Preasymptotic error bounds for multivariate approximation problems

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Based on joint works with T. Kühn, S. Mayer, W. Sickel

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Multivariate approximation in $L_2(\mathbb{T}^d)$

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- Approximate F_d via **linear algorithms** $A_{n,d}$ of rank n

$$f \approx A_{n,d}(f) := \sum_{j=1}^{n} \lambda_j(f) g_j \,,$$

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- Minimal worst-case error: **Approximation numbers** a_n
- This talk: **Estimates for small n**

A classical result

- **J.W. Jerome '67**

$$c_s(d)n^{-s/d} \leq a_n(H^s(\mathbb{T}^d), L_2(\mathbb{T}^d)) \leq C_s(d)n^{-s/d} \tag{1}$$

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- **There is still hope...**

Theorem (Kühn, Sickel, U. '14)

Let $s > 0$. It holds

$$\lim_{n\to\infty} n^{s/d}\, a_n(H^s(\mathbb{T}^d), L_2(\mathbb{T}^d)) = vol(B_2^d)^{s/d} \asymp d^{-s/2}$$

Complexity for small errors

IBC language

- Previous result gives a reliable bound on the complexity $n(\varepsilon, F_d)$ for errors $\varepsilon \approx 0$
- In practice moderate errors like $\varepsilon \approx 0.001$ are sufficient
- $\Longrightarrow$ We ask for **preasymptotic error bounds**

Isotropic Sobolev spaces

- Smoothness $m \in \mathbb{N}$
- Multivariate Fourier series

$$f(x) = \frac{1}{(2\pi)^{d/2}} \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{ik \cdot x}$$

$$\| f \|_{H^m}^2 \quad = \quad \sum_{|\alpha|_1 \le m} \| D^\alpha f \|_2^2$$

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$$\asymp_m \sum_{k \in \mathbb{Z}^d} \Big(1 + \sum_{i=1}^{d} |k_i|^2\Big)^m |\hat{f}(k)|^2$$

Equivalent norms – compressible frequencies

- $s > 0$ (fractional smoothness)

$$H^s(\mathbb{T}^d) := \Big\{ f \in L_2(\mathbb{T}^d) : \| f \|^2_{H^{s,p}} := \sum_{k \in \mathbb{Z}^d} \Big(1 + \sum_{i=1}^{d} |k_i|^p\Big)^{2s/p} |\hat{f}(k)|^2 < \infty \Big\}$$

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- Projections, $0 < p < \infty$

$$P_m f := \frac{1}{(2\pi)^{d/2}} \sum_{|k|_p \leq m} \hat{f}(k) e^{ik\cdot x} \quad , \quad n := \text{rank } P_m$$

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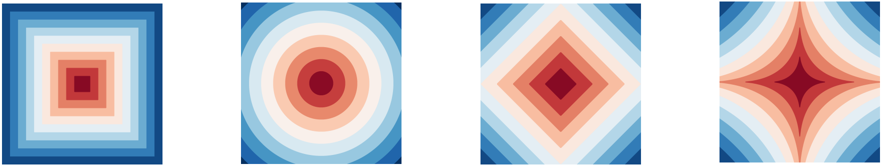
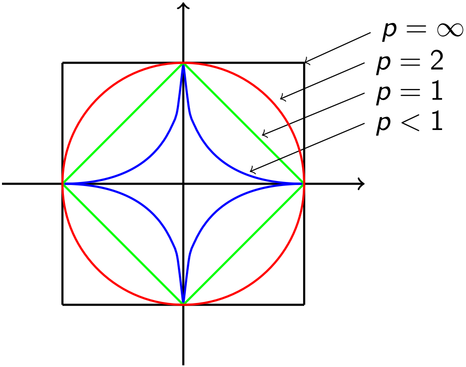


$p = \infty$ $\quad$ $p = 2$ (classical) $\quad$ $p = 1$ $\quad$ $p = \frac{1}{2}$

Unit balls in ℓ_p



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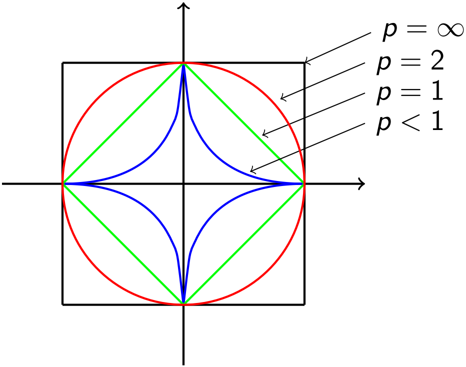

$$\mathrm{vol}(B_p^d) = 2^d \frac{\Gamma(1+\frac{1}{p})^d}{\Gamma(\frac{d}{p}+1)} \quad , \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

$$2\left[\frac{1}{e(p+d)}\right]^{\frac{1}{p}} \leq \mathrm{vol}(B_p^d)^{1/d} \leq 2\left[\frac{e(p+1)}{d}\right]^{\frac{1}{p}}$$

A "compressed" constant

Theorem (Kühn, Mayer, U. '15)

Let $s > 0$ *and* $0 < p < \infty$. *It holds*

$$\lim_{n\to\infty} n^{s/d}\, a_n(H^{s,p}(\mathbb{T}^d), L_2(\mathbb{T}^d)) = vol(B_p^d)^{s/d} \asymp d^{-s/p}$$

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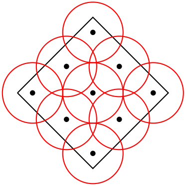
- Now: Preasymptotics...

Main tool: entropy numbers

- **Kolmogorov** entropy
- Let X be a (quasi-)Banach space
- and $M \subset X$ a bounded set
- $\{x_1, \ldots, x_k\} \subset X$ represents an ε-covering of M, if

$$M \subset \bigcup_{j=1}^{k} (x_j + \varepsilon B_X)$$



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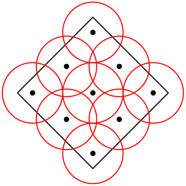

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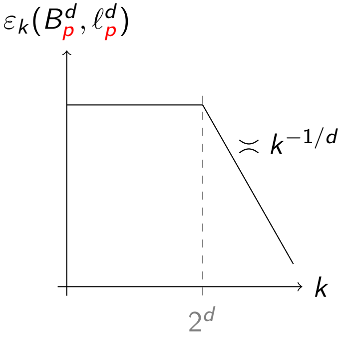
Definition (k-th entropy number)

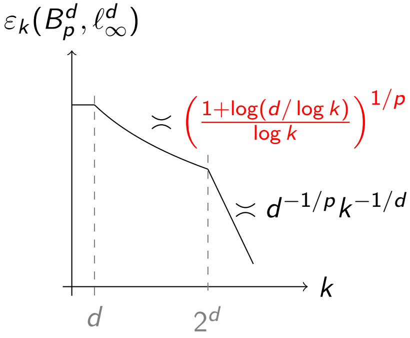
$\varepsilon_k(M, X) := \inf\{\varepsilon > 0 : \exists x_1, \dots, x_k \in X \text{ which } \varepsilon\text{-cover } M \}$

- dyadic entropy numbery $e_k := \varepsilon_{2^{k-1}}$

Entropy numbers of ℓ_p-balls

- **Schütt '84**, **Edmunds, Triebel '96**, **Kühn '01**,
- Lecture notes **T. Ullrich** "Metric entropy and n-widths in approximation theory"





Approximation numbers

Definition (n-te Approximationszahl)

X, Y (quasi-)Banachräume.

$$a_n(X,Y) := \inf_{\text{rank } A<n} \sup_{\|f\|_X \le 1} \|f - Af\|_Y$$
$$= \inf_{\text{rank } A<n} \|Id - A\|_{X\to Y}, \qquad n \in \mathbb{N}$$

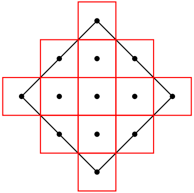
Counting via entropy

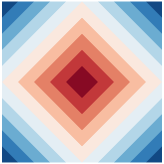
- $s > 0, 0 < p \leq \infty$

$$\| f \|^2_{H^{s,p}} \asymp \sum_{k \in \mathbb{Z}^d} \left(1 + \sum_{i=1}^{d} |k_i|^p\right)^{2s/p} |\hat{f}(k)|^2$$

- Projection operators

$$P_m f := \frac{1}{(2\pi)^{d/2}} \sum_{|k|_p \leq m} \hat{f}(k) e^{ik\cdot x} \quad , \quad n := \text{rank } P_m$$





$p = 1$

Approximation numbers via entropy

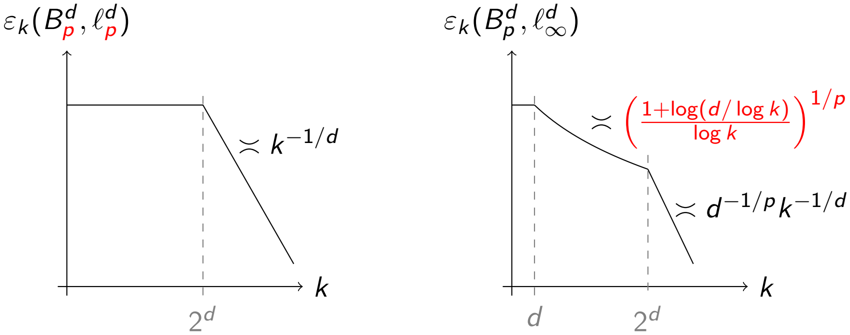
Theorem (Kühn, Mayer, U. '15)

Sei $s > 0$, $0 < p \leq \infty$. *Dann gilt für alle* $n \in \mathbb{N}$

$$a_n(H^{s,p}(\mathbb{T}^d), L_2(\mathbb{T}^d)) \asymp_{p,s} \varepsilon_n(B_p^d, \ell_\infty^d)^s$$

- Constants for Markus: lower bound: $c_{s,p} = 2^{-(1+1/p)s}$, upper bound: $C_s = 4^s$
- $p = \infty \Longrightarrow$ curse !

Entropy numbers of ℓ_p-balls



A first result on preasymptotics

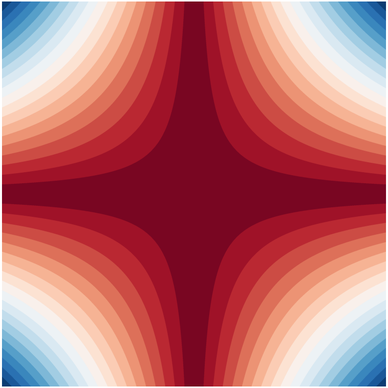
Theorem (Kühn, Mayer, U. '15)

Let $0 < s, p < \infty$ *and* $d \in \mathbb{N}$. *Then*

$$a_n(H^{s,p}(\mathbb{T}^d), L_2(\mathbb{T}^d)) \asymp_{p,s} \begin{cases} 1 & , \ 1 \le n \le d \\ \left(\frac{\log(1+d/\log n)}{\log n}\right)^{s/p} & , \ d \le n \le 2^d \\ d^{-s/p} n^{-s/d} & , \ n \ge 2^d \end{cases}$$

- Compressibility parameter p affects the preasymptotic behavior!

Hyperbolic crosses



- Decay of Fourier coefficients: product weight

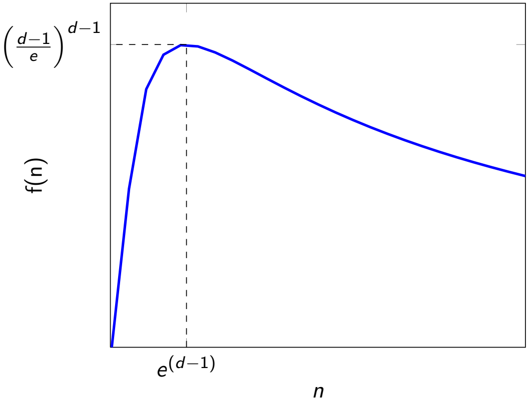
$$\|f\|^2_{H^r_{mix}} := \sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2 \prod_{i=1}^{d} (1+|k_i|^2)^r$$

- Projection on the hyperbolic cross

$$P_{\mathcal{H}_n} f := \frac{1}{(2\pi)^{d/2}} \sum_{k\in\mathcal{H}_n} \hat{f}(k) e^{ik\cdot x}$$

Classical asymptotic rate

- Isotropic: $a_n \asymp n^{-r/d}$
- Mixed: $a_n \asymp_{d,r} n^{-r}(\log n)^{(d-1)r}$ (**Babenko '63**)



The asymptotic constant

Theorem (Kühn, Sickel, U. '14)

Let $r > 0$ *and* $d \in \mathbb{N}$. *Then*

$$\lim_{n\to\infty} \frac{n^r}{(\log n)^{(d-1)r}} \cdot a_n(H^r_{mix}(\mathbb{T}^d), L_2(\mathbb{T}^d)) = \left[\frac{2^d}{(d-1)!}\right]^r$$

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For comparison: The isotropic situation:

Theorem

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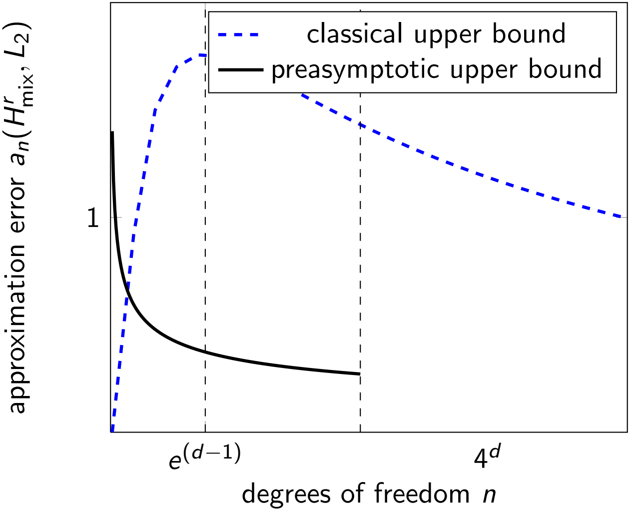
The “preasymptotic range”

Theorem (Kühn, Sickel, U. ’14)

Let $r > 0$, $d \in \mathbb{N}$ *und* $1 \leq n \leq \frac{d}{2} 4^d$. *Then*

$$a_n(I_d : H^r_{\text{mix}}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq \left(\frac{e^2}{n}\right)^{\frac{r}{2+\log_2 d}}$$

Preasymptotic bounds



Outlook

Anisotropic mixed spaces

- $s_1 = \ldots = s_\nu < s_{\nu+1} \le \ldots \le s_d$

$$\|f\|_{H^{\vec{s}}_{mix}} := \sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2 \prod_{i=1}^{d} (1+|k_i|)^{2s_i}$$

- Temlyakov '86: rate $n^{-s_1}(\log n)^{(\nu-1)s_1}$
- Relevant for the stationary Schrödinger equation (**Yserentant '10**)
- Constants and preasymptotic behavior unknown!

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Target space $\mathbf{H}^1$

- Relevant for Galerkin methods, energy norm
- Constants partly known (**Dũng, U. '11**)
- Preasymptotics unknown!
- Break of scale: involved combinatorics

Thank you for your attention!