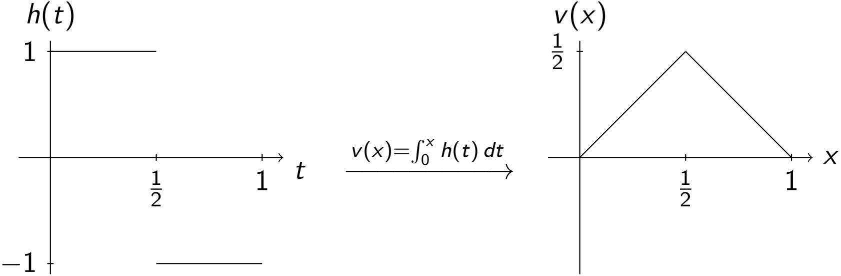
$$\sigma_m(S^r_pW, \mathcal{F}^d)_q \lesssim m^{-r}(\log m)^{(d-1)(r+1/2)}$$

- Is there a small smoothness effect?
- Faber basis for spaces in the red region?

Faber $= \int$ Haar



Faber $= \int$ Haar



- The Haar wavelet-system

$$h_{j,k}(x) := 2^{j/2} h(2^j x - k) \quad , \quad j,k \in \mathbb{Z}\,.$$

is an orthonormal basis in $L_2(\mathbb{R})$

100 years Haar and Faber basis

- **1908 Faber**, Über stetige Funktionen, Math. Ann. 66
- **1910 Haar**, Zur Theorie der orthogonalen Funktionensysteme, Math. Ann. 69
- **1910 Faber**, Über die Orthogonalfunktionen des Herrn Haar, DMV Jahresber.
- **1928 Schauder**, Eine Eigenschaft der Haarschen Orthog.-systeme, Math. Z. 28
- **1932 Paley**, A remarkable series of orthogonal functions, Proc. London Math. Soc. 34
- **1937 Marcinkiewicz**, Quelques théor. sur les sér. orthogonales, Ann. Sci. Math.
- **1972 Ciesielski**, Spline bases in function spaces, Proc. conf. Appr. Theory
- **1973 Triebel**, Über die Existenz von Schauderbasen in Sobolev-Räumen
- **1978 Triebel**, On Haar bases in Besov spaces, Serdica 4
- **2010 Triebel** Discrepancy, numerical integration, and hyperbolic cross approximation, EMS Zürich

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- Haar basis in the Sobolev space $W_p^r(\mathbb{R})$?

An unconditional basis in L_p

- In contrast to the trigonometric system $\{e^{ikx}\}_{k\in\mathbb{Z}}$ the Haar wavelet system is an unconditional basis in L_p for $p \neq 2$

Theorem (Marcinkiewicz '37, Paley '32)

Let $1 < p < \infty$.
(i) *The Haar system* $\{h_{j,k}\}_{j,k\in\mathbb{Z}}$ *is an unconditional basis in* $L_p(\mathbb{R})$.
(ii) *There exist constants* $0 < c < C$ *such that*

$$c\|f\|_p \leq \Big\| \Big(\sum_{j\in\mathbb{Z}} \Big| \sum_{k\in\mathbb{Z}} 2^j \langle f, h_{j,k}\rangle \chi_{j,k} \Big|^2 \Big)^{1/2} \Big\|_p \leq C\|f\|_p$$

- The statement does neither hold in case $p = 1$ nor $p = \infty$!

The Haar-basis in Sobolev spaces

Theorem (Seeger, U. '16)

Let $1 < p < \infty$. The Haar system $\{h_{j,k}\}_{j\in\mathbb{N}_{-1},k\in\mathbb{Z}}$ is an unconditional basis in $W_p^r(\mathbb{R})$ if and only if

$$\max\left\{\frac{1}{p}-1,-\frac{1}{2}\right\} < r < \min\left\{\frac{1}{p},\frac{1}{2}\right\}.$$

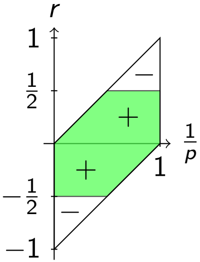


Figure: Unconditional basis in W_p^r

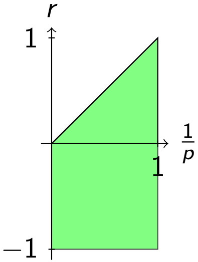


Figure: Haar functions in W_p^r

Unconditional bases

Definition

Let X be a Banach space. Any biorthogonal system (x_n, x_n^*) with a countable index set A is an unconditional basis in X if

- $\overline{\mathrm{span}}\{x_n\}_n = X$
- There exists a constant $C > 0$ such that for every $x \in X$ and every finite index set $B \subset A$

$$\Big\| \sum_{n \in B} x_n^*(x) x_n \Big\|_X \leq C \|x\|_X$$

- **In other words:** the norm of any projection

$$P_B(x) := \sum_{n \in B} x_n^*(x) x_n$$

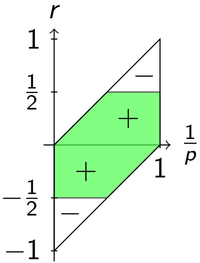
is uniformly bounded by C

Large projection norms

- Let $E \subset \{h_{j,k}\}_{j \in \mathbb{N}_{-1}, k \in \mathbb{Z}}$ a finite subset of the Haar system
- Associated projections

$$P_E(f) := \sum_{h_{j,k} \in E} 2^j \langle f, h_{j,k} \rangle h_{j,k}$$

"grow" with $\#E$ (in the "lower" und "upper" triangle)



- **Probabilistic** construction
- **Quantitative** version (sharp growing rates)

Figure: Unconditional basis in $S^r_p W$

A probabilistic argument

- $\#A \sim 2^{N+1}$
- Random operator

$$T_{A,\varepsilon} g(x) = \sum_{2^j \in A} \varepsilon_j \sum_{\mu=0}^{2^j-1} 2^j \langle h_{j,\mu}, g \rangle h_{j,\mu}(x)$$

- Random test function

$$f_{A,\varepsilon'} = \sum_{2^k \in A} \varepsilon'_k 2^{-(k+N)r} \sum_{\mu=1}^{2^k} \eta_{k+N,2^N\mu}$$

- Via **Khinchine's inequality**

$$\Big(\mathbb{E}_{\varepsilon'} \mathbb{E}_{\varepsilon} \big\| T_{A,\varepsilon} f_{A,\varepsilon'} \big\|_{H^r_p}^2 \Big)^{1/2} \geq c 2^{N(-r-1/2)}$$

Thank you for your attention!