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Research Article

Continuous Characterizations of Besov-Lizorkin-Triebel Spaces and New Interpretations as Coorbits

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We give characterizations for homogeneous and inhomogeneous Besov-Lizorkin-Triebel spaces (H. Triebel 1983, 1992, and 2006) in terms of continuous local means for the full range of parameters. In particular, we prove characterizations in terms of Lusin functions (tent spaces) and spaces involving the Peetre maximal function to apply the classical coorbit space theory according to Feichtinger and Gröchenig (H. G Feichtinger and K. Gröchenig 1988, 1989, and 1991). This results in atomic decompositions and wavelet bases for homogeneous spaces. In particular we give sufficient conditions for suitable wavelets in terms of moment, decay and smoothness conditions.

1. Introduction

This paper deals with Besov-Lizorkin-Triebel spaces $\dot{B}^s_{p,q}(\mathbb{R}^d)$ and $\dot{F}^s_{p,q}(\mathbb{R}^d)$ on the Euclidean space $\mathbb{R}^d$ and their interpretation as coorbits. For this purpose we prove a number of characterizations for homogeneous and inhomogeneous spaces for the full range of parameters. Classically introduced in Triebel's monograph [1, Section 2.3.1] by means of a dyadic decomposition of unity, we use more general building blocks and provide in addition continuous characterizations in terms of Lusin and maximal functions. Equivalent (quasi-) normings of this kind were first given by Triebel in [2]. His proofs use in an essential way the fact that the function under consideration belongs to the respective space. Therefore, the obtained equivalent (quasi-)norms could not yet be considered as a definition or characterization of the space. Later on, Triebel was able to solve this problem partly in his monograph [3, Sections 2.4.2, 2.5.1] by restricting to the Banach space case.

Afterwards, Rychkov [4] completed the picture by simplifying a method due to Bui et al. [5, 6]. However, [4] contains some problematic arguments. One aim of the present paper is to provide a complete and self-contained reference for general characterizations of discrete and continuous type by avoiding these arguments. We use a variant of a method from Rychkov's subsequent papers [7, 8], which is originally due to Strömberg and Torchinsky developed in their monograph [9, Chapter 5].

In a different language, the results can be interpreted in terms of the continuous wavelet transform (see Appendix A.1) belonging to a function space on the $ax + b$-group $\mathcal{G}$. Spaces on $\mathcal{G}$ considered here are mixed norm spaces like tent spaces [10] and Peetre-type spaces. The latter are indeed new and received their name from the fact that quantities related to the classical Peetre maximal function are involved. This leads to the main intention of the paper. We use the established characterizations for the homogeneous spaces in order to embed them in the abstract framework of coorbit space theory originally due to Feichtinger and Gröchenig [11–15] in the 1980s. This connection was already observed by them in [11, 14, 15]. They worked with Triebel's equivalent continuous normings from [2] and the results on tent spaces, which were introduced more or less at the same time by Coifman et al. [10] to interpret Lizorkin-Triebel spaces as coorbits. On the one hand the present paper gives a late justification, and on the other hand, we observe that Peetre-type spaces on $\mathcal{G}$ are a much better choice for this issue. Their two-sided translation invariance is immediate and much more transparent as we will show in Section 4.1. Furthermore, generalizations in different directions are now possible. In a forthcoming paper, we will show how to apply a generalized coorbit space theory due to Fornasier and Rauhut [16] in order to recover generalized inhomogeneous spaces based on the characterizations given here. Moreover, the extension of the results to quasi-Banach spaces using a theory developed by Rauhut in [17, 18] is possible.

Once we have interpreted classical homogeneous Besov-Lizorkin-Triebel spaces as certain coorbits, we are able to benefit from the achievements of the abstract theory in [11–15]. The main feature is a powerful discretization machinery which leads in an abstract universal way to atomic decompositions. We are now able to apply this method, which results in atomic decompositions and wavelet bases for homogeneous spaces. More precisely, sufficient conditions in terms of vanishing moments, decay, and smoothness properties of the respective wavelet function are given. Compact support of the used atoms does not play any role here. In particular, we specify the order of a suitable orthonormal spline wavelet system depending on the parameters of the respective space.

The paper is organized as follows. After giving some preliminaries, we start in Section 2 with the definition of classical Besov-Lizorkin-Triebel spaces and their characterization via continuous local means. In Section 3, we give a brief introduction to abstract coorbit space theory, which is applied in Section 4 on the $ax + b$-group $\mathcal{G}$. We recover the homogeneous spaces from Section 2 as coorbits of certain spaces on $\mathcal{G}$. Finally, several discretization results in terms of atomic decompositions and wavelet isomorphisms are established. The underlying decay result of the continuous wavelet transform and some basic facts about orthonormal wavelet bases are shifted to the appendix.

1.1. Notation

Let us first introduce some basic notation. The symbols $\mathbb{R}, \mathbb{C}, \mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ denote the real numbers, complex numbers, natural numbers, natural numbers including 0, and the integers. The dimension of the underlying Euclidean space for function spaces is denoted by d, its elements will be denoted by $x, y, z, \ldots$, and $|x|$ is used for the Euclidean norm. We will use $|k|_1$ for the ℓ_1^d-norm of a vector k. For a multi-index $\overline{\alpha}$ and $x \in \mathbb{R}^d$, we write

$$x^{\overline{\alpha}} = x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d} \tag{1.1}$$

and define the differential operators $D^{\overline{\alpha}}$ and Δ by

$$D^{\overline{\alpha}} = \frac{\partial^{|\overline{\alpha}|_1}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \qquad \Delta = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}. \tag{1.2}$$

If X is a (quasi-)Banach space and $f \in X$, we use $\|f \mid X\|$ or simply $\|f\|$ for its (quasi-)norm. Operator norms of linear mappings $A : X \to Y$ are denoted by $\|A : X \to Y\|$ or simply by $\|A\|$. As usual, the letter c denotes a constant, which may vary from line to line but is always independent of f, unless the opposite is explicitly stated. We also use the notation $a \lesssim b$ if there exists a constant $c > 0$ (independent of the context-dependent relevant parameters) such that $a \leq c\, b$. If $a \lesssim b$ and $b \lesssim a$, we will write $a \asymp b$.

2. Function Spaces on $\mathbb{R}^d$

2.1. Vector-Valued Lebesgue Spaces

The space $L_p(\mathbb{R}^d)$, $0 < p \leq \infty$, denotes the collection of complex-valued functions (equivalence classes) with finite (quasi-)norm

$$\left\| f \mid L_p\left(\mathbb{R}^d\right) \right\| = \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{1/p}, \tag{2.1}$$

with the usual modification if $p = \infty$. The Hilbert space $L_2(\mathbb{R}^d)$ plays a separate role, see for instance, Section 3. Having a sequence of complex-valued functions $\{f_k\}_{k \in I}$ on $\mathbb{R}^d$, where I is a countable index set, we put

$$\begin{aligned} \left\| f_k \mid \ell_q\left(L_p\left(\mathbb{R}^d\right)\right) \right\| &= \left(\sum_{k \in I} \left\| f_k \mid L_p\left(\mathbb{R}^d\right) \right\|^q \right)^{1/q}, \\ \left\| f_k \mid L_p\left(\ell_q, \mathbb{R}^d\right) \right\| &= \left\| \left(\sum_{k \in I} |f_k(x)|^q \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|, \end{aligned} \tag{2.2}$$

where we modify appropriately in the case $q = \infty$.

2.2. Maximal Functions

For a locally integrable function f, we denote by $Mf(x)$ the Hardy-Littlewood maximal function defined by

$$(Mf)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)| dy, \qquad x \in \mathbb{R}^d, \tag{2.3}$$

where the supremum is taken over all cubes centered at x with sides parallel to the coordinate axes. The following theorem is due to Fefferman and Stein [19].

Theorem 2.1. *For $1 < p < \infty$ and $1 < q \le \infty$, there exists a constant $c > 0$, such that*

$$\left\| Mf_k \mid L_p\left(\ell_q, \mathbb{R}^d\right) \right\| \le c \| f_k \mid L_p(\ell_q) \| \tag{2.4}$$

holds for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

Let us recall the classical Peetre maximal operator, introduced in [20]. Given a sequence of function $\{\Psi_k\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ and a positive number $a > 0$, we define the system of maximal functions

$$(\Psi_k^* f)_a(x) = \sup_{y\in\mathbb{R}^d} \frac{|(\Psi_k * f)(x+y)|}{(1 + 2^k |y|)^a}, \qquad x \in \mathbb{R}^d, \ k \in \mathbb{Z}. \tag{2.5}$$

Since $(\Psi_k * f)(y)$ makes sense pointwise (see the following paragraph), everything is well-defined. However, the value "∞" is also possible for $(\Psi_k^* f)_a(x)$. This was the reason for the problematic arguments in [4] mentioned in the introduction. We will often use dilates $\Psi_k(x) = 2^{kd}\Psi(2^k x)$ of a fixed function $\Psi \in \mathcal{S}(\mathbb{R}^d)$, where $\Psi_0(x)$ might be given by a separate function. Also continuous dilates are needed. Let the operator $\mathfrak{D}_t^{L_p}$, $t > 0$, generate the p-normalized dilates of a function Ψ given by $\mathfrak{D}_t^{L_p}\Psi := t^{-d/p}\Psi(t^{-1}\cdot)$. If $p = 1$, we omit the super index and use additionally $\Psi_t := \mathfrak{D}_t\Psi := \mathfrak{D}_t^{L_1}\Psi$. We define $(\Psi_t^* f)_a(x)$ by

$$(\Psi_t^* f)_a(x) = \sup_{y\in\mathbb{R}^d} \frac{|(\Psi_t * f)(x+y)|}{(1 + |y|/t)^a}, \qquad x \in \mathbb{R}^d, \ t > 0. \tag{2.6}$$

We will refer to this construction later on. It turned out that this maximal function construction can be used to interpret classical smoothness spaces as coorbits of certain function spaces on the group.

2.3. Tempered Distributions and Fourier Transform

As usual, $\mathcal{S}(\mathbb{R}^d)$ is used for the locally convex space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$, where the topology is generated by the family of seminorms

$$\|\varphi\|_{k,\ell} = \sup_{x\in\mathbb{R}^d, |\overline{\alpha}|_1\le\ell} \left|D^{\overline{\alpha}}\varphi(x)\right|(1+|x|)^k, \qquad \varphi\in\mathcal{S}\left(\mathbb{R}^d\right),\ k,\ell\in\mathbb{N}_0. \tag{2.7}$$

The space $\mathcal{S}'(\mathbb{R}^d)$ is called the set of all tempered distributions on $\mathbb{R}^d$ and defined as the topological dual of $\mathcal{S}(\mathbb{R}^d)$. Indeed, a linear mapping $f : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ if and only if there exist numbers $k,\ell\in\mathbb{N}_0$ and a constant $c = c_f$ such that

$$|f(\varphi)| \le c_f \sup_{x\in\mathbb{R}^d, |\overline{\alpha}|_1\le\ell} \left|D^{\overline{\alpha}}\varphi(x)\right|(1+|x|)^k \tag{2.8}$$

for all $\varphi\in\mathcal{S}(\mathbb{R}^d)$. $\mathcal{S}'(\mathbb{R}^d)$ is equipped with the weak*-topology.

The convolution $\varphi * \psi$ of two integrable functions φ,ψ is defined via the integral

$$(\varphi*\psi)(x) = \int_{\mathbb{R}^d}\varphi(x-y)\psi(y)dy. \tag{2.9}$$

If $\varphi,\psi \in \mathcal{S}(\mathbb{R}^d)$, then (2.9) still belongs to $\mathcal{S}(\mathbb{R}^d)$. The convolution can be generalized to $\mathcal{S}(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d)$ via $(\varphi * f)(x) = f(\varphi(x-\cdot))$, makes sense pointwise, and is a C^∞-function in $\mathbb{R}^d$ of at most polynomial growth.

As usual, the Fourier transform defined on both $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ is given by $(\mathcal{F}f)(\varphi) := f(\mathcal{F}\varphi)$, where $f\in\mathcal{S}'(\mathbb{R}^d)$, $\varphi\in\mathcal{S}(\mathbb{R}^d)$, and

$$\mathcal{F}\varphi(\xi) := (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}\varphi(x)dx. \tag{2.10}$$

The mapping $\mathcal{F}$ is a bijection (in both cases) and its inverse is given by $\mathcal{F}^{-1}\varphi = \mathcal{F}\varphi(-\cdot)$.

In order to deal with homogeneous spaces, we need to define the subset $\mathcal{S}_0(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$. Following [1, Chapter 5], we put

$$\mathcal{S}_0\left(\mathbb{R}^d\right) = \left\{\varphi\in\mathcal{S}\left(\mathbb{R}^d\right) : D^{\overline{\alpha}}(\mathcal{F}\varphi)(0) = 0 \text{ for every multi-index } \overline{\alpha}\in\mathbb{N}_0^d\right\}. \tag{2.11}$$

The set $\mathcal{S}_0'(\mathbb{R}^d)$ denotes the topological dual of $\mathcal{S}_0(\mathbb{R}^d)$. If $f \in \mathcal{S}'(\mathbb{R}^d)$, the restriction of f to $\mathcal{S}_0(\mathbb{R}^d)$ clearly belongs to $\mathcal{S}_0'(\mathbb{R}^d)$. Furthermore, if $P(x)$ is an arbitrary polynomial in $\mathbb{R}^d$, we have $(f + P(\cdot))(\varphi) = f(\varphi)$ for every $\varphi\in\mathcal{S}_0(\mathbb{R}^d)$. Conversely, if $f \in \mathcal{S}_0'(\mathbb{R}^d)$, then f can be extended from $\mathcal{S}_0(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$, that is, to an element of $\mathcal{S}'(\mathbb{R}^d)$. However, this fact is not trivial and makes use of the Hahn-Banach theorem in locally convex topological vector spaces. We may identify $\mathcal{S}_0'(\mathbb{R}^d)$ with the factor space $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}(\mathbb{R}^d)$, since two different extensions differ by a polynomial.

2.4. Besov-Lizorkin-Triebel Spaces

Let us first introduce the concept of a dyadic decomposition of unity, see also [1, Section 2.3.1].

Definition 2.2. (a) Let $\Phi(\mathbb{R}^d)$ be the collection of all systems $\{\varphi_j(x)\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^d)$ with the following properties:

(i) $\varphi_j(x) = \varphi(2^{-j}x),\ j \in \mathbb{N}$,

(ii) $\operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^d : |x| \leq 2\},\ \operatorname{supp} \varphi \subset \{x \in \mathbb{R}^d : 1/2 \leq |x| \leq 2\}$,

(iii) $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for every $x \in \mathbb{R}^d$.

(b) Moreover, the system $\dot{\Phi}(\mathbb{R}^d)$ denotes the collection of all systems $\{\varphi_j(x)\}_{j\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^d)$ with the following properties:

(i) $\varphi_j(x) = \varphi(2^{-j}x),\ j \in \mathbb{Z}$,

(ii) $\operatorname{supp} \varphi = \{x \in \mathbb{R}^d : 1/2 \leq |x| \leq 2\}$,

(iii) $\sum_{j=-\infty}^{\infty} \varphi_j = 1$ for every $x \in \mathbb{R}^d \setminus \{0\}$.

Remark 2.3. If we take $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$\varphi_0(x) = \begin{cases} 1: & |x| \leq 1 \\ 0: & |x| > 2 \end{cases} \tag{2.12}$$

and define $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$, then the system $\{\varphi_j(x)\}_{j\in\mathbb{N}_0}$ belongs to $\Phi(\mathbb{R}^d)$ and the system $\{\varphi_j(x)\}_{j\in\mathbb{Z}}$ with $\varphi_0 := \varphi$ belongs to $\dot{\Phi}(\mathbb{R}^d)$.

Now we are ready for the definition of the Besov and Lizorkin-Triebel spaces. See for instance [1, Section 2.3.1] for details and further properties.

Definition 2.4. Let $\{\varphi_j(x)\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^d)$ and $\Phi_j = \mathcal{F}^{-1}\varphi_j,\ j \in \mathbb{N}_0$. Let further $-\infty < s < \infty$ and $0 < q \leq \infty$.

(i) If $0 < p \leq \infty$, then

$$B^s_{p,q}\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^d\right) : \left\| f \mid B^s_{p,q}\left(\mathbb{R}^d\right)\right\| = \left(\sum_{j=0}^{\infty} 2^{jsq}\left\|\Phi_j * f \mid L_p\left(\mathbb{R}^d\right)\right\|^q\right)^{1/q} \right\} < \infty. \tag{2.13}$$

(ii) If $0 < p < \infty$, then

$$F^s_{p,q}\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^d\right) : \left\| f \mid F^s_{p,q}\left(\mathbb{R}^d\right) \right\| = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left| (\Phi_j * f)(x) \right|^q \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\| < \infty \right\}. \tag{2.14}$$

In case $q = \infty$, we replace the sum by a supremum in both cases.

The homogeneous counterparts are defined as follows. For details, further properties and how to deal with ocurring technicalities we refer to [1, Chapter 5].

Definition 2.5. Let $\{\varphi_j(x)\}_{j\in\mathbb{Z}} \in \dot{\Phi}(\mathbb{R}^d)$ and $\Phi_j = \mathcal{F}^{-1}\varphi_j$. Let further $-\infty < s < \infty$ and $0 < q \le \infty$.

(i) If $0 < p \le \infty$, then

$$\dot{B}^s_{p,q}\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}_0'\left(\mathbb{R}^d\right) : \left\| f \mid \dot{B}^s_{p,q}\left(\mathbb{R}^d\right) \right\| = \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \left\| \Phi_j * f \mid L_p\left(\mathbb{R}^d\right) \right\|^q \right)^{1/q} < \infty \right\}. \tag{2.15}$$

(ii) If $0 < p < \infty$, then

$$\dot{F}^s_{p,q}\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}_0'\left(\mathbb{R}^d\right) : \left\| f \mid \dot{F}^s_{p,q}\left(\mathbb{R}^d\right) \right\| = \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \left| (\Phi_j * f)(x) \right|^q \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\| < \infty \right\}. \tag{2.16}$$

In case $q = \infty$, we replace the sum by a supremum in both cases.

2.5. Inhomogeneous Spaces

Essential for the sequel are functions $\Phi_0, \Phi \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$|(\mathcal{F}\Phi_0)(x)| > 0 \quad \text{on } \{|x| < 2\varepsilon\},$$
$$|(\mathcal{F}\Phi)(x)| > 0 \quad \text{on } \left\{\frac{\varepsilon}{2} < |x| < 2\varepsilon\right\}, \tag{2.17}$$

for some $\varepsilon > 0$, and

$$D^{\overline{\alpha}}(\mathcal{F}\Phi)(0) = 0 \quad \forall |\overline{\alpha}| \le R. \tag{2.18}$$

We will call the functions Φ_0 and Φ kernels for local means. Recall that $\Phi_k = 2^{kd}\Phi(2^k\cdot)$, $k \in \mathbb{N}$, and $\Psi_t = \mathfrak{D}_t\Psi$.

Theorem 2.6. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, $a > d/\min\{p,q\}$ and $R+1 > s$. Let further $\Phi_0, \Phi \in \mathcal{S}(\mathbb{R}^d)$ be given by* (2.17) *and* (2.18). *Then the space $F^s_{p,q}(\mathbb{R}^d)$ can be characterized by*

$$F^s_{p,q}\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^d\right) : \left\| f \mid F^s_{p,q}\left(\mathbb{R}^d\right)\right\|_i < \infty \right\}, \quad i = 1,\dots,5, \tag{2.19}$$

where

$$\left\| f \mid F^s_{p,q}\right\|_1 = \left\|\Phi_0 * f \mid L_p\left(\mathbb{R}^d\right)\right\| + \left\| \left(\int_0^1 t^{-sq} |(\Phi_t * f)(x)|^q \frac{dt}{t}\right)^{1/q} \mid L_p\left(\mathbb{R}^d\right)\right\|, \tag{2.20}$$

$$\left\| f \mid F^s_{p,q}\right\|_2 = \left\|\Phi_0 * f \mid L_p\left(\mathbb{R}^d\right)\right\| + \left\| \left(\int_0^1 t^{-sq} \left[\sup_{z\in\mathbb{R}^d} \frac{|(\Phi_t * f)(x+z)|}{(1+|z|/t)^a}\right]^q \frac{dt}{t}\right)^{1/q} \mid L_p\left(\mathbb{R}^d\right)\right\|, \tag{2.21}$$

$$\left\| f \mid F^s_{p,q}\right\|_3 = \left\|\Phi_0 * f \mid L_p\left(\mathbb{R}^d\right)\right\| + \left\| \left(\int_0^1 t^{-sq} \int_{|z|<t} |(\Phi_t * f)(x+z)|^q \frac{dt}{t^{d+1}}\right)^{1/q} \mid L_p\left(\mathbb{R}^d\right)\right\|, \tag{2.22}$$

$$\left\| f \mid F^s_{p,q}\right\|_4 = \left\| \left(\sum_{k=0}^{\infty} 2^{skq} \left[\sup_{z\in\mathbb{R}^d} \frac{|(\Phi_k * f)(x+z)|}{(1+2^k|z|)^a}\right]^q \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right)\right\|, \tag{2.23}$$

$$\left\| f \mid F^s_{p,q}\right\|_5 = \left\| \left(\sum_{k=0}^{\infty} 2^{skq} |(\Phi_k * f)(x)|^q \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right)\right\|, \tag{2.24}$$

with the usual modification in case $q = \infty$. Furthermore, all quantities $\|f \mid F^s_{p,q}(\mathbb{R}^d)\|_i$, $i = 1, \ldots, 5$, are equivalent (quasi-)norms in $F^s_{p,q}(\mathbb{R}^d)$.

For the inhomogeneous Besov spaces, we have the following characterizations.

Theorem 2.7. *Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$, $a > d/p$, and $R+1 > s$. Let further $\Phi_0, \Phi \in \mathcal{S}(\mathbb{R}^d)$ be given by* (2.17) *and* (2.18). *Then the space $B^s_{p,q}(\mathbb{R}^d)$ can be characterized by*

$$B^s_{p,q}\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^d\right) : \left\| f \mid B^s_{p,q}\left(\mathbb{R}^d\right) \right\|_i < \infty \right\}, \quad i = 1, \ldots, 4, \tag{2.25}$$

where

$$\begin{aligned}
\left\| f \mid B^s_{p,q} \right\|_1 &= \left\| \Phi_0 * f \mid L_p\left(\mathbb{R}^d\right) \right\| + \left(\int_0^1 t^{-sq} \left\| (\Phi_t * f)(x) \mid L_p\left(\mathbb{R}^d\right) \right\|^q \frac{dt}{t} \right)^{1/q}, \\
\left\| f \mid B^s_{p,q} \right\|_2 &= \left\| \Phi_0 * f \mid L_p\left(\mathbb{R}^d\right) \right\| + \left(\int_0^1 t^{-sq} \left\| \sup_{z \in \mathbb{R}^d} \frac{|(\Phi_t * f)(x+z)|}{(1+|z|/t)^a} \mid L_p\left(\mathbb{R}^d\right) \right\|^q \frac{dt}{t} \right)^{1/q}, \\
\left\| f \mid B^s_{p,q} \right\|_3 &= \left(\sum_{k=0}^{\infty} 2^{skq} \left\| \sup_{z \in \mathbb{R}^d} \frac{|(\Phi_k * f)(x+z)|}{(1+2^k|z|)^a} \mid L_p\left(\mathbb{R}^d\right) \right\|^q \right)^{1/q}, \\
\left\| f \mid B^s_{p,q} \right\|_4 &= \left(\sum_{k=0}^{\infty} 2^{skq} \left\| (\Phi_k * f)(x) \mid L_p\left(\mathbb{R}^d\right) \right\|^q \right)^{1/q},
\end{aligned} \tag{2.26}$$

with the usual modification if $q = \infty$. Furthermore, all quantities $\|f | B^s_{p,q}(\mathbb{R}^d)\|_i$, $i = 1, \ldots, 4$, are equivalent quasinorms in $B^s_{p,q}(\mathbb{R}^d)$.

2.6. Homogeneous Spaces

The homogeneous spaces can be characterized in a similar way. Here we do not have a separate function Φ_0 anymore. We put $\Phi_0 = \Phi$.

Theorem 2.8. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $a > d/\min\{p, q\}$, and $R+1 > s$. Let further $\Phi \in \mathcal{S}(\mathbb{R}^d)$ be given by* (2.17) *and* (2.18). *Then the space $\dot{F}^s_{p,q}(\mathbb{R}^d)$ can be characterized by*

$$\dot{F}^s_{p,q}\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}'_0\left(\mathbb{R}^d\right) : \left\| f \mid \dot{F}^s_{p,q}\left(\mathbb{R}^d\right) \right\|_i < \infty \right\}, \quad i = 1, \ldots, 5, \tag{2.27}$$

where

$$\left\| f \mid \dot{F}^s_{p,q} \right\|_1 = \left\| \left(\int_0^\infty t^{-sq} \left| (\Phi_t * f)(x) \right|^q \frac{dt}{t} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|,$$

$$\left\| f \mid \dot{F}^s_{p,q} \right\|_2 = \left\| \left(\int_0^\infty t^{-sq} \left[\sup_{z \in \mathbb{R}^d} \frac{\left| (\Phi_t * f)(x+z) \right|}{(1+|z|/t)^a} \right]^q \frac{dt}{t} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|,$$

$$\left\| f \mid \dot{F}^s_{p,q} \right\|_3 = \left\| \left(\int_0^\infty t^{-sq} \int_{|z|<t} \left| (\Phi_t * f)(x+z) \right|^q \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|, \tag{2.28}$$

$$\left\| f \mid \dot{F}^s_{p,q} \right\|_4 = \left\| \left(\sum_{k=-\infty}^{\infty} 2^{skq} \left[\sup_{z \in \mathbb{R}^d} \frac{\left| (\Phi_k * f)(x+z) \right|}{(1+2^k|z|)^a} \right]^q \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|,$$

$$\left\| f \mid \dot{F}^s_{p,q} \right\|_5 = \left\| \left(\sum_{k=-\infty}^{\infty} 2^{skq} \left| (\Phi_k * f)(x) \right|^q \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|$$

with the usual modification if $q = \infty$. *Furthermore, all quantities* $\|f|\dot{F}^s_{p,q}(\mathbb{R}^d)\|_i$, $i = 1, \ldots, 5$, *are equivalent quasinorms in* $\dot{F}^s_{p,q}(\mathbb{R}^d)$.

For thing characterizations.

Theorem 2.9. *Let* $s \in \mathbb{R}$, $0 < p, q \le \infty$, $a > d/p$, *and* $R + 1 > s$. *Let further* $\Phi \in \mathcal{S}(\mathbb{R}^d)$ *be given by* (2.17) *and* (2.18). *Then the space* $\dot{B}^s_{p,q}(\mathbb{R}^d)$ *can be characterized by*

$$\dot{B}^s_{p,q}\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{S}'_0\left(\mathbb{R}^d\right) : \left\| f \mid \dot{B}^s_{p,q}\left(\mathbb{R}^d\right) \right\|_i < \infty \right\}, \quad i = 1, \ldots, 4, \tag{2.29}$$

where

$$\left\| f \mid \dot{B}^s_{p,q} \right\|_1 = \left(\int_0^\infty t^{-sq} \left\| (\Phi_t * f)(x) \mid L_p\left(\mathbb{R}^d\right) \right\|^q \frac{dt}{t} \right)^{1/q},$$

$$\left\| f \mid \dot{B}^s_{p,q} \right\|_2 = \left(\int_0^\infty t^{-sq} \left\| \sup_{z \in \mathbb{R}^d} \frac{\left| (\Phi_t * f)(x+z) \right|}{(1+|z|/t)^a} \mid L_p\left(\mathbb{R}^d\right) \right\|^q \frac{dt}{t} \right)^{1/q},$$

$$\left\| f \mid \dot{B}^s_{p,q} \right\|_3 = \left(\sum_{k=-\infty}^{\infty} 2^{skq} \left\| \sup_{z \in \mathbb{R}^d} \frac{\left| (\Phi_k * f)(x+z) \right|}{(1+2^k|z|)^a} \mid L_p\left(\mathbb{R}^d\right) \right\|^q \right)^{1/q}, \tag{2.30}$$

$$\left\| f \mid \dot{B}^s_{p,q} \right\|_4 = \left(\sum_{k=-\infty}^{\infty} 2^{skq} \left\| (\Phi_k * f)(x) \mid L_p\left(\mathbb{R}^d\right) \right\|^q \right)^{1/q},$$

with the usual modification if $q = \infty$. *Furthermore, all quantities* $\|f \mid \dot{B}^s_{p,q}(\mathbb{R}^d)\|_i$, $i = 1, \ldots, 4$, *are equivalent quasinorms in* $\dot{B}^s_{p,q}(\mathbb{R}^d)$.

Remark 2.10. Observe, that the (quasi-)norms $\|\cdot \mid \dot{F}^s_{p,q}(\mathbb{R}^d)\|_3$ and $\|\cdot \mid F^s_{p,q}(\mathbb{R}^d)\|_3$ are characterizations via Lusin functions, see [3, Section 2.4.5] and [1, Section 2.12.1] and the references given there. We will return to it later when defining tent spaces, see Definition 4.1 and (4.3).

2.7. Particular Kernels

For more details concerning particular choices for the kernels Φ_0 and Φ, we refer mainly to Triebel [3, Section 3.3].

The most prominent nontrivial examples (besides the one given in Remark 2.3) of functions Φ_0 and Φ satisfying (2.17) and (2.18) are the classical local means. The name comes from the compact support of Φ_0, Φ, which is admitted in the following statement.

Corollary 2.11. *Let p, q, s as in Theorem 2.6. Let further $k_0, k^0 \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\mathcal{F}k_0(0),\ \mathcal{F}k^0(0) \neq 0 \tag{2.31}$$

and define

$$\Phi_0 = k_0, \qquad \Phi = \Delta^N k^0, \tag{2.32}$$

where $N \in \mathbb{N}$ such that $2N > s$. Then (2.20), (2.21), (2.22), (2.23), *and* (2.24) *characterize $F^s_{p,q}(\mathbb{R}^d)$.*

Corollary 2.12. *Let p, q, s as in Theorem 2.6. Let further $\Phi_0 \in \mathcal{S}(\mathbb{R}^d)$ be a radial function such that $\mathcal{F}\Phi_0$ is non-increasing and atisfying*

$$\mathcal{F}\Phi_0(0) \neq 0, \qquad D^{\overline{\alpha}}\mathcal{F}\Phi_0(0) = 0 \tag{2.33}$$

for $1 \leq |\overline{\alpha}|_1 \leq R$, where $R + 1 > s$. Define

$$\Phi(x) = \Phi_0(x) - \frac{1}{2^d}\Phi_0\left(\frac{x}{2}\right). \tag{2.34}$$

Then (2.23), *and* (2.24) *characterize $F^s_{p,q}(\mathbb{R}^d)$.*

2.8. Proofs

We give the proof for Theorem 2.6 in full detail. The proof of Theorem 2.8 is more or less the same, even a bit simpler. We refer to the next paragraph for the necessary modifications. The proofs in the Besov scale are analogous, so we omit them completely. The strategy is a modification of Rychkov [4], where he proved the discrete case, that is, that (2.23) and (2.24) characterize $F^s_{p,q}(\mathbb{R}^d)$. However, Hansen [21, Remark 3.2.4] recently observed that the arguments used for proving (23) and (23′) in [4] are somehow problematic. The finiteness of the Peetre maximal function is assumed. But this is not true in general under the stated assumptions. Consider for instance in dimension $d = 1$ the functions

$$\Psi_0(t) = \Psi_1(t) = e^{-t^2} \tag{2.35}$$

and, if $a > 0$ is given, the tempered distribution $f(t) = |t|^n$ with $a < n \in \mathbb{N}$. Then $(\Psi_k^* f)_a(x)$ is infinite in every point $x \in \mathbb{R}$. The mentioned incorrect argument was inherited to some subsequent papers dealing with similar topics, for instance [22–24]. Anyhow, the stated results hold true. An alternative strategy, in order to avoid the problematic Lemma 3 in [4], is given in Rychkov [7] as well as [8]. A variant of this method, which is originally due to Strömberg/Torchinsky [9, Chapter V], is also used in our proof below.

We start with a convolution-type inequality which will be often needed below. The following lemma is essentially [4, Lemma 2].

Lemma 2.13. *Let $0 < p, q \le \infty$ and, $\delta > 0$. Let $\{g_k\}_{k\in\mathbb{N}_0}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^d$ and put*

$$G_\ell(x) = \sum_{k\in\mathbb{Z}} 2^{-|k-\ell|\delta} g_k(x), \qquad x \in \mathbb{R}^d,\ \ell \in \mathbb{Z}. \tag{2.36}$$

Then there is some constant $C = C(p, q, \delta)$, such that

$$\begin{gathered} \left\| \{G_\ell\}_\ell \mid \ell_q\left(L_p\left(\mathbb{R}^d\right)\right) \right\| \le C \left\| \{g_k\}_k \mid \ell_q\left(L_p\left(\mathbb{R}^d\right)\right) \right\|, \\ \left\| \{G_\ell\}_\ell \mid L_p\left(\ell_q, \mathbb{R}^d\right) \right\| \le C \left\| \{g_k\}_k \mid L_p\left(\ell_q, \mathbb{R}^d\right) \right\| \end{gathered} \tag{2.37}$$

hold true.

Proof of Theorem 2.6. The strategy of the proof is as follows. First, we prove the equivalence of the “continuous” characterizations (2.20) and (2.21). The next step is to build the bridge between the “continuous” (2.21) and the “discrete” characterization (2.23) and to change from the system (Φ_0, Φ) to a system (Ψ_0, Ψ). The equivalence of (2.23) and (2.24) goes parallel to (2.20) and (2.21). This was the original proof by Rychkov in [4]. So, up to this point, we have that (2.20), (2.21), (2.23), and (2.24) generate the same space for every chosen functions (Φ_0, Φ) satisfying (2.17) and (2.18), namely, $F^s_{p,q}(\mathbb{R}^d)$. Indeed, Definition 2.4 can be seen as a special case of (2.24).

Step 1. We are going to prove the following inequalities:

$$\left\| f \mid F^s_{p,q} \right\|_2 \lesssim \left\| f \mid F^s_{p,q} \right\|_1 \lesssim \left\| f \mid F^s_{p,q} \right\|_2 \tag{2.38}$$

for every $f \in \mathcal{S}'(\mathbb{R}^d)$.

Substep 1.1. Put $\varphi_0 = \mathcal{F}\Phi_0$ and $\varphi_\ell = (\mathcal{F}\Phi)(2^{-\ell}\cdot)$ if $\ell \geq 1$. Because of (2.17), it is possible to find functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^d)$ with supp $\psi_0 \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2\varepsilon\}$, supp $\psi \subset \{\xi \in \mathbb{R}^d : \varepsilon/2 \leq |\xi| \leq 2\varepsilon\}$ and $\psi_\ell(x) = \psi(2^{-\ell}x)$ such that

$$\sum_{\ell\in\mathbb{N}_0} \varphi_\ell(\xi) \cdot \psi_\ell(\xi) = 1. \tag{2.39}$$

We need a bit more. Fix a $1 \leq t \leq 2$. Clearly, we have also

$$\sum_{\ell\in\mathbb{N}_0} \varphi_\ell(t\xi) \cdot \psi_\ell(t\xi) = 1 \tag{2.40}$$

for all $\xi \in \mathbb{R}^d$. With $\Psi_0 = \mathcal{F}^{-1}\psi_0$ and $\Psi = \mathcal{F}^{-1}\psi$, we obtain then

$$g = \sum_{m\in\mathbb{N}_0} (\Psi_m)_t * (\Phi_m)_t * g. \tag{2.41}$$

The 2^ℓ dilation gives then

$$g_\ell = \sum_{m\in\mathbb{N}_0} (\Psi_m)_{t2^{-\ell}} * (\Phi_m)_{t2^{-\ell}} * g_\ell \tag{2.42}$$

for every $g \in \mathcal{S}'(\mathbb{R}^d)$, where $g_\ell(\eta) = g(\eta_{-\ell})$, $\eta \in \mathcal{S}(\mathbb{R}^d)$. Obviously, we can rewrite (2.42) to obtain

$$g = \sum_{m\in\mathbb{N}_0} (\Psi_m)_{t2^{-\ell}} * (\Phi_m)_{t2^{-\ell}} * g \tag{2.43}$$

for all $g \in \mathcal{S}'(\mathbb{R}^d)$. Let us now choose $g = (\Phi_\ell)_t * f$. This gives the final version of the convolution identity

$$(\Phi_\ell)_t * f = \sum_{m\in\mathbb{N}_0} (\Phi_\ell)_t * (\Psi_m)_{t2^{-\ell}} * (\Phi_m)_{t2^{-\ell}} * f. \tag{2.44}$$

For $m, \ell \in \mathbb{N}_0$ we define

$$\Lambda_{m,\ell}(x) = \begin{cases} 2^{\ell d}\Phi_0(2^\ell x): & m = 0, \\ \Phi_\ell(x): & m > 0, \end{cases} \quad x \in \mathbb{R}^d. \tag{2.45}$$

Clearly, we have

$$(\Phi_\ell)_t * (\Phi_m)_{t2^{-\ell}} = (\Lambda_{m,\ell})_t * (\Phi_{m+\ell})_t. \tag{2.46}$$

Plugging this into (2.44), we end up with the pointwise representation

$$\begin{aligned}((\Phi_\ell)_t * f)(y) &= \sum_{m\in\mathbb{N}_0} ((\Psi_m)_{2^{-\ell}t} * (\Lambda_{m,\ell})_t * (\Phi_{m+\ell})_t * f)(y) \\ &= \sum_{m\in\mathbb{N}_0} [(\Psi_m)_{2^{-\ell}t} * (\Lambda_{m,\ell})_t] * ((\Phi_{m+\ell})_t * f)(y) \\ &= \sum_{m\in\mathbb{N}_0} \int_{\mathbb{R}^d} [(\Psi_m)_{2^{-\ell}t} * (\Lambda_{m,\ell})_t](y-z) \cdot ((\Phi_{m+\ell})_t * f)(z)dz\end{aligned} \tag{2.47}$$

for all $y \in \mathbb{R}^d$.

Substep 1.2. Let us prove the following important inequality first. For every $r > 0$ and every $N \in \mathbb{N}_0$, we have

$$\left|((\Phi_\ell)_t * f)(x)\right|^r \leq c \sum_{k\in\mathbb{N}_0} 2^{-kNr} 2^{(k+\ell)d} \int_{\mathbb{R}^d} \frac{\left|((\Phi_{k+\ell})_t * f)(y)\right|^r}{\left(1+2^\ell|x-y|\right)^{Nr}} dy, \tag{2.48}$$

where c is independent of $f \in \mathcal{S}'(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, and $\ell \in \mathbb{N}_0$.

The representation (2.47) will be the starting point to prove (2.48). Namely, we have for $y \in \mathbb{R}^d$

$$\begin{aligned}\left|((\Phi_\ell)_t * f)(y)\right| &\leq \sum_{m\in\mathbb{N}_0} \int_{\mathbb{R}^d} \left|((\Psi_m)_{t2^{-\ell}} * (\Lambda_{m,\ell})_t)(y-z)\right| \cdot \left|((\Phi_{m+\ell})_t * f)(z)\right| dz \\ &\leq \sum_{m\in\mathbb{N}_0} S_{m,\ell,t} \int_{\mathbb{R}^d} \frac{\left|((\Phi_{m+\ell})_t * f)(z)\right|}{\left(1+2^\ell|y-z|\right)^N} dz,\end{aligned} \tag{2.49}$$

where

$$S_{m,\ell,t} = \sup_{x\in\mathbb{R}^d} \left|[(\Psi_m)_{2^{-\ell}t} * (\Lambda_{m,\ell})_t](x)\right| \cdot \left(1+2^\ell|x|\right)^N. \tag{2.50}$$

Elementary properties of the convolution yield (compare with (2.84))

$$\begin{aligned}S_{m,\ell,t} &= \frac{2^{\ell d}}{t^d} \sup_{x\in\mathbb{R}^d} \left|(\Psi_m * (\Lambda_{m,\ell})_{2^\ell})\left(\frac{x2^\ell}{t}\right)\right| \cdot \left(1+2^\ell|x|\right)^N \\ &= \frac{2^{\ell d}}{t^d} \sup_{x\in\mathbb{R}^d} \left|(\varphi_m * \eta_{m,\ell})(x)\right| \cdot (1+|tx|)^N,\end{aligned} \tag{2.51}$$

where

$$\eta_{m,\ell}(x) = \begin{cases} \Phi(x): & \ell \geq 0,\ m > 0, \\ \Phi_0(x): & \text{otherwise.}\end{cases} \tag{2.52}$$

Lemma A.3 yields

$$S_{m,\ell,t} \le c_N 2^{\ell d} 2^{-mN}, \tag{2.53}$$

which we put into (2.49) to obtain

$$\left|\left((\Phi_\ell)_t * f\right)(y)\right| \le C_N \sum_{m\in\mathbb{N}_0} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}\left|\left((\Phi_{m+\ell})_t * f\right)(z)\right|}{\left(1+2^\ell|y-z|\right)^N} dz. \tag{2.54}$$

We prefer the strategy used by Rychkov in [7, Theorem 3.2] and [8, Lemma 2.9], which is a variant of the Strömberg/Torchinsky technique introduced in [9, Chapter V].

Let us continue by replacing ℓ by $k+\ell$ in (2.54) and multiply on both sides with 2^{-kN}. Then we can estimate

$$\begin{aligned}
2^{-kN}\left|\left((\Phi_{k+\ell})_t * f\right)(y)\right| &\le C_N \sum_{m\in\mathbb{N}_0} 2^{-kN} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+k+\ell)d}\left|\left((\Phi_{m+k+\ell})_t * f\right)(z)\right|}{\left(1+2^{k+\ell}|y-z|\right)^N} dz \\
&\le C_N \sum_{m\in\mathbb{N}_0} 2^{-(m+k)N} \int_{\mathbb{R}^d} \frac{2^{(m+k+\ell)d}\left|\left((\Phi_{m+k+\ell})_t * f\right)(z)\right|}{\left(1+2^{\ell}|y-z|\right)^N} dz \\
&= C_N \sum_{m\in k+\mathbb{N}_0} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}\left|\left((\Phi_{m+\ell})_t * f\right)(z)\right|}{\left(1+2^{\ell}|y-z|\right)^N} dz,
\end{aligned} \tag{2.55}$$

$$\le C_N \sum_{m\in\mathbb{N}_0} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}\left|\left((\Phi_{m+\ell})_t * f\right)(z)\right|}{\left(1+2^{\ell}|y-z|\right)^N} dz. \tag{2.56}$$

Next, we apply the elementary inequalities

$$\begin{aligned}
&\left(1+2^\ell|y-z|\right)\cdot\left(1+2^\ell|x-y|\right) \ge \left(1+2^\ell|x-z|\right), \\
&\left|\left((\Phi_{m+\ell})_t * f\right)(z)\right| \le \left|\left((\Phi_{m+\ell})_t * f\right)(z)\right|^r \left(1+2^\ell|x-z|\right)^{N(1-r)} \\
&\qquad \times \sup_{y\in\mathbb{R}^d} \frac{\left|\left((\Phi_{m+\ell})_t * f\right)(y)\right|^{1-r}}{\left(1+2^\ell|x-y|\right)^{N(1-r)}},
\end{aligned} \tag{2.57}$$

where $0 < r \le 1$. We define the maximal function

$$M_{\ell,N}(x,t) = \sup_{k\in\mathbb{N}_0} \sup_{y\in\mathbb{R}^d} 2^{-kN} \frac{\left|\left((\Phi_{k+\ell})_t * f\right)(y)\right|}{\left(1+2^\ell|x-y|\right)^N}, \quad x \in \mathbb{R}^d, \tag{2.58}$$

and estimate

$$M_{\ell,N}(x,t) \le C_N \sum_{m\in\mathbb{N}_0} 2^{-mN} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}\left|\left((\Phi_{m+\ell})_t * f\right)(z)\right|}{\left(1+2^{\ell}|x-z|\right)^N} dz \tag{2.59}$$

$$\begin{aligned} &\le C_N \sum_{m\in\mathbb{N}_0} 2^{-mNr} \left(2^{-mN} \sup_{y\in\mathbb{R}^d} \frac{\left|\left((\Phi_{m+\ell})_t * f\right)(y)\right|}{\left(1+2^{\ell}\left|x-y\right|\right)^N} \right)^{1-r} \\ &\quad \times \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}\left|\left((\Phi_{m+\ell})_t * f\right)(z)\right|^r}{\left(1+2^{\ell}|x-z|\right)^{Nr}} dz. \end{aligned} \tag{2.60}$$

Observe that we can estimate the term $(\cdots)^{1-r}$ in the right-hand side of (2.60) by $M_{\ell,N}(x,t)^{1-r}$. Hence, if $M_{\ell,N}(x,t) < \infty$ we obtain from (2.60)

$$M_{\ell,N}(x,t)^r \le C_N \sum_{m\in\mathbb{N}_0} 2^{-mNr} \int_{\mathbb{R}^d} \frac{2^{(m+\ell)d}\left|\left((\Phi_{m+\ell})_t * f\right)(z)\right|^r}{\left(1+2^{\ell}|x-z|\right)^{Nr}} dz, \tag{2.61}$$

where C_N is independent of x, f, ℓ, and $t \in [1,2]$. We claim that there exists $N^f \in \mathbb{N}_0$ such that $M_{\ell,N}(x,t) < \infty$ for all $N \ge N^f$. Indeed, we use that $f \in \mathcal{S}'(\mathbb{R}^d)$, that is, there is an $M \in \mathbb{N}_0$ and $c_f > 0$ such that

$$\left|\left((\Phi_{k+\ell})_t * f\right)(y)\right| \le c_f \sup_{|\bar{\alpha}|_1\le M} \sup_{z\in\mathbb{R}^d} \left| D^{\bar{\alpha}} \Phi_{k+\ell}(z) \right| \cdot \left(1+\left|y-z\right|\right)^M. \tag{2.62}$$

Assuming $N > M$, we estimate as follows:

$$\begin{aligned} \left|\left((\Phi_{\ell})_t * f\right)(x)\right| &\le M_{\ell,N}(x,t) \\ &\le c \sup_{k\in\mathbb{N}_0} \sup_{y\in\mathbb{R}^d} 2^{-kN} \frac{\left|\left((\Phi_{k+\ell})_t * f\right)(y)\right|}{\left(1+\left|x-y\right|^2\right)^{N/2}} \\ &\le c \sup_{k\in\mathbb{N}_0} \sup_{y\in\mathbb{R}^d} 2^{-kN} 2^{(k+\ell)(M+d)} \sup_{z\in\mathbb{R}^d} \sup_{|\bar{\alpha}|_1\le M} \frac{\left|D^{\bar{\alpha}}\gamma_{k+\ell}(z)\right| \cdot \left(1+\left|y-z\right|\right)^M}{\left(1+\left|x-y\right|\right)^N} \\ &\le c\, 2^{\ell(M+d)} \sup_{k\in\mathbb{N}_0} \sup_{z\in\mathbb{R}^d} \sup_{|\bar{\alpha}|_1\le M} \left|D^{\bar{\alpha}}\gamma_{k+\ell}(z)\right| (1+|x-z|)^N, \end{aligned} \tag{2.63}$$

where we again used the inequality (compare with (2.57))

$$1+\left|y-z\right| \le \left(1+\left|x-y\right|\right)(1+|x-z|) \tag{2.64}$$

and have set

$$\gamma_\ell(t) = \begin{cases} \Phi_0(t) : & \ell = 0, \\ \Phi(t) : & \ell > 0. \end{cases} \tag{2.65}$$

Hence $\gamma_{k+\ell}$ gives us only two different functions from $\mathcal{S}(\mathbb{R}^d)$. This implies the boundedness of $M_{\ell,N}(x,t)$ for $x \in \mathbb{R}^d$ if $N > M = N^f$. Therefore, (2.61) together with (2.63) yield (2.48) with $c = C_N$, independent of x, f, and ℓ, for all $N \geq N^f$. But this is not yet what we want. Observe that the right-hand side of (2.48) decreases as N increases. Therefore, we have (2.48) *for all* $N \in \mathbb{N}_0$ but with $c = c(f) = C_{N^f}$ *depending on* f. This is still not yet what we want. Now we argue as follows: starting with (2.48) where $c = c(f)$ and $N \in \mathbb{N}_0$ arbitrary, we apply the same arguments as used from (2.55) to (2.56), switch to the maximal function (2.58) with the help of (2.57), and finish with (2.61) instead of (2.59) but with a constant that depends on f. But this does not matter now. It is Important, that a finite right-hand side of (2.61) (which is the same as rhs(2.48)) implies $M_{\ell,N}(x,t) < \infty$.

We assume rhs (2.48) $< \infty$. Otherwise, there is nothing to prove in (2.48). Returning to (2.60) and having in mind that now $M_{\ell,N}(x,t) < \infty$, we end up with (2.61) for all N and C_N independent of f. Finally, from (2.61) we obtain (2.48) and are done in case $0 < r \leq 1$.

Of course, (2.48) also holds true for $r > 1$ with a much simpler proof. In that case, we use (2.54) with $N+1$ instead of N and apply Hölder's inequality with respect to $1/r + 1/r' = 1$ first for integrals and then for sums.

Substep 1.3. The inequality (2.48) implies immediately a stronger version of itself. Using (2.57) again, we obtain for $a \leq N$ and $\ell \in \mathbb{N}$

$$\left(\Phi^*_{2^{-\ell}t} f\right)_a(x)^r \leq c \sum_{k\in\mathbb{N}_0} 2^{-kNr} 2^{(k+\ell)d} \int_{\mathbb{R}^d} \frac{\left|\left((\Phi_{k+\ell})_t * f\right)(y)\right|^r}{\left(1 + 2^\ell |x-y|\right)^{ar}} dy. \tag{2.66}$$

We proved that the inequality (2.66) holds for all $t \in [1,2]$ where $c > 0$ is independent of t. If we choose $r < \min\{p,q\}$, we can apply the norm

$$\left(\int_1^2 |\cdot|^{q/r} \frac{dt}{t}\right)^{r/q}, \tag{2.67}$$

on both sides and use Minkowski's inequality for integrals, which yields

$$\left(\int_1^2 \left|\left(\Phi^*_{2^{-\ell}t} f\right)_a(x)\right|^q \frac{dt}{t}\right)^{r/q} \leq c \sum_{k\in\mathbb{N}_0} 2^{-kNr} 2^{(k+\ell)d} \int_{\mathbb{R}^d} \frac{\left(\int_1^2 \left|\left((\Phi_{k+\ell})_t * f\right)(y)\right|^q \,(dt/t)\right)^{r/q}}{\left(1 + 2^\ell |x-y|\right)^{ar}} dy. \tag{2.68}$$

If $ar > d$, then we have

$$g_\ell(y) = \frac{2^{d\ell}}{(1+2^\ell|y|)^{ar}} \in L_1\left(\mathbb{R}^d\right), \tag{2.69}$$

and we observe

$$\begin{aligned}
&\left(\int_1^2 \left|2^{\ell s}(\Phi^*_{2^{-\ell}t}f)_a(x)\right|^q \frac{dt}{t}\right)^{r/q} \\
&\quad\le c\sum_{k\in\mathbb{N}_0} 2^{-kNr}2^{kd}2^{\ell sr}\left[g_\ell * \left(\int_1^2 |2^{\ell s}((\Phi_{k+\ell})_t * f)(\cdot)|^q \frac{dt}{t}\right)^{r/q}\right](x).
\end{aligned} \tag{2.70}$$

Now we use the majorant property of the Hardy-Littlewood maximal operator (see Section 2.2 and [25, Chapter 2]) and continue estimating

$$\left(\int_1^2 \left|2^{\ell s}(\Phi^*_{2^{-\ell}t}f)_a(x)\right|^q \frac{dt}{t}\right)^{r/q} \le c\sum_{k\in\mathbb{N}_0} 2^{\ell rs}2^{k(-Nr+d)}M\left[\left(\int_1^2 |((\Phi_{k+\ell})_t * f)(\cdot)|^q \frac{dt}{t}\right)^{r/q}\right](x). \tag{2.71}$$

An index shift on the right-hand side gives

$$\begin{aligned}
&\left(\int_1^2 \left|2^{\ell s}(\Phi^*_{2^{-\ell}t}f)_a(x)\right|^q \frac{dt}{t}\right)^{r/q} \\
&\quad= c\sum_{k\in\ell+\mathbb{N}_0} 2^{\ell rs}2^{(k-\ell)(-Nr+d)}M\left[\left(\int_1^2 |((\Phi_k)_t * f)(\cdot)|^q \frac{dt}{t}\right)^{r/q}\right](x) \\
&\quad= c\sum_{k\in\ell+\mathbb{N}_0} 2^{(\ell-k)(Nr-d+rs)}2^{krs}M\left[\left(\int_1^2 |((\Phi_k)_t * f)(\cdot)|^q \frac{dt}{t}\right)^{r/q}\right](x).
\end{aligned} \tag{2.72}$$

Choose now $1/a < r < \min\{p,q\}$, $N > \max\{0,-s\} + a$ and put

$$\delta = N + s - \frac{d}{r} > 0. \tag{2.73}$$

We obtain for $\ell \in \mathbb{N}$

$$\left(\int_1^2 \left|2^{\ell s}(\Phi^*_{2^{-\ell}t}f)_a(x)\right|^q \frac{dt}{t}\right)^{r/q} \le c\sum_{k\in\mathbb{N}} 2^{-\delta r|\ell-k|}2^{krs}M\left[\left(\int_1^2 |((\Phi_k)_t * f)(\cdot)|^q \frac{dt}{t}\right)^{r/q}\right](x). \tag{2.74}$$

Now we apply Lemma 2.13 in $L_{p/r}(\ell_{q/r},\mathbb{R}^d)$, which yields

$$\left\| \left(\int_1^2 \left| 2^{\ell s} (\Phi^*_{2^{-\ell}t} f)_a(x) \right|^q \frac{dt}{t} \right)^{r/q} \mid L_{p/r}(\ell_{q/r}) \right\| \\ \leq c \left\| M \left[\left(\int_1^2 \left| 2^{ks} ((\Phi_k)_t * f)(\cdot) \right|^q \frac{dt}{t} \right)^{r/q} \right] \mid L_{p/r}(\ell_{q/r}) \right\|. \tag{2.75}$$

The Fefferman-Stein inequality (see Section 2.2 /Theorem 2.1, having in mind that $p/r, q/r > 1$) gives

$$\begin{aligned} &\left\| \left(\int_1^2 \left| 2^{\ell s} (\Phi^*_{2^{-\ell}t} f)_a(x) \right|^q \frac{dt}{t} \right)^{1/q} \mid L_p(\ell_q) \right\|^r \\ &= \left\| M \left[\left(\int_1^2 \left| 2^{\ell s} (\Phi^*_{2^{-\ell}t} f)_a(x) \right|^q \frac{dt}{t} \right)^{r/q} \right] \mid L_{p/r}(\ell_{q/r}) \right\| \\ &\lesssim \left\| \left(\int_1^2 |2^{ks} ((\Phi_k)_t * f)(\cdot)|^q \frac{dt}{t} \right)^{r/q} \mid \ell_{q/r}(\ell_{q/r}) \right\| \\ &= \left\| \left(\int_1^2 \left| 2^{ks} ((\Phi_k)_t * f)(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \mid L_p(\ell_q) \right\|^r. \end{aligned} \tag{2.76}$$

Hence, we obtain

$$\begin{aligned} \left\| \left(\int_0^1 |\lambda^{-sq} (\Phi^*_\lambda f)_a(x)|^q \frac{d\lambda}{\lambda} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\| &\asymp \left\| \left(\sum_{\ell=1}^{\infty} \int_1^2 \left| 2^{\ell s} (\Phi^*_{2^{-\ell}t} f)_a(x) \right|^q \frac{dt}{t} \right)^{1/q} \mid L_p \right\| \\ &= \left\| \left(\int_1^2 \left| 2^{\ell s} (\Phi^*_{2^{-\ell}t} f)_a(x) \right|^q \frac{dt}{t} \right)^{1/q} \mid L_p(\ell_q) \right\| \\ &\lesssim \left\| \left(\int_0^1 |\lambda^{-sq} (\Phi_\lambda * f)(x)|^q \frac{d\lambda}{\lambda} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|. \end{aligned} \tag{2.77}$$

This proves $\|f \mid F^s_{p,q}(\mathbb{R}^d)\|_2 \lesssim \|f \mid F^s_{p,q}(\mathbb{R}^d)\|_1$. Since the reverse inequality is trivial, this finishes Step 1.

Step 2. Let $\Psi_0, \Psi \in \mathcal{S}'(\mathbb{R}^d)$ be functions satisfying (2.18). In fact, the condition (2.17) for Ψ_0, Ψ is not necessary for what follows.

Substep 2.1. We are going to prove

$$\left\| f \mid F^s_{p,q}\left(\mathbb{R}^d\right)\right\|^{\Psi}_4 \lesssim \left\| f \mid F^s_{p,q}\left(\mathbb{R}^d\right)\right\|^{\Phi}_2 \tag{2.78}$$

for all $f \in \mathcal{S}'(\mathbb{R}^d)$. We decompose f similar as in Step 1. Exploiting the property (2.17) for the system Φ, we find $\mathcal{S}(\mathbb{R}^d)$-functions $\lambda_0, \lambda \in \mathcal{S}(\mathbb{R}^d)$ such that supp $\lambda_0 \subset \{\xi \in \mathbb{R}^d : |\xi| \le 2\varepsilon\}$ and supp $\lambda \subset \{\xi \in \mathbb{R}^d : \varepsilon/2 \le |\xi| \le 2\varepsilon\}$ and

$$\sum_{\ell \in \mathbb{N}_0} \lambda_\ell(tx) \cdot \varphi_\ell(tx) = 1 \tag{2.79}$$

for $x \in \mathbb{R}^d$ and $t \in [1,2]$ fix. Putting $\Lambda_0 = \mathcal{F}\lambda_0$ and $\Lambda = \mathcal{F}\Lambda$, we obtain the decomposition

$$g = \sum_{\ell \in \mathbb{N}_0} (\Lambda_\ell)_t * (\Phi_\ell)_t * g \tag{2.80}$$

for every $g \in \mathcal{S}'(\mathbb{R}^d)$. We put $g = \Psi_\ell * f$ and see

$$\Psi_\ell * f = \sum_{k \in \mathbb{N}_0} \Psi_\ell * (\Lambda_k)_t * (\Phi_k)_t * f. \tag{2.81}$$

Now, we estimate as follows:

$$\begin{aligned} &\left|\left((\Psi_\ell * (\Lambda_k)_t) * \left((\Phi_k)_t * f\right)\right)(y)\right| \\ &\quad \le \int_{\mathbb{R}^d} |(\Psi_\ell * (\Lambda_k)_t)(z)| \cdot \left|\left((\Phi_k)_t * f\right)(y-z)\right| dz \\ &\quad \le \left(\Phi^*_{2^{-k}t} f\right)_a(y) \int_{\mathbb{R}^d} |(\Phi_\ell * (\Lambda_k)_t)(z)| \cdot \left(1 + 2^k|z|\right)^a dz \\ &\quad \le \left(\Phi^*_{2^{-k}t} f\right)_a(y) J_{\ell,k}, \end{aligned} \tag{2.82}$$

where

$$J_{\ell,k} = \int_{\mathbb{R}^d} |(\Psi_\ell * (\Lambda_k)_t)(z)| \left(1 + 2^k|z|\right)^a dz. \tag{2.83}$$

We first observe that for $x \in \mathbb{R}^d$ and functions $\mu, \eta \in \mathcal{S}(\mathbb{R}^d)$, the following identity holds true for $u, v > 0$

$$(\mu_u * \eta_v)(z) = \frac{1}{u^d}\left[\mu * \eta_{v/u}\right]\left(\frac{z}{u}\right) = \frac{1}{v^d}\left[\mu_{u/v} * \eta\right]\left(\frac{z}{v}\right). \tag{2.84}$$

This yields in case $\ell \geq k$ (with a minor change if $k = 0$ or $\ell = 0$)

$$\begin{aligned} J_{\ell,k} &= \int_{\mathbb{R}^d} \left| \left[(\Psi_{\ell-k})_{1/t} * \Lambda \right](z) \right| (1 + |tz|)^a \\ &\lesssim \sup_{z\in\mathbb{R}^d} \left| \left[(\Psi_{\ell-k})_{1/t} * \Lambda \right](z) \right| (1 + |z|)^{a+d+1} \\ &\lesssim 2^{(k-\ell)(R+1)}, \end{aligned} \tag{2.85}$$

where we used Lemma A.3 for the last estimate.

If $k > l$, we change the roles of Ψ and Λ to obtain again with Lemma A.3

$$\begin{aligned} J_{\ell,k} &= \int_{\mathbb{R}^d} |[\Psi * (\Lambda_{k-\ell})](z)| \left(1 + \left|2^{k-\ell} z\right|\right)^a dx \\ &\lesssim 2^{(k-\ell)^a} \sup_{z\in\mathbb{R}^d} |[\Psi * (\Lambda_{k-\ell})_t](z)| (1 + |z|)^{a+d+1} \\ &\lesssim 2^{(\ell-k)(L+1-a)}, \end{aligned} \tag{2.86}$$

where L can be chosen arbitrary large since Λ satisfies (M_L) for every $L \in \mathbb{N}$ according to its construction. Let us further use the estimate

$$\begin{aligned} \left(\Psi_k^* f\right)_a(y) &\leq \left(\Psi_k^* f\right)_a(x) \left(1 + 2^k |x - y|\right)^a \\ &\lesssim \left(\Psi_k^* f\right)_a(x) \left(1 + 2^\ell |x - y|\right)^a \max\left\{1, 2^{(k-\ell)a}\right\}. \end{aligned} \tag{2.87}$$

Consequently,

$$\begin{aligned} \sup_{y\in\mathbb{R}^d} \frac{2^{s\ell} \left| \left(\Psi_\ell * (\Lambda_k)_t * \left((\Phi_k)_t * f\right)\right)(y) \right|}{\left(1 + 2^\ell |x - y|\right)^a} &\lesssim 2^{ks} \left(\Phi_{2^{-k}t}^* f\right)_a(x) 2^{(\ell-k)s} \max\left\{1, 2^{(k-\ell)a}\right\} J_{\ell,k} \\ &\leq 2^{ks} \left(\Phi_{2^{-k}t}^* f\right)_a(x) \begin{cases} 2^{(\ell-k)(L+1-a+s)} : & k > l, \\ 2^{(k-\ell)(R+1-s)} : & \ell \geq k. \end{cases} \end{aligned} \tag{2.88}$$

Plugging this into (2.81) and choosing $L \geq a + |s|$ and $\delta = \min\{1, R + 1 - s\}$, we obtain the inequality

$$2^{\ell s} \left(\Psi_\ell^* f\right)_a(x) \lesssim \sum_{k\in\mathbb{N}_0} 2^{-|k-\ell|\delta} 2^{ks} \left(\Phi_{2^{-k}t}^* f\right)_a(x) \tag{2.89}$$

for all $x \in \mathbb{R}^d$ and all $t \in [1,2]$. Suppose first that $q \geq 1$. Then we take on both sides $(\int_1^2 |\cdot|^q dt/t)^{1/q}$, which gives

$$2^{\ell s}(\Psi_\ell^* f)_a(x) \lesssim \left(\sum_{k\in\mathbb{N}_0} 2^{-|k-\ell|\delta} 2^{ks} \int_1^2 \left|(\Phi_{2^{-k}t}^* f)_a(x)\right|^q \frac{dt}{t}\right)^{1/q}. \tag{2.90}$$

Applying Lemma 2.13 yields

$$\left\|2^{\ell s}(\Psi_\ell^* f)_a(x) L_p\left(\ell_q,\ \mathbb{R}^d\right)\right\| \lesssim \left\|\left(\sum_{k=1}^{\infty} 2^{ksq} \int_1^2 \left|(\Phi_{2^{-k}t}^* f)_a(x)\right|^q \frac{dt}{t}\right)^{1/q} \mid L_p\left(\mathbb{R}^d\right)\right\|, \tag{2.91}$$

which gives the desired result.

In case $q < 1$, we argue as follows. The quantity $(\int_1^2 |\cdot|^q dt/t)^{1/q}$ is not longer a norm, but a q-norm. This gives

$$2^{\ell s}(\Psi_\ell^* f)_a(x)^q \lesssim \sum_{k\in\mathbb{N}_0} 2^{-|k-\ell|\delta q} 2^{ksq} \int_1^2 \left|(\Phi_{2^{-k}t}^* f)_a(x)\right|^q \frac{dt}{t}. \tag{2.92}$$

Notice that the right-hand side is nothing more than a convolution $(\gamma * \beta)_\ell$ of the sequences

$$\gamma_k = 2^{-|k|\delta q}, \qquad \beta_k = 2^{ksq} \int_1^2 \left|(\Phi_{2^{-k}t}^* f)_a(x)\right|^q \frac{dt}{t}. \tag{2.93}$$

Now we apply the ℓ_1-norm to both sides and get for all $x \in \mathbb{R}^d$

$$\begin{aligned} \left\|2^{\ell s}(\Psi_\ell^* f)_a(x) \mid \ell_q\right\|^q &\leq \|\gamma \mid \ell_1\| \cdot \|\beta| \ \ell_1\| \\ &\lesssim \sum_{k=1}^{\infty} 2^{ksq} \int_1^2 \left|(\Phi_{2^{-k}t}^* f)_a(x)\right|^q \frac{dt}{t}. \end{aligned} \tag{2.94}$$

We take both sides to the power $(\cdots)^{1/q}$ and apply the $L_p(\mathbb{R}^d)$-norm. This gives (2.78).

Substep 2.2. With similar arguments and obvious modifications of Substep 2.1, we obtain for all $f \in \mathcal{S}'(\mathbb{R}^d)$

$$\left\|f \mid F_{p,q}^s\left(\mathbb{R}^d\right)\right\|_2^{\Psi} \lesssim \left\|f \mid F_{p,q}^s\left(\mathbb{R}^d\right)\right\|_4^{\Phi}. \tag{2.95}$$

Step 3. Choosing $t = 1$ in Step 1 and omitting the integration over t, we see immediately

$$\left\|f \mid F_{p,q}^s\left(\mathbb{R}^d\right)\right\|_5 \lesssim \left\|f \mid F_{p,q}^s\left(\mathbb{R}^d\right)\right\|_4 \lesssim \left\|f \mid F_{p,q}^s\left(\mathbb{R}^d\right)\right\|_5. \tag{2.96}$$

Step 4. What remains is to show that (2.22) is equivalent to the rest.

Substep 4.1. Let us prove

$$\left\| f \mid F^s_{p,q}\left(\mathbb{R}^d\right)\right\|_2 \lesssim \left\| f \mid F^s_{p,q}\left(\mathbb{R}^d\right)\right\|_3. \tag{2.97}$$

We return to (2.66) in Substep 1.3. If $|z| < 2^{-(\ell+k)}t$, formula (2.66) implies by shift in the integral the following:

$$\left(\Phi^*_{2^{-\ell}t} f\right)_a(x)^r \le C_N \sum_{k\in\mathbb{N}_0} 2^{-kNr} 2^{(k+\ell)d} \int_{\mathbb{R}^d} \frac{\left|\left((\Phi_{k+\ell})_t * f\right)(y+z)\right|^r}{\left(1+2^\ell|x-y|\right)^{ar}}\, dy. \tag{2.98}$$

Indeed, we have

$$\begin{aligned} 1+2^\ell|x-y| &\le 1+2^\ell\left(\left|x-(y+z)\right|+|z|\right) \\ &\lesssim 1+2^\ell\left(\left|x-(y+z)\right|\right)+2^{-k} \\ &\lesssim 1+2^\ell\left(\left|x-(y+z)\right|\right), \end{aligned} \tag{2.99}$$

where the last estimate follows from the fact that $k \in \mathbb{N}_0$ in the sum. Instead of the integral $(\int_1^2 |\cdot|^{q/r} dt/t)^{r/q}$ (see Step 3), we take now on both sides of (2.98) the norm

$$\left(\int_1^2 \int_{|z|<t} |\cdot|^{q/r} dz \frac{dt}{t^{d+1}}\right)^{r/q}. \tag{2.100}$$

The integration over z does not influence the left-hand side. Instead of (2.68), we obtain

$$\begin{aligned} &\left(\int_1^2 \left|\left(\Phi^*_{2^{-\ell}t} f\right)_a(x)\right|^q \frac{dt}{t}\right)^{r/q} \\ &\quad \le c \sum_{k\in\mathbb{N}_0} 2^{-kNs} 2^{(k+\ell)d} \int_{\mathbb{R}^d} \frac{\left(\int_1^2 \int_{|z|<t} \left|\left((\Phi_{k+\ell})_t * f\right)(y)\right|^q dz\left(dt/t^{d+1}\right)\right)^{r/q}}{\left(1+2^\ell|x-y|\right)^{ar}} dy. \end{aligned} \tag{2.101}$$

We continue with analogous arguments as after (2.68) and end up with (2.97).

Substep 4.2. We prove $\|f \mid F^s_{p,q}(\mathbb{R}^d)\|_3 \lesssim \|f \mid F^s_{p,q}(\mathbb{R}^d)\|_2$. Indeed, it is easy to see, that we have for all $t > 0$

$$\begin{aligned} \frac{1}{t^d}\int_{|z|<t}\left|(\Phi_t * f)(x+z)\right|dz &\lesssim \sup_{|z|<t}\frac{\left(\left|(\Phi_t * f)(x+z)\right|\right.}{(1+1/t|z|)^a} \\ &\lesssim (\Phi_t^* f)_a(x), \end{aligned} \tag{2.102}$$

and we are done. The proof is complete. □

Proof of Theorem 2.8. The proof of Theorem 2.8 works almost analogously to the previous one. It is even a bit simpler, since we do not have to deal with a separate function Φ_0, which causes some technical difficulties. However, there are still some technical obstacles which have to be discussed. Although we are in the homogeneous world, we use the same decomposition as used in (2.43), even with the inhomogeneity Φ_0. In the definition of $\Lambda_{m,\ell}(x)$ in (2.45), we have to add $\Phi(x)$ if $\ell = 0$ and $m > 0$. The consequence is (2.47) for every $\ell \in \mathbb{Z}$, where $\Phi_0 = \Phi$. Hence, the inhomogeneity is buried in $\Lambda_{m,\ell}$. This yields (2.66) for all $\ell \in \mathbb{Z}$, where k still runs trough $\mathbb{N}_0$. We need this for the argument in Substep 4.1. In contrast to the previous decomposition, we use (2.80) now for $\ell \in \mathbb{Z}$, where $\Phi_0 = \Phi$, $\Lambda_0 = \Lambda$, and $k \in \mathbb{Z}$. This works since we assume $g \in \mathcal{S}_0'(\mathbb{R}^d)$. The rest is clear. □

Proof of Corollaries 2.11 and 2.12. Corollary 2.11 is more or less clear. We know that Δ^N gives $(\sum_{k=1}^d |\xi_k|^2)^N$ as factor on the Fourier side. This gives (2.18) immediately, and together with (2.31) we have (2.17) for $\varepsilon > 0$ small enough.

In the case of Corollary 2.12, the situation is a bit delicate. Clearly, Condition (2.18) holds true. But the problem here is, that (2.17) may be violated for all $\varepsilon > 0$. However, we argue as follows. In Step 2 in the proof above we have seen, that we do not need (2.17) for the system (Ψ_0, Ψ). Hence, we can estimate (2.20), (2.21), (2.22), (2.23), and (2.24) from above by a different characterization of $F^s_{p,q}(\mathbb{R}^d)$. For the remaining estimates we apply Theorem 2.6 with the system $(\Phi_0, \widetilde{\Phi})$, where

$$\widetilde{\Phi} = \Phi_0(x) - \frac{1}{2^{kd}}\Phi_0\left(\frac{x}{2^k}\right) \tag{2.103}$$

and $k \in \mathbb{N}$ is chosen in such a way that (2.17) is satisfied. What remains is a simple consequence of the fact that

$$\widetilde{\Phi} = \Phi + \Phi_{-1} + \cdots + \Phi_{-(k-1)} \tag{2.104}$$

and the triangle inequality. This type of argument is due to Triebel [3, Section 3.3.3]. □

3. Classical Coorbit Space Theory

In [11–13, 15], a general theory of Banach spaces related to integrable group representations was developed. The ingredients are a locally compact group $\mathcal{G}$ with identity e, a Hilbert space $\mathcal{H}$, and an irreducible, unitary, and continuous representation π on $\mathcal{H}$, which is at least integrable. Then one can associate a Banach space $\mathrm{Co}Y$ to any solid, translation-invariant Banach space Y of functions on the group $\mathcal{G}$. The main achievement of this abstract theory is a powerful discretization machinery for $\mathrm{Co}Y$, that is, a universal approach to atomic decompositions and Banach frames. It allows to transfer certain questions concerning Banach space or interpolation theory from the function space to the associated sequence space level, see [12, 13, 26]. In connection with smoothness spaces of Besov-Lizorkin-Triebel type, the philosophy of this approach is to measure smoothness of a function in decay properties of the continuous wavelet transform $W_g f$, see the appendix for details. Indeed, homogeneous Besov and Lizorkin-Triebel-type spaces turn out to be coorbits of properly chosen spaces Y on the $ax+b$-group G.

The are many more examples according to this abstract theory. One main class of examples refers to the Heisenberg group $\mathbb{H}$ and the short-time Fourier transform and leads to the well-known modulation spaces as coorbits of weighted $L_p(\mathbb{H})$ spaces, see [11, Section 7.1] and also [27].

3.1. Function Spaces on $\mathcal{G}$

Integration on $\mathcal{G}$ will always be with respect to the left Haar measure $d\mu(x)$. The Haar module on $\mathcal{G}$ is denoted by Δ. We define further $L_xF(y) = F(x^{-1}y)$ and $R_xF(y) = F(yx)$, $x, y \in \mathcal{G}$, the left and right translation operators. A Banach function space Y on the group $\mathcal{G}$ is supposed to have the following properties:

(i) Y is continuously embedded in $L_1^{\mathrm{loc}}(\mathcal{G})$,

(ii) Y is invariant under left and right translation L_y and R_y, which represent continuous operators on Y,

(iii) Y is solid, that is, $H \in Y$ and $|F(x)| \le |H(x)|$ a.e. imply $F \in Y$ and $\|F \mid Y\| \le \|H|Y\|$.

The continuous weight w is called submultiplicative if $w(xy) \le w(x)w(y)$ for all $x, y \in \mathcal{G}$. Further, another weight m is called w-moderate if $m(xyz) \le w(x)m(y)w(z)$, $x, y, z \in \mathcal{G}$. The space $L_p^w(\mathcal{G})$ of functions F on the group $\mathcal{G}$ is defined via the norm

$$\left\| F \mid L_p^w(\mathcal{G}) \right\| = \left(\int_{\mathcal{G}} |F(x)w(x)|^p d\mu(x) \right)^{1/p}, \tag{3.1}$$

where $1 \le p \le \infty$ (modification if $p = \infty$). If $w \equiv 1$, then we simply write $L_p(\mathcal{G})$. These spaces provide left and right translation invariance, which is easy to show. Later, in Section 4.1, we are going to introduce certain mixed norm spaces where the translation invariance is not longer automatic.

3.2. Sequence Spaces

Definition 3.1. Let $X = \{x_i\}_{i\in I}$ be some discrete set of points in $\mathcal{G}$ and V a relatively compact neighborhood of $e \in \mathcal{G}$.

(i) X is called V-dense if $\mathcal{G} = \bigcup_{i\in I} x_i V$.

(ii) X is called relatively separated if for all compact sets $K \subset \mathcal{G}$, there exists a constant C_K such that

$$\sup_{j\in I} \sharp\{i \in I : x_i K \cap x_j K \neq \emptyset\} \leq C_K. \tag{3.2}$$

(iii) X is called V-well-spread (or simply well-spread) if it is both relatively separated and V-dense for some V.

Definition 3.2. For a family $X = \{x_i\}_{i\in I}$ which is V-well-spread with respect to a relatively compact neighborhood V of $e \in \mathcal{G}$ we define the sequence space Y^b and $Y^\sharp$ associated to Y as

$$\begin{aligned} Y^b &= \left\{ \{\lambda_i\}_{i\in I} : \left\| \{\lambda_i\}_{i\in I} \mid Y^b \right\| = \left\| \sum_{i\in I} |\lambda_i| |x_i V|^{-1} \chi_{x_i V} \mid Y \right\| < \infty \right\}, \\ Y^\sharp &= \left\{ \{\lambda_i\}_{i\in I} : \left\| \{\lambda_i\}_{i\in I} \mid Y^\sharp \right\| = \left\| \sum_{i\in I} |\lambda_i| \chi_{x_i V} \mid Y \right\| < \infty \right\}. \end{aligned} \tag{3.3}$$

Remark 3.3. For a well-spread family X, the spaces Y^b and $Y^\sharp$ do not depend on the choice of V, that is, different sets V define equivalent norms on Y^b and $Y^\sharp$, respectively. For more facts on these sequence spaces, confer [12].

3.3. Coorbit Spaces

The starting point is a Hilbert space $\mathcal{H}$ and an integrable, irreducible, unitary and continuous representation π of $\mathcal{G}$ on $\mathcal{H}$. Then the general voice transform $V_g f$ is a function on the group $\mathcal{G}$ given by

$$V_g f(x) = \langle \pi(x) g, f \rangle, \tag{3.4}$$

where the brackets denote the inner product in $\mathcal{H}$.

The continuous wavelet transform $W_g f$ (appendix) is a voice transform for the situation $\mathcal{H} = L_2(\mathbb{R}^d)$ and the $ax + b$-group.

Definition 3.4. For a weight $w(\cdot) \geq 1$ on $\mathcal{G}$, we define the space $A_w \subset \mathcal{H}$ of admissible vectors by

$$A_w = \{g \in \mathcal{H} : V_g g \in L_1^w(\mathcal{G})\}. \tag{3.5}$$

If $A_w \neq \{0\}$ and $g \in A_w$, we define further

$$\mathcal{H}_w^1\left(\mathbb{R}^d\right) = \left\{ f \in \mathcal{H} : \left\| f \mid \mathcal{H}_w^1 \right\| = \left\| V_g f \mid L_1^w(\mathcal{G}) \right\| < \infty \right\}. \tag{3.6}$$

Finally, we denote by $(\mathcal{H}_w^1)^\sim$ the canonical antidual of $\mathcal{H}_w^1$, that is, the space of conjugate linear functionals on $\mathcal{H}_w^1$.

We see immediately that $A_w \subset \mathcal{H}_w^1 \subset \mathcal{H}$. The voice transform (3.4) can now be extended to $\mathcal{H}_w \times (\mathcal{H}_w^1)^{\sim}$ by the usual dual pairing. The space $\mathcal{H}_w^1$ can be considered as the space of test functions, whereas the space $(\mathcal{H}_w^1)^{\sim}$ serves as reservoir, that is, distributions.

Let now Y be a space on $\mathcal{G}$ such that (i)–(iii) in Section 3.1 hold true. We define further

$$w_Y(x) = \max\left\{\|L_x\|, \|L_{x^{-1}}\|, \|R_x\|, \Delta\left(x^{-1}\right)\|R_{x^{-1}}\|\right\}, \quad x \in \mathcal{G}, \tag{3.7}$$

where the operator norms are considered from Y to Y.

Definition 3.5. Let Y be a space on $\mathcal{G}$ satisfying (i)–(iii) and $w_Y(x)$ be given by (3.7). Let further $g \in A_w$. We define the space $\mathrm{Co}Y$, called coorbit space of Y, through

$$\mathrm{Co}Y = \left\{f \in \left(\mathcal{H}_w^1\right)^{\sim} : V_g f \in Y\right\} \qquad \text{with } \|f \mid \mathrm{Co}Y\| = \|V_g f \mid Y\|. \tag{3.8}$$

The following result is of crucial importance. See [14] for details.

Lemma 3.6 (Correspondence principle). *Let $G = V_g g \in L_1^w(\mathcal{G})$ for a fixed analyzing vector $g \in A_w$ with proper normalization. Then a function $F \in Y$ is of the form $V_g f$ for some $f \in \mathrm{Co}Y$ if and only if*

$$F(x) = (F * G)(x) := \int_{\mathcal{G}} F(y) L_y G(x) d\mu(y), \quad x \in \mathcal{G}. \tag{3.9}$$

In other words, CoY is isometrically isomorphic to the closed subspace $Y * G$ of Y.

The following basic properties are proved for instance, in [28, Theorem 4.5.13].

Theorem 3.7. (i) *The space CoY is independent of the analyzing vector $g \in A_w$.*

(ii) *The definition of the space CoY is independent of the reservoir in the following sense. Assume that $S \subset \mathcal{H}_w^1$ is a nontrivial locally convex vector space, which is invariant under π. Assume further that there exists a nonzero vector $g \in S \cap A_w$ for which the reproducing formula*

$$V_g f = V_g g * V_g f \tag{3.10}$$

holds for all $f \in S^{\sim}$. Then we have

$$CoY = \left\{f \in \left(\mathcal{H}_w^1\right)^{\sim} : V_g f \in Y\right\} = \{f \in S^{\sim} : V_g f \in Y\}. \tag{3.11}$$

3.4. Discretizations

This section collects briefly the basic facts concerning atomic (frame) decompositions in coorbit spaces. We are interested in atoms of type $\{\pi(x_i)g\}_{i\in I}$, where $\{x_i\}_{i\in I} \subset \mathcal{G}$ represents a discrete subset, whereas g denotes a fixed admissible analyzing vector.

Definition 3.8. A family $\{g_i\}_{i\in I}$ in a Banach space B is called an atomic decomposition for B if there exists a family of bounded linear functionals $\{\lambda_i\}_{i\in I} \subset B'$ (not necessary unique) and a Banach sequence space $B^\sharp = B^\sharp(I)$ such that

(a) $\{\lambda_i(f)\}_{i\in I} \in B^\sharp$ for all $f \in B$ and there exists a constant $C_1 > 0$ with

$$\left\| \{\lambda_i(f)\}_{i\in I} \mid B^\sharp \right\| \le C_1 \|f \mid B\|. \tag{3.12}$$

(b) For all $f \in B$, we have

$$f = \sum_{i\in I} \lambda_i(f) g_i \tag{3.13}$$

in some suitable topology.

(c) If $\{\lambda_i\}_{i\in I} \in B^\sharp$, then $\sum_{i\in I} \lambda_i g_i \in B$ and there exists a constant $C_2 > 0$ such that

$$\left\| \sum_{i\in I} \lambda_i g_i \mid B \right\| \le C_2 \left\| \{\lambda_i\}_{i\in I} \mid B^\sharp \right\|. \tag{3.14}$$

Definition 3.9. A family $\{h_i\}_{i\in I} \subset B'$ is called a Banach frame for B if there exists a Banach sequence space $B^b = B^b(I)$ and a linear bounded reconstruction operator $\Theta : B^b \to B$ such that

(a) $\{h_i(f)\}_{i\in I} \in B^b$ for all $f \in B$ and there exist constants C_1, C_2 such that

$$C_1 \|f \mid B\| \le \left\| \{h_i(f)\}_{i\in I} \mid B^b \right\| \le C_2 \|f \mid B\|, \tag{3.15}$$

(b) $\Theta(\{h_i(f)\}_{i\in I}) = f$.

Remark 3.10. This setting differs slightly from the understanding of Triebel in [3, 29].

The following abstract result for the atomic decomposition in $\mathrm{Co}Y$ is due to Feichtinger and Gröchenig (see [12, Theorem 6.1]).

Theorem 3.11. *Let Y be a function space on the group $\mathcal{G}$ which satisfies the hypotheses* (i)–(iii). *Let further*

$$w_Y(x) := \max\left\{ \|L_x\|, \|L_{x^{-1}}\|, \|R_x\|, \Delta\left(x^{-1}\right) \|R_{x^{-1}}\| \right\}, \tag{3.16}$$

where the operator norms are taken from Y to$Y$, and $g \in A_w$ such that

$$\int_{\mathcal{G}} \left(\sup_{y \in xV} |\langle \pi(y) g, g \rangle| \right) w(x,t) d\mu(x) < \infty. \tag{3.17}$$

Then there exists a neighborhood U of $e \in \mathcal{G}$ and constants $C_0, C_1 > 1$ such that for every U-well-spread discrete set $X = \{x_i\}_{i \in I} \subset \mathcal{G}$, the following is true.

(i) *(Analysis) Every $f \in CoY$ has a representation*

$$f = \sum_{i \in I} \lambda_i \pi(x_i) g \tag{3.18}$$

with coefficients $\{\lambda_i\}_{i \in I}$ depending linearly on f and satisfying the estimate

$$\left\| \{\lambda_i\}_{i \in I} \mid Y^{\sharp} \right\| \leq C_0 \| f \mid Y \|. \tag{3.19}$$

(ii) *(Synthesis) Conversely, for any sequence $\{\lambda_i\}_{i \in I} \in Y^{\sharp}$, the element $f = \sum_{i \in I} \lambda_i \pi(x_i) g$ is in CoY and one has*

$$\| f \mid CoY \| \leq C_1 \left\| \{\lambda_i\}_{i \in I} \mid Y^{\sharp} \right\|. \tag{3.20}$$

In both cases, convergence takes place in the norm of CoY if the finite sequences are norm dense in $Y^{\sharp}$, and in the weak* -sense of $(\mathscr{H}_w^1)^{\sim}$ otherwise.

Remark 3.12. According to Definition 3.8, the family $\{\pi(x_i) g\}_{i \in I}$ represents an atomic decomposition for CoY.

Theorem 3.13. *Under the same conditions as in Theorem 3.11, the system $\{\pi(x_i) g\}_{i \in I}$ represents a Banach frame for CoY, that is,*

$$\| f \mid CoY \| \asymp \left\| \{ \langle \pi(x_i) g, f \rangle \} \mid Y^b \right\|. \tag{3.21}$$

The following powerful result goes back to Gröchenig [14] and was generalized by Rauhut [17].

Theorem 3.14. *Suppose that the functions g_r, γ_r, $r = 1, \ldots, n$, satisfy* (3.17). *Let $X = \{x_i\}_{i \in I}$ be a well-spread set such that*

$$f = \sum_{r=1}^{n} \sum_{i \in I} \langle \pi(x_i) \gamma_r, f \rangle \pi(x_i) g_r \tag{3.22}$$

for all $f \in H$. Then expansion (3.22) *extends to all $f \in CoY$. Moreover, $f \in (\mathscr{H}_w^1)^{\sim}$ belongs to CoY if and only if $\{\langle \pi(x_i) \gamma_r, f \rangle\}_{i \in I}$ belongs to $Y^{\sharp}$ for each $r = 1, \ldots, n$. The convergence is considered in CoY if the finite sequences are dense in $Y^{\sharp}$. In general, we have weak*-convergence.*

Proof. The proof of this result relies on the fact that there exists an atomic decomposition $\{\pi(y_i)g\}_{i\in I}$ by Theorem 3.11 with a certain g satisfying (3.17) and a corresponding sequence of points $Z = \{y_i\}_{i\in I}$. This has to be combined with Theorem 3.13 and Theorem 3.11/(ii) and we are done. See [14] for the details. □

4. Coorbit Spaces on the $ax + b$-Group

Let $\mathcal{G} = \mathbb{R}^d \rtimes \mathbb{R}^*_+$ the d-dimensional $ax + b$-group. Its multiplication is given by

$$(x,t)(y,s) = (x + ty, st). \tag{4.1}$$

The left Haar measure μ on $\mathcal{G}$ is given by dt/t^{d+1}, the Haar module is $\Delta(x,t) = t^{-d}$. Giving a function F on $\mathcal{G}$, the left and right translation $L_y = L_{(y,r)}$ and $R_y = R_{(y,r)}$ are given by

$$\begin{aligned} L_{(y,r)}F(x,t) &= F\Big((y,r)^{-1}(x,t)\Big) = F\left(\frac{x-y}{r}, \frac{t}{r}\right), \\ R_{(y,r)}F(x,t) &= F\big((x,t)(y,r)\big) = F(x+ty, rt). \end{aligned} \tag{4.2}$$

4.1. Peetre-Type Spaces on $\mathcal{G}$

The present paragraph is devoted to the definition of certain mixed norm spaces on the group. Such spaces were considered in various papers, see [10, 11, 14, 15]. Especially tent spaces have some nice applications in harmonic analysis. In particular, they were used by many authors in order to apply coorbit space theory to Lizorkin-Triebel spaces.

Here we use a different approach and define a new scale of function spaces on the group $\mathcal{G}$. We call them Peetre-type spaces since the Peetre maximal function is involved in the definition. It turned out that they are very easy to handle in connection with translation invariance. Compared with the tent space approach, they are the more natural choice for considering Lizorkin-Triebel spaces as coorbit spaces. Additionally they seem to be the suitable choice for inhomogeneous spaces and more general situations like weighted spaces, which will be studied in a subsequent contribution to the subject.

Definition 4.1. Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$, and $a > 0$. We define the spaces $\dot{L}^s_{p,q}(\mathcal{G})$, $\dot{T}^s_{p,q}(\mathcal{G})$, and $\dot{P}^{s,a}_{p,q}(\mathcal{G})$ on the group $\mathcal{G}$ via the finiteness of the following (quasi-)norms:

$$\begin{aligned} \left\| F \mid \dot{L}^s_{p,q}(\mathcal{G}) \right\| &= \left(\int_0^\infty t^{-sq} \left\| F(\cdot,t) \mid L_p\left(\mathbb{R}^d\right) \right\|^q \frac{dt}{t^{d+1}} \right)^{1/q}, \\ \left\| F \mid \dot{T}^s_{p,q}(\mathcal{G}) \right\| &= \left\| \left(\int_0^\infty t^{-sq} \int_{B(0,t)} |F(x+z,t)|^q \, dz \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|, \\ \left\| F \mid \dot{P}^{s,a}_{p,q}(\mathcal{G}) \right\| &= \left\| \left(\int_0^\infty t^{-sq} \left[\sup_{y\in\mathbb{R}^d} \frac{|F(x+y,t)|}{(1+|y|/t)^a} \right]^q \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|, \end{aligned} \tag{4.3}$$

using the usual modification in case $q = \infty$.

Proposition 4.2. *The spaces $\dot{L}^s_{p,q}(\mathcal{G}), \dot{T}^s_{p,q}(\mathcal{G})$ and $\dot{P}^s_{p,q}(\mathcal{G})$, are left and right translation invariant. Precisely, we have*

$$\begin{aligned}
&\left\| L_{(z,r)} : \dot{L}^s_{p,q}(\mathcal{G}) \longrightarrow \dot{L}^s_{p,q}(\mathcal{G}) \right\| = r^{d(1/p-1/q)-s}, \\
&\left\| R_{(z,r)} : \dot{L}^s_{p,q}(\mathcal{G}) \longrightarrow \dot{L}^s_{p,q}(\mathcal{G}) \right\| = r^{s+d/q}, \\
&\left\| L_{(z,r)} : \dot{T}^s_{p,q}(\mathcal{G}) \longrightarrow \dot{T}^s_{p,q}(\mathcal{G}) \right\| = r^{d/q-s}, \\
&\left\| R_{(z,r)} : \dot{T}^s_{p,q}(\mathcal{G}) \longrightarrow \dot{T}^s_{p,q}(\mathcal{G}) \right\| \leq C r^{d/q+s} \max\left\{ 1, r^{-b}(1+|z|)^b \right\},
\end{aligned} \tag{4.4}$$

where $b > 0$ is a constant depending on d, p, and q, and

$$\begin{aligned}
&\| L_{(z,r)} : \dot{P}^{s,a}_{p,q}(\mathcal{G}) \longrightarrow \dot{P}^{s,a}_{p,q}(\mathcal{G}) \| = r^{d(1/p-1/q)-s}, \\
&\| R_{(z,r)} : \dot{P}^{s,a}_{p,q}(\mathcal{G}) \longrightarrow \dot{P}^{s,a}_{p,q}(\mathcal{G}) \| \leq r^{s+d/q} \max\{ 1, r^{-a} \} (1+|z|)^a.
\end{aligned} \tag{4.5}$$

Proof.

Step 1. The left and right translation invariance of $\dot{L}^s_{p,q}(\mathcal{G})$ and $\dot{T}^s_{p,q}(\mathcal{G})$ was shown in [28, Lemma. 4.7.10].

Step 2. Let us consider $\dot{P}^{s,a}_{p,q}(\mathcal{G})$. Clearly, we have for $F \in \dot{P}^{s,a}_{p,q}(\mathcal{G})$

$$\begin{aligned}
\left\| L_{(z,r)} F \mid \dot{P}^s_{p,q}(\mathcal{G}) \right\| &= \left\| \left(\int_0^\infty t^{-sq} \left[\sup_{y \in \mathbb{R}^d} \frac{\left| F((x+y-z)/r, t/r) \right|}{(1+|y|/t)^a} \right]^q \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\| \\
&= r^{d/p} \left\| \left(\int_0^\infty t^{-sq} \left[\sup_{y \in \mathbb{R}^d} \frac{\left| F(x+y, t/r) \right|}{(1+r|y|/t)^a} \right]^q \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\| \\
&= r^{d(1/p-1/q)-s} \left\| \left(\int_0^\infty t^{-sq} \left[\sup_{y \in \mathbb{R}^d} \frac{\left| F(x+y, t) \right|}{(1+|y|/t)^a} \right]^q \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_{p(\mathbb{R}^d)} \right\|.
\end{aligned} \tag{4.6}$$

Hence, we obtain

$$\left\| L_{(z,r)} : \dot{P}^{s,a}_{p,q}(\mathcal{G}) \longrightarrow \dot{P}^s_{p,q}(\mathcal{G}) \right\| = r^{d(1/p-1/q)-s}. \tag{4.7}$$

The right translation invariance is obtained by

$$\begin{aligned}\left\| R_{(z,r)}F \mid \dot{P}^{s,a}_{p,q}(\mathcal{G})\right\| &= \left\| \left(\int_0^\infty t^{-sq} \left[\sup_{y\in\mathbb{R}^d} \frac{\left|F(x+tz+y,tr)\right|}{\left(1+\left|y\right|/t\right)^a} \right]^q \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\| \\ &= \left\| \left(\int_0^\infty t^{-sq} \left[\sup_{y\in\mathbb{R}^d} \frac{\left|F(x+y,tr)\right|}{\left(1+|y-tz|/t\right)^a} \right]^q \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\| \\ &= r^{s+d/q} \left\| \left(\int_0^\infty t^{-sq} \left[\sup_{y\in\mathbb{R}^d} \frac{\left|F(x+y,t)\right|}{\left(1+\left|y-tz\right|r/t\right)^a} \right]^q \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|.\end{aligned} \tag{4.8}$$

Observe that

$$\begin{gathered}\sup_{y\in\mathbb{R}^d} \frac{\left|F(x+y,t)\right|}{\left(1+\left|y-tz\right|r/t\right)^a} = \sup_{y\in\mathbb{R}^d} \left[\frac{\left|F(x+y,t)\right|}{\left(1+\left|y\right|/t\right)^a} \cdot \frac{\left(1+\left|y\right|/t\right)^a}{\left(1+\left|y-tz\right|r/t\right)^a} \right], \\ \frac{\left(1+\left|y\right|/t\right)^a}{\left(1+\left|y-tz\right|r/t\right)^a} \le \frac{\left(1+\left|y-tz\right|/t+|z|\right)^a}{\left(1+\left|y-tz\right|r/t\right)^a} = \frac{\left(1+|y-tz|/t\right)^a\right)(1+|z|)^a}{\left(1+\left|y-tz\right|r/t\right)^a}.\end{gathered} \tag{4.9}$$

This yields

$$\sup_{y\in\mathbb{R}^d} \frac{\left|F(x+y,t)\right|}{\left(1+\left|y-tz\right|r/t\right)^a} \le \max\left\{1,r^{-a}\right\}(1+|z|)^a \sup_{y\in\mathbb{R}^d} \frac{\left|F(x+y,t)\right|}{\left(1+\left|y\right|/t\right)^a} \tag{4.10}$$

and consequently

$$\left\| R_{(z,r)} : \dot{P}^{s,a}_{p,q}(\mathcal{G}) \longrightarrow \dot{P}^{s,a}_{p,q}(\mathcal{G}) \right\| \le r^{s+d/q} \max\left\{1,r^{-a}\right\}(1+|z|)^a. \tag{4.11}$$

□

Remark 4.3. Note that we did neither use the translation invariance of the Lebesgue measure nor any change of variable in order to prove the right translation invariance *of* $\dot{P}^{s}_{p,q}(\mathcal{G})$. This gives room for generalizations, that is, replacing the space $L_p(\mathbb{R}^d)$ by some weighted Lebesgue space $L_p(\mathbb{R}^d,\omega)$ in the definition.

4.2. New Old Coorbit Spaces

We start with $\mathcal{H} = L_2(\mathbb{R}^d)$ and the representation

$$\pi(x,t) = T_x \mathfrak{D}_t^{L_2}, \tag{4.12}$$

which is unitary continuous on $\mathcal{H}$. This representation is not irreducible. However, if we restrict to radial functions $g \in L_2(\mathbb{R}^d)$, then span $\{\pi(x,t)g : (x,t) \in \mathcal{G}\}$ is dense in $L_2(\mathbb{R}^d)$.

Another possibility to overcome this obstacle is to extend the group by $SO(d)$, which is somehow equivalent. See [11, 12] for details.

The voice transform in this special situation is represented by the so-called continuous wavelet transform $W_g f$, see Appendix A.1 in the appendix. Recall the abstract definition of the space $\mathscr{H}_w^1$ and $\mathscr{A}_w$ from Definition 3.4. The following lemma is proved for instance in [28, Lemma 4.7.11] and is also a consequence of our Lemma A.3 on the decay of the continuous wavelet transform. It states under which conditions on the weight w the space $\mathscr{H}_w^1$ is nontrivial.

Lemma 4.4. *If the weight function $w(x,t) \geq 1$ satisfies the condition*

$$w(x,t) \leq (1+|x|)^r \left(t^s + t^{-s'}\right) \tag{4.13}$$

for some $r, s, s' \geq 0$, then

$$\mathcal{S}_0\left(\mathbb{R}^d\right) \hookrightarrow \mathscr{H}_w^1. \tag{4.14}$$

This is a kind of minimal condition which we need in order to define coorbit spaces in a reasonable way. Instead of $(H_w^1)^{\sim}$, one may use $\mathcal{S}_0'(\mathbb{R}^d)$ as reservoir and a radial $g \in \mathcal{S}_0(\mathbb{R}^d)$ as analyzing vector. Considering (3.7), we have to restrict to such function spaces Y on $\mathcal{G}$ satisfying (i),(ii),(iii) in Section 3.1 where additionally (recall the definition of w_Y in (3.16))

(iv)

$$w_Y(x,t) \lesssim (1+|x|)^r \left(t^s + t^{-s'}\right) \tag{4.15}$$

for some $r, s, s' \geq 0$. The following theorem shows how the spaces of Besov-Lizorkin-Triebel type from Section 2 can be recovered as coorbit spaces.

Theorem 4.5. (i) *For $1 \leq p,q \leq \infty$, and $s \in \mathbb{R}$, we have $\dot{B}^s_{p,q}(\mathbb{R}^d) = Co\dot{L}^{s+d/2-d/q}_{p,q}(\mathcal{G})$,*

(ii) *for $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$ we have*

$$\dot{F}^s_{p,q}\left(\mathbb{R}^d\right) = Co\dot{T}^{s+d/2}_{p,q}(\mathcal{G}) \tag{4.16}$$

(iii) *and if additionally $a > d/\min\{p,q\}$, we obtain*

$$\dot{F}^s_{p,q}\left(\mathbb{R}^d\right) = Co\dot{P}^{s+d/2-d/q,a}_{p,q}(\mathcal{G}). \tag{4.17}$$

Proof. Theorem 4.5 is a direct consequence of Proposition 4.2, Theorems 2.8, 2.9, and the abstract result in Theorem 3.7. □

Remark 4.6. (a) The assertions (i) and (ii) are not new. They appear for instance in [11, 14, 15]. They rely on the characterizations given by Triebel in [2] and [3, Sections 2.4, 2.5], see in particular, [3, Section 2.4.5] for the variant in terms of tent spaces, which were invented in [10]. From the deep result in [10, Proposition 4], it follows that $\dot{T}^{s}_{p,q}(\mathcal{G})$ are translation invariant Banach function spaces on $\mathcal{G}$, which makes them feasible for coorbit space theory.

(b) Assertion (iii) is indeed new and makes the rather complicated tent spaces $\dot{T}^{s}_{p,q}(\mathcal{G})$ obsolete for this issue. We showed that $Y = P^{s,a}_{p,q}(\mathcal{G})$ is a much better choice since the right translation invariance is immediate and gives more transparent estimates for its norm. Once we are interested in reasonable conditions for atomic decompositions, this is getting important, see Section 4.5.

4.3. Sequence Spaces

In the following, we consider a compact neighborhood of the identity element in $\mathcal{G}$ given by $\mathcal{U} = [0,\alpha]^d \times [\beta^{-1},1]$, where $\alpha > 0$ and $1 < \beta$. Furthermore, we consider the discrete set of points

$$\left\{x_{j,k} = \left(\alpha k \beta^{-j}, \beta^{-j}\right) : j \in \mathbb{Z},\ k \in \mathbb{Z}^d\right\}. \tag{4.18}$$

Then the family $\{x_{j,k}\mathcal{U}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ defines a partition of $\mathcal{G}$. Indeed,

$$x_{j,k}\mathcal{U} = Q_{j,k} \times \left[\beta^{-(j+1)}, \beta^{-j}\right], \tag{4.19}$$

where

$$Q_{j,k} = \left[\alpha k_1 \beta^{-j}, \alpha(k_1+1)\beta^{-j}\right] \times \cdots \times \left[\alpha k_d \beta^{-j}, \alpha(k_d+1)\beta^{-j}\right]. \tag{4.20}$$

Note that in this case the spaces $Y^{\sharp}$ and Y^{b} coincide. We will further use the notation

$$\chi_{j,k}(x) = \begin{cases} 1: & x \in Q_{j,k} \\ 0: & \text{otherwise.} \end{cases} \tag{4.21}$$

Definition 4.7. Let Y be a function space on $\mathcal{G}$ satisfying Section 3/(i), (ii), (iii). We put

$$Y^{\sharp}(\alpha,\beta) = \left\{ \left\{\lambda_{j,k}\right\}_{j,k} : \left\| \left\{\lambda_{j,k}\right\}_{j,k} \mid Y^{\sharp}(\alpha,\beta) \right\| < \infty \right\}, \tag{4.22}$$

where

$$\left\| \left\{\lambda_{j,k}\right\}_{j,k} \mid Y^{\sharp}(\alpha,\beta) \right\| = \left\| \sum_{j,k} \left|\lambda_{j,k}\right| \chi_{j,k}(x) \chi_{[\beta^{j},\beta^{j+1}]}(t) \mid Y \right\|. \tag{4.23}$$

Theorem 4.8. *Let* $1 \le p, q \le \infty$, $s \in \mathbb{R}$, *and* $a > d/\min\{p,q\}$. *Then*

$$\left\| \{\lambda_{j,k}\}_{j,k} \mid (\dot{P}^{s,a}_{p,q})^{\sharp}(\alpha,\beta) \right\| \backsim \left\| \left(\sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \beta^{\ell(s+d/q)q} |\lambda_{\ell,k}|^q \chi_{\ell,k}(x) \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|,$$
$$\left\| \{\lambda_{j,k}\}_{j,k} \mid \left(\dot{L}^{s}_{p,q}\right)^{\sharp}(\alpha,\beta) \right\| \backsim \left(\sum_{\ell\in\mathbb{Z}} \beta^{\ell(s+d/q-d/p)q} \left(\sum_{k\in\mathbb{Z}^d} |\lambda_{\ell,k}|^p \right)^{q/p} \right)^{1/q}. \tag{4.24}$$

Proof. We prove the first statement. The proof for the second one is even simpler. Let

$$F(x,t) = \sum_{j,k} |\lambda_{j,k}| \chi_{j,k}(x) \cdot \chi_{[\beta^{-(j+1)},\beta^{-j}]}(t). \tag{4.25}$$

Discretizing the integral over t by $t \sim \beta^{-\ell}$, we obtain

$$\|F \mid Y\| = \left\| \left(\int_0^\infty t^{-sq} \left[\sup_w \frac{|F(x+w,t)|}{(1+|w|/t)^a} \right]^q \frac{dt}{t^{d+1}} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|$$
$$\asymp \left\| \left(\sum_{\ell\in\mathbb{Z}} \beta^{\ell(s+d/q)q} \int_{\beta^{-(\ell+1)}}^{\beta^{-\ell}} \left[\sup_w \frac{|F(x+w,t)|}{(1+\beta^\ell|w|)^a} \right]^q \frac{dt}{t} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|. \tag{4.26}$$

With $t \in [\beta^{-(\ell+1)}, \beta^{-\ell}]$, we observe

$$F(x,t) = \sum_k |\lambda_{\ell,k}| \chi_{\ell,k}(x) \tag{4.27}$$

and estimate

$$\|F \mid Y\| \le \left\| \left(\sum_{\ell\in\mathbb{Z}} \beta^{\ell(s+d/q)q} \sup_{w\in\mathbb{R}^d} \frac{1}{(1+\beta^\ell|w|)^a} \left[\sum_k |\lambda_{\ell,k}| \chi_{\ell,k}(x+w) \right]^q \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|. \tag{4.28}$$

In order to include also the situation $\min\{p,q\} \le 1$, we use the following trick. Obviously, we can rewrite and estimate (4.28) with $0 < r < 1$ in the following way:

$$\|F \mid Y\| \le \left\| \left(\sum_{\ell\in\mathbb{Z}} \left[\sum_k \beta^{\ell(s+d/q)r} |\lambda_{\ell,k}|^r \sup_{w\in\mathbb{R}^d} \frac{\chi_{\ell,k}(x+w)}{(1+\beta^\ell|w|)^{ar}} \right]^{q/r} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|. \tag{4.29}$$

The next observation is the useful estimate

$$\sup_w \frac{|\chi_{\ell,k}(x+w)|}{(1+\beta^\ell|w|)^{ar}} \lesssim \frac{1}{(1+\beta^\ell|x-k\beta^{-\ell}|)^{ar}} \lesssim \left(\chi_{\ell,k}(\cdot) * \frac{\beta^{\ell d}}{(1+\beta^\ell|\cdot|)^{ar}} \right)(x). \tag{4.30}$$

Indeed, the first estimate is obvious. Let us establish the second one

$$\begin{aligned}
\left(\chi_{\ell,k}(\cdot) * \frac{1}{(1+\beta^{\ell}|\cdot|)^{ar}}\right)(x) &= \int_{\substack{|y_i - k_i\beta^{-\ell}|\le\beta^{-\ell} \\ i=1,\dots,d}} \frac{1}{(1+\beta^{\ell}|x-y|)^{ar}}\,dy \\
&\gtrsim \int_{|y|\le c\beta^{-\ell}} \frac{1}{(1+\beta^{\ell}|x-k\beta^{-\ell}-y|)^{ar}}\,dy \\
&\gtrsim \int_{|y|\le c\beta^{-\ell}} \frac{1}{(1+\beta^{\ell}|x-k\beta^{-\ell}|+\beta^{\ell}|y|)^{ar}}\,dy \\
&\gtrsim \beta^{-\ell d}\int_0^1 \frac{u^{d-1}}{(1+\beta^{\ell}|x-k\beta^{-\ell}|+u)^{ar}}\,du \\
&\gtrsim \frac{\beta^{-\ell d}}{(1+\beta^{\ell}|x-k\beta^{-\ell}|)^{ar}}.
\end{aligned} \tag{4.31}$$

Note, that the functions

$$g_{\ell}(x) = \frac{\beta^{\ell d}}{(1+\beta^{\ell}|\cdot|)^{ar}} \tag{4.32}$$

belong to $L_1(\mathbb{R}^d)$ with uniformly bounded norm where we need $ar > d$. Putting (4.31) and (4.30) into (4.29), we obtain

$$\|F \mid Y\|^r \le \left\| \left(\sum_{\ell\in\mathbb{Z}} \left[g_{\ell} * \sum_{k\in\mathbb{Z}^d} \beta^{\ell(s+d/q)r} |\lambda_{\ell,k}|^r \chi_{\ell,k} \right)(x) \right]^{q/r} \right)^{r/q} \mid L_{p/r}\left(\mathbb{R}^d\right) \right\|. \tag{4.33}$$

Now we are in a position to use the majorant property of the Hardy/Littlewood maximal operator (see Section 2.2 and [25, Chapter 2]), which states that a convolution of a function f with an $L_1(\mathbb{R}^d)$-function (having norm one) can be estimated from above by the Hardy/Littlewood maximal function of f. We choose $r < \{\min\{p, q\}$ and apply Theorem 2.1 for the $L_{p/r}(\ell_{q/r})$ situation. This gives

$$\begin{aligned}
\|F \mid Y\|^r &\lesssim \left\| \left(\sum_{\ell\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}^d} \beta^{\ell(s+d/q)r} |\lambda_{\ell,k}|^r \chi_{\ell,k}(x) \right]^{q/r} \right)^{r/q} \mid L_{p/r}\left(\mathbb{R}^d\right) \right\| \\
&= \left\| \left(\sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \beta^{\ell(s+d/q)q} |\lambda_{\ell,k}|^q \chi_{\ell,k}(x) \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|^r
\end{aligned} \tag{4.34}$$

and finishes the upper estimate. Both conditions, $ar > d$ and $r < min\{p, q\}$, are compatible if $a > d/\min\{p, q\}$ is assumed at the beginning.

For the estimate from below, we go back to (4.26) and observe

$$\sup_{w} \frac{|F(x+w,t)|}{\left(1+\beta^{\ell}|w|\right)^{a}} \geq |F(x,t)|, \tag{4.35}$$

which results in

$$\|F \mid Y\| \gtrsim \left\| \left(\sum_{\ell\in\mathbb{Z}} \beta^{\ell(s+d/q)q} \int_{\beta^{-(\ell+1)}}^{\beta^{-\ell}} |F(x,t)|^{q} \frac{dt}{t} \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|. \tag{4.36}$$

A further use of (4.27) gives finally

$$\|F \mid Y\| \gtrsim \left\| \left(\sum_{\ell\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \beta^{\ell(s+d/q)q} |\lambda_{\ell,k}|^{q} \chi_{\ell,k}(x) \right)^{1/q} \mid L_p\left(\mathbb{R}^d\right) \right\|. \tag{4.37}$$

The proof is complete. □

4.4. Atomic Decompositions

The following theorem is a direct consequence of the abstract results in Theorems 3.11, and 3.13.

Theorem 4.9. *Let $1 \leq p, q \leq \infty$, $a > d/\min\{p,q\}$, and $s \in \mathbb{R}$. Let further $g \in \mathcal{S}_0(\mathbb{R}^d)$ be a radial function. Then there exist numbers $\alpha_0 > 0$ and $\beta_0 > 1$ such that for all $0 < \alpha \leq \alpha_0$ and $1 < \beta \leq \beta_0$ the family*

$$\{g_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d} = \left\{ T_{\alpha k \beta^j} \mathfrak{D}^{L_2}_{\beta^j} g \right\} \tag{4.38}$$

has the following properties.

(i) *$\{g_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ forms a Banach frame for $Co\dot{L}^{s}_{p,q}(\mathcal{G})$ and $Co\dot{P}^{s,a}_{p,q}(\mathcal{G})$, that is, we have a dual frame $\{e_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \subset \mathcal{S}_0(\mathbb{R}^d)$ with $f = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \langle g_{j,k}, f\rangle e_{j,k}$ and the norm equivalences*

$$\left\| f \mid Co\dot{L}^{s}_{p,q}(\mathcal{G}) \right\| \asymp \left\| \langle g_{j,k}, f\rangle \mid \left(\dot{L}^{s}_{p,q}(\mathcal{G}) \right)^{\sharp}(\alpha,\beta) \right\|, \quad f \in Co\dot{L}^{s}_{p,q}(\mathcal{G}) \tag{4.39}$$

as well as

$$\| f \mid Co\dot{P}^{s,a}_{p,q}(\mathcal{G}) \| \asymp \left\| \langle g_{j,k}, f\rangle \mid (\dot{P}^{s,a}_{p,q}(\mathcal{G}))^{\sharp}(\alpha,\beta) \right\|, \quad f \in Co\dot{P}^{s,a}_{p,q}(\mathcal{G}). \tag{4.40}$$

(ii) $\{g_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is an atomic decomposition, that is, for* $f \in Co\dot{P}^{s,a}_{p,q}(\mathcal{G})$, *we have a (not necessary unique) decomposition* $\sum_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \lambda_{j,k}(f) g_{j,k}$ *such that*

$$\left\| \{\lambda_{j,k}(f)\}_{j,k} \mid (\dot{P}^{s,a}_{p,q}(\mathcal{G}))^{\sharp}(\alpha,\beta) \right\| \lesssim \| f \mid Co\dot{P}^{s,a}_{p,q}(\mathcal{G}) \|. \tag{4.41}$$

Conversely, if $\{\lambda_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \in (\dot{P}^{s,a}_{p,q}(\mathcal{G}))^{\sharp}(\alpha,\beta)$, *then* $f = \sum_{j\in\mathbb{Z},k\in\mathbb{Z}^d} \lambda_{j,k} g_{j,k}$ *converges and belongs to* $Co\dot{P}^{s,a}_{p,q}(\mathcal{G})$ *and moreover,*

$$\| f \mid Co\dot{P}^{s,a}_{p,q}(\mathcal{G}) \| \lesssim \left\| \{\lambda_{j,k}\}_{j,k} \mid (\dot{P}^{s,a}_{p,q}(\mathcal{G}))^{\sharp}(\alpha,\beta) \right\| \tag{4.42}$$

(analogously for $Co\dot{L}^{s}_{p,q}(\mathcal{G})$*). Convergence is considered in the strong topology if the finite sequences are dense in* $(\dot{P}^{s,a}_{p,q}(\mathcal{G}))^{\sharp}(\alpha,\beta)$ *and in the weak***-topology otherwise.*

Remark 4.10. (i) Since the analyzing function or atom g can be chosen arbitrarily, we allow more flexibility here than in the results given i*n Frazier/Jawerth* [30] and *Triebel* [3, 29].

(ii) Instead of regular families of sampling points $(\alpha\beta^{-j}k, \beta^{-j})$ rather irregular families of points in $\mathcal{G}$ are allowed as long as they are distributed sufficiently dense, see Theorem 3.11.

4.5. Wavelet Frames

In the sequel, we consider wavelet bases on $\mathbb{R}^d$ in the sense of Lemma A.5 from Appendix A.2 in the appendices. We have given an orthonormal scaling function Ψ^0 and the associated wavelet Ψ^1 on $\mathbb{R}$ and consider the tensor products Ψ^c, $c \in E$. Our aim is to specify, that is, give sufficient conditions to Ψ^0, Ψ^1, such that (A.19) represents an unconditional basis in $B^s_{p,q}(\mathbb{R}^d)$ and $F^s_{p,q}(\mathbb{R}^d)$. We will apply the abstract Theorem 3.14.

In order to do so we need to have (3.17) for all functions Ψ^c. We will state certain smoothness, decay, and moment conditions to Ψ^1 and Ψ^0, see Definition A.1 in the appendix, to ensure this. Let us fix the neighborhood $V = [-1,1]^d \times (1/2, 1] \subset \mathcal{G}$ of $e \in \mathcal{G}$.

Proposition 4.11. *Let* $L \in \mathbb{N}$, $K > 0$, *and* Ψ^0 *be an orthogonal scaling function with associated wavelet* Ψ^1 *on* $\mathbb{R}$. *The function* Ψ^0 *is supposed to satisfy* (D) *and* (S_K) *and* Ψ_1 *is supposed to satisfy* (D), (S_K), *and* (M_{L-1}). *For* $r_1, r_2 \in \mathbb{R}$, *the weight* $w(x,t)$ *is given by*

$$w(x,t) = (1+|x|)^{v}\left(t^{r_2} + t^{-r_1}\right), \quad (x,t) \in \mathcal{G}. \tag{4.43}$$

If now

$$r_1 < \min\{L, K\} - \frac{d}{2}, \qquad r_2 < \min\{L, K\} + \frac{d}{2} - v, \tag{4.44}$$

then we have

$$\int_{\mathbb{R}^d} \int_0^{\infty} \sup_{(y,s)\in(x,t)V} \left| \langle \pi(y,s)\Psi^c, \Psi^c \rangle \right| w(x,t) \frac{dt}{t^{d+1}} dx < \infty. \tag{4.45}$$

Proof. With Lemma A.3, we obtain for $W_{\Psi^1}\Psi^1$ the following estimates:

$$\left|\left(W_{\Psi^1}\Psi^1\right)(s,t)\right| \lesssim \frac{t^{\min\{L,K\}+1/2}}{(1+t)^{2\min\{L,K\}+1}} \cdot \frac{1}{(1+|s|/(1+t))^N}. \tag{4.46}$$

And in addition

$$\left|\left(W_{\Psi^i}\Psi^i\right)(s,t)\right| \lesssim \frac{t^{1/2}}{(t+1)} \frac{1}{(1+|s|/(1+t))^N}, \qquad i = 1,2. \tag{4.47}$$

Hence, for any $c \in E$, the tensor product structure gives (assume without restriction that $c_d = 1$)

$$|W_{\Psi^c}\Psi^c(x,t)| \lesssim \frac{t^{\min\{L,K\}}}{(1+t)^{2\min\{L,K\}}} \cdot \frac{t^{d/2}}{(1+t)^d} \prod_{i=1}^{d} \frac{1}{(1+|x_i|/(1+t))^N}. \tag{4.48}$$

The expression $\sup_{(y,s)\in(x,t)V}|W_{\Psi^c}\Psi^c(y,s)|$ can be estimated similar

$$\begin{aligned}
\sup_{(y,s)\in(x,t)V} \left|W_{\Psi^c}\Psi^c(y,s)\right| &= \sup_{\substack{|y_i-x_i|\le t\\ t/2\le s\le t}} \left|W_{\Psi^c}\Psi^c(y,s)\right| \\
&\lesssim \frac{t^{\min\{L,K\}}}{(1+t)^{2\min\{L,K\}}} \frac{t^{d/2}}{(1+t)^d} \prod_{i=1}^{d} \frac{1}{(1+|x_i|/(1+t))^N} \\
&\lesssim \frac{t^{\min\{L,K\}}}{(1+t)^{2\min\{L,K\}}} \frac{t^{d/2}}{(1+t)^d} \frac{1}{(1+|x|/(1+t))^N}.
\end{aligned} \tag{4.49}$$

Fubini's theorem and a change of variable yields

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_0^\infty \sup_{(y,s)\in(x,t)V} \left|\langle \pi(y,s)\Psi^c, \Psi^c\rangle\right| w(x,t) \frac{dt}{t^{d+1}}\, dx &\lesssim \int_0^\infty \frac{t^{\min\{L,K\}}}{(1+t)^{2\min\{L,K\}}} t^{d/2} \\
&\quad \cdot \left(t^{r_2} + t^{-r_1}\right) \frac{dt}{t^{d+1}}.
\end{aligned} \tag{4.50}$$

Finally it is easy to see that the latter is finite if the conditions in (4.44) are valid. This proves Proposition 4.11. □

Theorem 4.12. *Let $L \in \mathbb{N}$, $K > 0$, and Ψ^0 be an orthogonal scaling function with associated wavelet Ψ^1 on $\mathbb{R}$. The function Ψ^0 is supposed to satisfy (D) and (S_K), and Ψ_1 is supposed to satisfy (D), (S_K) and (M_{L-1}).*

(a) *If $1 \le p, q \le \infty$ and*

$$-\min\{L,K\} + \frac{d}{p} < s < \min\{L,K\} - d\left(1 - \frac{1}{p}\right), \tag{4.51}$$

then (A.19) *is a Banach frame for $\dot{B}^s_{p,q}(\mathbb{R}^d)$ in the sense of* (3.22).

(b) *If* $1 \leq p, q < \infty$ *and*

$$-\min\{L,K\} + 2d\max\left\{\frac{1}{p},\frac{1}{q}\right\} < s < \min\{L,K\} - d\max\left\{\frac{1}{p},\frac{1}{q},1-\frac{1}{p}\right\}, \tag{4.52}$$

then (A.19) *is a Banach frame for* $\dot{F}^s_{p,q}(\mathbb{R}^d)$ *in the sense of* (3.22).

Proof. Let us prove (a). First of all, we apply Theorem 4.5/(i). Afterwards, we use Proposition 4.2 in order to estimate the weight $w_Y(y,t)$ for $Y = \dot{L}^{s+d/2-d/q}_{p,q}(\mathcal{G})$. We obtain

$$\begin{aligned} w_Y(x,t) &= \max\left\{t^{d(1/p-1/2)-s}, t^{s-d(1/p-1/2)}, t^{s+d/2}, t^{-s+d/2}\right\} \\ &\leq \begin{cases} t^{-r_1}: & 0 < t < 1, \\ t^{r_2}: & t \geq 1. \end{cases} \end{aligned} \tag{4.53}$$

Let us distinguish the cases, $s \geq 0$ and $s < 0$. In the first case we can put $r_1 = \max\{s - d(1/p - 1/2), -s + d(1/p - 1/2), s - d/2\}$, $r_2 = \max\{s + d/2, -s + d(1/p - 1/2)\}$, and $v = 0$. Now we apply first Proposition 4.11. This gives the condition

$$0 \leq s < \min\{L,K\} - d(1 - 1/p). \tag{4.54}$$

In the second case we put $r_1 = \max\{s - d(1/p - 1/2), -s + d(1/p - 1/2), -s - d/2\}$, $r_2 = \max\{-s + d/2, -s + d(1/p - 1/2)\}$ and $v = 0$. With Proposition 4.11, we obtain the condition

$$-\min\{L,K\} + d/p < s < 0. \tag{4.55}$$

Finally (4.54), (4.55), and Theorem 3.14 yield (a).

Step 2. We prove (b). We apply Theorem 4.5/(iii) and afterwards Proposition 4.2 and obtain for $Y = \dot{P}^{s+d/2-d/q,a}_{p,q}(\mathcal{G})$

$$\begin{aligned} w_Y(x,t) &= \max\Big\{t^{d(1/p-1/2)-s}, t^{s-d(1/p-1/2)}, \\ &\qquad t^{s+d/2}\max\{1,t^{-a}\}(1+|x|^a), t^{-s+d/2}\max\{t^{-a},t^a\}(1+|x|)^a\Big\} \\ &\leq (1+|x|)^a \begin{cases} t^{-r_1}: & 0 < t < 1, \\ t^{r_2}: & t \geq 1. \end{cases} \end{aligned} \tag{4.56}$$

First, we consider the case $s \geq 0$. We can put $r_1 = \max\{s + a - d/2, s + d/2 - d/p\}$, $r_2 = \max\{s + d/2, -s + d/2 + a\}$, and $v = a$. Proposition 4.11 gives the condition

$$0 \leq s < \min\{L, K\} - \max\left\{a, d\left(1 - \frac{1}{p}\right)\right\} \tag{4.57}$$

which can be rewritten to

$$0 \leq s < \min\{L, K\} - d\max\left\{\frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{p}\right\} \tag{4.58}$$

since a can be chosen arbitrarily greater than $d\max\{1/p, 1/q\}$. This gives the upper bound in (b). Now we consider $s < 0$. We put $r_1 = \max\{-s + d/p - d/2, s + d/2 - d/p\}$, $r_2 = -s + d/2 + a$. This yields

$$-\min\{L, K\} + 2a < s < 0 \tag{4.59}$$

and can be rewritten to

$$-\min\{L, K\} + 2d\max\{1/p, 1/q\} < s < 0. \tag{4.60}$$

This yields the lower bound in (b) and we are done. □

The following corollary is a direct consequence of Theorem 4.12 and the facts in Appendix A.2.

Corollary 4.13. *Let $m > 0$ and $(\Psi_0, \Psi_1) = (\varphi_m, \psi_m)$ the spline wavelet system of order m. Then*

(a) If $1 \leq p, q \leq \infty$ and $-m + 1 + \dfrac{d}{p} < s < m - 1 - d\left(1 - \dfrac{1}{p}\right)$,

then (A.19) *is a Banach frame for $\dot{B}^s_{p,q}(\mathbb{R}^d)$ in the sense of* (3.22)*;*

(b) If $1 \leq p, q \leq \infty$ and

$$-m + 1 + 2d\max\left\{\frac{1}{p}, \frac{1}{q}\right\} < s < m - 1 - d\max\left\{\frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{p}\right\}, \tag{4.61}$$

then (A.19) *is a Banach frame for $\dot{F}^s_{p,q}(\mathbb{R}^d)$ in the sense of* (3.22).

Remark 4.14. The (optimal) smoothness conditions in [31] are slightly weaker than (a) in case $d = 1$. However, compared to the approach of Triebel [29, 32], we admit some more degree of freedom. The wavelet or atom does not have to be compactly supported. Additionally, in case $d = 1$, we do not need that $\psi \in C^u(\mathbb{R})$ where $u > s$. Indeed, the conditions in (a) and (b) are slightly weaker.

Remark 4.15. More examples can be obtained by using compactly supported Daubechies' wavelets of order N or Meyer's wavelets. Based on the underlying abstract result in Theorem 3.14 even biorthogonal wavelet systems which provide sufficiently high smoothness and vanishing moments are suitable for this issue.

Appendix

A. Wavelets

A.1. The Continuous Wavelet Transform

The vector g is said to be the analyzing vector for a function $f \in L_2(\mathbb{R}^d)$. The continuous wavelet transform $W_g f$ is then defined through

$$W_g f(x,t) = \left\langle T_x \mathfrak{D}_t^{L_2} g, f \right\rangle, \quad x \in \mathbb{R}^d,\ t > 0, \tag{A.1}$$

where the bracket $\langle \cdot, \cdot \rangle$ denotes the inner product in $L_2(\mathbb{R}^d)$. We call g an admissible wavelet if

$$c_g := \int_{\mathbb{R}^d} \frac{\left|\mathcal{F} g(\xi)\right|^2}{|\xi|^d} d\xi < \infty. \tag{A.2}$$

If this is the case, then the family $\{T_x \mathfrak{D}_t^{L_2} g\}_{t>0, x\in\mathbb{R}^d}$ represents a tight continuous frame in $L_2(\mathbb{R})$. In particular, this means that the above family is dense in $L_2(\mathbb{R}^d)$. For a proof we refer to Theorem 1.5.1 in [28].

Let us now specify the conditions (M_L), (D), and (S_K), which we intend to impose on functions $\Phi, \Psi \in L_2(\mathbb{R}^d)$ in order to obtain a good decay of the continuous wavelet transform $|W_\Psi \Phi(x,t)|$.

Definition A.1. Let $L+1 \in \mathbb{N}_0$, $K > 0$, and fix the conditions (D), (M_L), and (S_K) for a function $\Psi \in L_2(\mathbb{R}^d)$.

(D) For every $N \in \mathbb{N}$, there exists a constant c_N such that

$$|\Psi(x)| \le \frac{c_N}{(1+|x|)^N}. \tag{A.3}$$

(M_L) We have vanishing moments

$$D^{\overline{\alpha}} \mathcal{F} \Psi(0) = 0 \tag{A.4}$$

for all $|\overline{\alpha}|_1 \le L$.

(S_K) The function

$$(1+|\xi|)^K \left| D^{\overline{\alpha}} \mathcal{F} \Psi(\xi) \right| \tag{A.5}$$

belongs to $L_1(\mathbb{R}^d)$ for every multi-index $\overline{\alpha} \in \mathbb{N}_0^d$.

Remark A.2. If a function $g \in L_2(\mathbb{R}^d)$ satisfies (S_K) for some $K > 0$ then by well-known properties of the Fourier transform we have $g \in C^{\lfloor K \rfloor}(\mathbb{R}^d)$.

The following lemma provides a useful decay result for the continuous wavelet transform under certain smoothness, decay, and moment conditions, see also [4, 30, 33] for similar results in a different language. It represents a continuation of [4, Lemma 1] where one deals with $\mathcal{S}(\mathbb{R}^d)$ -functions.

Lemma A.3. *Let $L \in \mathbb{N}_0$, $K > 0$, and $\Phi, \Psi, \Phi_0 \in L_2(\mathbb{R}^d)$.*

(i) *Let Φ satisfy (D), (M_{L-1}), and let Φ_0 satisfy (D), (S_K). Then for every $N \in \mathbb{N}$ there exists a constant C_N such that the estimate*

$$|(W_\Phi \Phi_0)(x,t)| \le C_N \frac{t^{\min\{L,K\}+d/2}}{(1+|x|)^N} \tag{A.6}$$

holds true for $x \in \mathbb{R}^d$ and $0 < t < 1$.

(ii) *Let Φ, Ψ satisfy (D), (M_{L-1}) and (S_K). For every $N \in \mathbb{N}$ there exists a constant C_N such that the estimate*

$$|(W_\Phi \Psi)(x,t)| \le C_N \frac{t^{\min\{L,K\}+d/2}}{(1+t)^{2\min\{L,K\}+d}} \left(1 + \frac{|x|}{1+t}\right)^{-N} \tag{A.7}$$

holds true for $x \in \mathbb{R}^d$ and $0 < t < \infty$.

Proof.

Step 1. Let us prove (i). We follow the proof of Lemma 1 in [4]. This reference deals with $\mathcal{S}(\mathbb{R}^d)$-functions, which makes the situation much more easy. We argue as follows. Clearly,

$$|(W_\Phi \Phi_0)(x,t)| = t^{d/2} |[(\mathfrak{D}_t \Phi) * \Phi_0](x)|. \tag{A.8}$$

Fix $0 < t < 1$. Obviously, the convolution $(\mathfrak{D}_t\Phi) * \Phi_0$ satisfies (D). By well-known properties of the Fourier transform, the derivative $D^{\overline{\alpha}}\mathcal{F}((\mathfrak{D}_t\Phi) * \Phi_0))(\xi)$ exists for every multi-index $\overline{\alpha} \in \mathbb{N}_0^d$. For fixed $\overline{\alpha}$, we estimate by using Leibniz' formula

$$\begin{aligned} \left| D^{\overline{\alpha}} \mathcal{F}((\mathfrak{D}_t\Phi) * \Phi_0))(\xi) \right| &= \left| D^{\overline{\alpha}} (\mathcal{F}\Phi(t\xi) \cdot \mathcal{F}\Phi_0(\xi)) \right| \\ &\le c_{\overline{\alpha}} \sum_{\overline{\beta} \le \overline{\alpha}} t^{|\overline{\beta}|_1} \left| D^{\overline{\beta}} \mathcal{F}\Phi(t\xi) \cdot D^{\overline{\alpha}-\overline{\beta}} \mathcal{F}\Phi_0(\xi) \right| \\ &\le c'_{\overline{\alpha}} t^L (1+|\xi|)^L \sum_{\overline{\beta} \le \overline{\alpha}} \left| D^{\overline{\alpha}-\overline{\beta}} \mathcal{F}\Phi_0(\xi) \right|. \end{aligned} \tag{A.9}$$

In the last step we used property (M_{L-1}). Assuming $K \geq L$ and exploiting (S_K), we obtain that the left-hand side of (A.9) belongs to $L_1(\mathbb{R}^d)$ and

$$\left\| D^{\overline{\alpha}} \mathcal{F}((\mathfrak{D}_t \Phi) * \Phi_0))(\xi) \mid L_1\left(\mathbb{R}^d\right) \right\| \leq c''_{\overline{\alpha}} t^L. \tag{A.10}$$

We proceed as follows:

$$\begin{aligned} \max_{|\overline{\alpha}|_1 \leq N+1} \left\| D^{\overline{\alpha}} \mathcal{F}((\mathfrak{D}_t \Phi) * \Phi_0))(\xi) \mid L_1\left(\mathbb{R}^d\right) \right\| &\geq \max_{|\overline{\alpha}|_1 \leq N+1} \left\| \mathcal{F}^{-1} \left[D^{\overline{\alpha}} \mathcal{F}((\mathfrak{D}_t \Phi) * \Phi_0) \right] \mid L_\infty\left(\mathbb{R}^d\right) \right\| \\ &\geq c_N \left\| (1 + |x|)^N [(\mathfrak{D}_t \Phi) * \Phi_0](x) \mid L_\infty\left(\mathbb{R}^d\right) \right\|. \end{aligned} \tag{A.11}$$

This estimate together with (A.8) and (A.10) yields (A.6).

Let us finally assume $K < L$ and return to (A.9). Clearly, the resulting inequality keeps valid if we replace the exponent L by $L' \in \mathbb{N}_0$ with $L' \leq L$. It is even possible to extend (A.9) to every $0 \leq L'' < L$ by the following argument. Let $L'' \notin \mathbb{N}$. We have on the one hand

$$\text{LHS (A.9)} \leq c'_{\overline{\alpha}} t^{\lfloor L'' \rfloor} (1 + |\xi|)^{\lfloor L'' \rfloor} G(\xi) \tag{A.12}$$

and on the other hand

$$\text{LHS (A.9)} \leq c'_{\overline{\alpha}} t^{\lfloor L'' \rfloor + 1} (1 + |\xi|)^{\lfloor L'' \rfloor + 1} G(\xi), \tag{A.13}$$

where $G(\xi) = \sum_{\overline{\beta} \leq \overline{\alpha}} |D^{\overline{\alpha} - \overline{\beta}} \mathcal{F} \Phi_0(\xi)|$. Choosing $0 < \theta < 1$ such that $L'' = (1 - \theta) \lfloor L'' \rfloor + \theta(\lfloor L'' \rfloor + 1)$, we obtain by a kind of interpolation argument

$$\begin{aligned} \text{LHS(A.9)} &= \text{LHS(A.9)}^{\,1-\theta}\ \text{LHS(A.9)}^{\theta} \\ &\leq c'_{\overline{\alpha}} t^{L''} (1 + |\xi|)^{L''} G(\xi). \end{aligned} \tag{A.14}$$

In particular, we obtain instead of (A.9)

$$\left| D^{\overline{\alpha}} \mathcal{F}\left((\mathfrak{D}_t \Phi) * \Phi_0) \right)(\xi) \right| \leq c'_{\overline{\alpha}} t^K (1 + |\xi|)^K \sum_{\overline{\beta} \leq \overline{\alpha}} \left| D^{\overline{\alpha} - \overline{\beta}} \mathcal{F} \Phi_0(\xi) \right|, \quad \xi \in \mathbb{R}^d. \tag{A.15}$$

We exploit property (S_K) for Φ_0 and proceed analogously as above. This proves (A.6).

Step 2. The estimate in (ii) is an immediate consequence of (A.6) and the fact

$$(W_\Phi \Psi)(x, t) = (W_\Psi \Phi)\left(\frac{-x}{t}, \frac{1}{t} \right). \tag{A.16}$$

This completes the proof. □

Corollary A.4. *Let Φ, Ψ belong to the Schwartz space $\mathcal{S}_0(\mathbb{R}^d)$. By Lemma A.3/(ii) for every $L, N \in \mathbb{N}$ there is a constant $C_{L,N} > 0$ such that*

$$|(W_\Phi \Psi)(x,t)| \le C_{L,N} \frac{t^{L+d/2}}{(1+t)^{2L+d}} \left(1 + \frac{|x|}{1+t}\right)^{-N}, \qquad x \in \mathbb{R}^d,\ t > 0. \tag{A.17}$$

Additionally, we obtain for $\Phi \in \mathcal{S}_0(\mathbb{R}^d)$ and $\Phi_0 \in \mathcal{S}(\mathbb{R}^d)$ such that

$$|(W_\Phi \Phi_0)(x,t)| \le C_{L,N} \frac{t^{L+d/2}}{(1+|x|)^N}, \qquad x \in \mathbb{R}^d,\ 0 < t < 1. \tag{A.18}$$

A.2. Orthonormal Wavelet Bases

The following Lemma is proved in Wojtaszczyk [34, Section 5.1].

Lemma A.5. *Suppose we have a multiresolution analysis in $L_2(\mathbb{R})$ with scaling functions $\Psi^0(t)$ and associated wavelets $\Psi^1(t)$. Let $E = \{0,1\}^d \setminus \{(0,\ldots,0)\}$. For $c = (c_1,\ldots,c_d) \in E$, let $\Psi^c = \otimes_{j=1}^d \Psi^{c_j}$. Then the system*

$$\left\{2^{jd/2}\Psi^c\left(2^j x - k\right)\right\}_{c\in E, j\in\mathbb{Z}, k\in\mathbb{Z}^d} \tag{A.19}$$

is an orthonormal basis in $L_2(\mathbb{R}^d)$.

Spline Wavelets

As a main example, we will consider the spline wavelet system. The normalized cardinal B-spline of order $m+1$ is given by

$$\mathcal{N}_{m+1}(x) := \mathcal{N}_m * \mathcal{X}(x), \quad x \in \mathbb{R},\ m \in \mathbb{N}, \tag{A.20}$$

beginning with $\mathcal{N}_1 = \mathcal{X}$, the characteristic function of the interval $(0,1)$. By

$$\varphi_m(x) := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left[\frac{\mathcal{F}\mathcal{N}_m(\xi)}{\left(\sum_{k=-\infty}^{\infty} |\mathcal{F}\mathcal{N}_m(\xi + 2\pi k)|^2\right)^{1/2}}\right](x), \qquad x \in \mathbb{R}, \tag{A.21}$$

we obtain an orthonormal scaling function, which is again a spline of order m. Finally, by

$$\varphi_m(x) := \sum_{k=-\infty}^{\infty} \left\langle \varphi_m\left(\frac{t}{2}\right), \varphi_m(t-k) \right\rangle (-1)^k\ \varphi_m(2x + k + 1) \tag{A.22}$$

the generator of an orthonormal wavelet system is defined. For $m = 1$, it is easily checked that $-\psi_1(x-1)$ is the Haar wavelet. In general, these functions ψ_m have the following properties:

(i) ψ_m restricted to intervals $[k/2, (k+1)/2]$, $k \in \mathbb{Z}$, is a polynomial of degree at most $m-1$.

(ii) $\psi_m \in C^{m-2}(\mathbb{R})$ if $m \geq 2$.

(iii) $\psi_m^{(m-2)}$ is uniformly Lipschitz continuous on $\mathbb{R}$ if $m \geq 2$.

(iv) The function ψ_m satisfies a moment condition of order $m-1$, that is,

$$\int_{-\infty}^{\infty} x^{\ell} \psi_m(x) dx = 0, \qquad \ell = 0, 1, \ldots, m-1. \tag{A.23}$$

In particular, ψ_m satisfies (D), (M_L) for $0 < L \leq m$ and (S_K) for $K < m-1$.

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