Optimal Cubature in Besov Spaces with Dominating Mixed Smoothness on the Unit Square

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Dedicated to J.F. Traub and G.W. Wasilkowski
on the occasion of their 80th and 60th birthdays

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Abstract

We prove new optimal bounds for the error of numerical integration in bivariate Besov spaces with dominating mixed order r. The results essentially improve on the so far best known upper bound achieved by using cubature formulas taking points from a sparse grid. Motivated by Hinrichs' observation that Hammersley type point sets provide optimal discrepancy estimates in Besov spaces with mixed smoothness on the unit square, we directly study quasi-Monte Carlo integration on such point sets. As the main tool we prove the representation of a bivariate periodic function in a piecewise linear tensor Faber basis. This allows for optimal worst case estimates of the QMC integration error with respect to Besov spaces with dominating mixed smoothness up to order $r < 2$. The results in this paper are a first step towards sharp results for spaces with arbitrarily large mixed order on the d-dimensional unit cube. In fact, in contrast to Fibonacci lattice rules, which are also practicable in this context, the QMC methods used in this paper have a proper counterpart in d dimensions.

Key Words: Cubature formula, Mixed Besov spaces, Faber basis, Hammersley point sets.

AMS Subject classification: 41A25; 41A63; 42B99; 46E35; 65D05; 65D07.

1 Introduction

Optimal cubature formulas play an important role for the treatment of multivariate functions in practice. Many real world problems, for instance, from finance, quantum physics, meteorology, etc., require the computation of integrals of d-variate functions where d may be very large. This can almost never be done analytically since often the available information of the signal or function f is highly incomplete. A general cubature formula $\Lambda_N(X, f) = \sum_{x_i \in X} \lambda_i f(x_i)$ aims at computing a good approximation $\Lambda_N(f)$ of the integral $I(f) = \int_Q f(x)\,dx$ within a

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reasonable computing time (assume $|Q|=1$). The discrete set X of "knots" and the vector $\Lambda=(\lambda_1,...,\lambda_N)$ of "weights" are fixed in advance for a class F of functions f. The condition $\sum_{i=1}^N \lambda_i = 1$ often occurs since then constant functions are integrated exactly. A special case is given by formulas with constant weight vector $\Lambda=(1/N,...,1/N)$ which are commonly referred to as quasi-Monte Carlo (QMC) methods and are denoted by $I_N(X,f)$. The optimal worst case error with respect to the class F is given by

$$\mathrm{Int}_N(F) := \inf_{X,\Lambda} \sup_{\|f|F\|\le 1} |I(f)-\Lambda_N(X,f)|\,.$$

In this paper we aim at sharp estimates for the asymptotic of the quantity $\mathrm{Int}_N(F)$ as N goes to infinity for a class of functions F with bounded mixed derivatives or differences, so-called Besov-Nikolskij classes $S^r_{p,q}B(\mathbb{T}^2)$ with $1\le p,q\le\infty$ and $r>1/p$, see Definition 2.7 below. Spaces of this type have a certain history in the former Soviet Union, see [1, 16] and the references therein, and continued attracting significant interest until recently [26, 24]. The by now classical research topic of numerically integrating such functions goes back to the work of Korobov [10], Hlawka [9], and Bakhvalov [2] in the 1960s to mention just a few. In contrast to the quadrature of univariate functions, where equidistant point grids lead to optimal formulas, the multivariate problem is much more involved. In fact, the choice of proper sets $X\subset Q_d$ of integration knots in a multidimensional domain, say $Q_d=[0,1]^d$, is connected with deep problems in number theory, already for $d=2$.

Recently, Triebel [21, 22] and, independently, Dinh [4] brought up the framework of tensor Faber bases for functions of the above type. The main feature is the fact that the basis coefficients are linear combinations of function values. The corresponding series expansion is thus extremely useful for sampling and integration issues. In [21, Chapt. 5] cubature formulas with non-equal weights and knots from a sparse grid were used to obtain the relation

$$N^{-r}(\log N)^{1-1/q} \lesssim \mathrm{Int}_N(S^r_{p,q}B(Q_2)) \lesssim N^{-r}(\log N)^{r+1-1/q} \tag{1.1}$$

if $1\le p,q\le\infty$ and $1/p<r<1+1/p$. In fact, $S^r_{p,q}B(Q_2)$ is the canonical restriction of $S^r_{p,q}B(\mathbb{R}^2)$, see [16, Chapt. 2], to the unit cube Q_2. Note, that there is a gap between upper and lower bound in (1.1). This gap has recently been closed for a subclass of $S^r_{p,q}B(Q_2)$ with $1/p<r\le 1$, namely those functions $S^r_{p,q}B(Q_2)^\urcorner$ with vanishing boundary values on the upper and right boundary line, by showing that the lower bound in (1.1) is sharp. It turned out that there is an intimate relation between optimal integration and the discrepancy [25] of discrete point sets, the Hlawka-Zaremba duality [8]. Triebel's adaption [21, Thm. 6.11] to Besov spaces of mixed smoothness together with Hinrichs' [7] sharp results on the discrepancy of Hammersley points imply the optimality of the associated QMC method in spaces $S^r_{p,q}B(Q_2)^\urcorner$ with $1/p<r\le 1$.

In this paper we go even further and provide sharp results for the original classes $S^r_{p,q}B(Q_2)$ with the less restrictive smoothness conditions, namely $1/p<r\le 2$. In a first step we mainly consider periodic bivariate functions on $\mathbb{T}^2:=\mathbb{R}^2/\mathbb{Z}^2$. This means that we deal with bivariate functions being 1-periodic in each component. Temlyakov [18] studied optimal cubature in the related Sobolev spaces $S^r_pW(\mathbb{T}^2)$ and Nikolskij spaces $S^r_{p,\infty}B(\mathbb{T}^2)$ by using QMC methods based on Fibonacci lattice rules. This highly nontrivial idea goes back to Bakhvalov [2] and indicates once more the deep connection to number theoretical problems. If $1<p<\infty$ and $r>\max\{1/p,1/2\}$ this approach yields the sharp result

$$\mathrm{Int}_N(S^r_pW(\mathbb{T}^2)) \asymp N^{-r}(\log N)^{1/2}\ , \quad N\in\mathbb{N}\,. \tag{1.2}$$

Moreover, if $1 \leq p \leq \infty$ and $r > 1/p$ then

$$\mathrm{Int}_N(S^r_{p,\infty}B(\mathbb{T}^2)) \asymp N^{-r}\log N\;, \quad N \in \mathbb{N}\,. \tag{1.3}$$

In a forthcoming paper by Dinh and the author Fibonacci QMC methods will be used to integrate functions from spaces $S^r_{p,q}B(\mathbb{T}^2)$. The results are sharp, as we will show, and match with the ones given in this paper.

Unfortunately, Fibonacci lattice rules do so far not have a proper counterpart in arbitrary dimensions d, however working for all $r > 1/p$. Aiming for both the parameters r and d arbitrary, we follow Hinrichs' observation [7] and study QMC methods on Hammersley type point sets

$$\mathcal{H}_n = \Big\{\Big(\frac{t_n}{2}+\frac{t_{n-1}}{2^2}+...+\frac{t_1}{2^n},\frac{s_1}{2}+\frac{s_2}{2^2}+...+\frac{s_n}{2^n}\Big) : t_i \in \{0,1\}, i=1,...,n\Big\} \tag{1.4}$$

in $d = 2$ as a first step. Here, $s_i = t_i$ or $s_i = 1 - t_i$ are chosen depending on i. In particular, every set $\mathcal{H}_n$ contains $N = 2^n$ points. The original van der Corput point set [25] is given by putting $s_i = t_i$ for all $i = 1, ..., n$. Clearly, there is a whole zoo of Hammersley type point sets which one might consider. Due to the periodicity of the functions under consideration our approach works for every point set of the above type. As a main result we obtain the relation

$$\mathrm{Int}_N(S^r_{p,q}B(\mathbb{T}^2)) \lesssim N^{-r}(\log N)^{1-1/q} \tag{1.5}$$

if $1 \leq p, q \leq \infty$ and $1/p < r < 2$. This is complemented by the sharp lower bound

$$\mathrm{Int}_N(S^r_{p,q}B(\mathbb{T}^2)) \gtrsim N^{-r}(\log N)^{1-1/q}$$

in case $1/p < r < 1+1/p$. In a second step we deal with the non-periodic situation $S^r_{p,q}B(Q_2)$. We still use (arbitrary) Hammersley points in the interior of Q_2. Nevertheless it seems to be necessary to use additional function values on the boundary of Q_2 with non-equal associated weights. The optimal non-periodic cubature formula presented here is not longer a QMC rule.

What concerns the d-variate problem we can easily obtain a cubature formula with knots from a sparse grids (and non-equal weights) by simply integrating the approximant in [17, Cor. 3]. This results in the one-sided relation

$$\mathrm{Int}_N(S^r_{p,q}B(\mathbb{T}^d)) \lesssim N^{-r}(\log N)^{(d-1)(r+1-1/q)} \tag{1.6}$$

which, compared with (1.5), apparently does not reflect the correct behavior of $\mathrm{Int}_N(S^r_{p,q}B(\mathbb{T}^2))$. Moreover, we will show in the subsequent paper, that any cubature formula using knots from a sparse grid produces a worst case error at least as big as the right hand side of (1.6). There is the strong conjecture that our results can be extended to the multivariate situation by using a d-dimensional variant of the Hammersley points, the explicit construction of Chen and Skriganov [3] which achieve the best possible asymptotic behavior for the L_p-discrepancy on $[0,1]^d$. We expect the power $(d-1)(1-1/q)$ in the logarithm in (1.5) for the same range of r. In fact, it has been recently observed by Markhasin [13, 12, 11] that Hinrichs' results have a direct counterpart in d dimensions. With an eye on the curse of dimensionality, it is even more interesting to consider the case $q = 1$ in the multivariate situation.

We will present a direct analysis here and do not use the connection to the discrepancy function established by the mentioned Hlawka-Zaremba type duality. Instead we prove a periodic tensor Faber basis representation in order to decompose the function of interest. We then shift the integration problem to the building blocks which are comparably simple tensor products of univariate hat functions. Due to the use of the piecewise linear Faber tensor basis we can not expect to get beyond $1/p < r < 2$. However, this restriction is technical and does not seem to be natural. Indeed, based on the results in this paper, the author and Dinh currently work on the problem whether the Faber basis can be replaced by the B-spline quasi-interpolant representation [4] in order to get rid of the restriction $r < 2$.

The paper is organized as follows. After briefly introducing the setting of periodic function spaces with dominating mixed smoothness in Section 2 we state a characterization by iterated differences, Lemma 2.9, suitable for our purpose. In Section 3 we define a tensor Faber system on the 2-torus and prove that it is a basis in $S^r_{p,q}B(\mathbb{T}^2)$ for $1/p < r < 1 + 1/p$. Moreover, in Proposition 3.4 we show even more, namely the boundedness of the coefficient mapping for $1/p < r < 2$ which is not the case for the opposite direction, Proposition 3.6. The main tools for the proof of the periodic Faber basis representation are rather classical, namely a multivariate Marcinkiewicz-Zygmund inequality, Lemma 2.3, as well as a periodic Bernstein-Nikolskij inequality, Lemma 2.1, and the characterization by differences, Lemma 2.9. Section 4 contains our main results for the periodic spaces, Theorem 4.7, which implies the upper bound in (1.5). Finally, Sections 5 and 6 deal with the non-periodic problem. The main result, Theorem 6.3 states the direct counterpart of (1.5) for the spaces $S^r_{p,q}B(Q_2)$. Applying the Hlawka-Zaremba duality "backwards" it has consequences for optimal discrepancy (discrepancy numbers) in spaces $S^r_{p,q}B(\mathbb{T}^2)$ with negative smoothness r, see Theorem 6.7, a problem recently pointed out in [7].

Notation. As usual $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. With $\mathbb{T}$ we denote the torus represented by the interval $[0,1]$. For $0 < p \leq \infty$ and $x \in \mathbb{R}^d$ we denote $|x|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ with the usual modification in the case $p = \infty$. If X and Y are two (quasi-)normed spaces, the norm of an element x in X will be denoted by $\|x|X\|$. The symbol $X \hookrightarrow Y$ indicates that the identity operator is continuous.

2 Periodic Besov spaces with dominating mixed smoothness

2.1 Preliminaries

Let $\mathbb{T}^2$ denote the 2-torus, represented in the Euclidean space $\mathbb{R}^2$ by the cube $\mathbb{T}^2 = [0,1]^2$, where opposite points are identified. That means $x, y \in \mathbb{T}^2$ are identified if and only if $x - y = k$, where $k = (k_1, k_2) \in \mathbb{Z}^2$. In particular one has $f(x) = f(y)$ if $x - y = k$ and $f \in D(\mathbb{T}^2)$, where $D(\mathbb{T}^2)$ denotes the collection of all complex-valued infinitely differentiable functions on $\mathbb{T}^2$. Its topology is generated by the family of norms

$$\|\varphi\|_N = \sum_{|\alpha|_1 \leq N} \sup_{x \in \mathbb{T}^2} \left|D^\alpha \varphi(x)\right| \quad , \quad N \in \mathbb{N}_0 \,.$$

A linear functional $f : D(\mathbb{T}^2) \to \mathbb{C}$ belongs to $D'(\mathbb{T}^d)$, if and only if there is a constant $c_N > 0$ such that $|f(\varphi)| \leq c_N \|\varphi\|_N$ holds for all $\varphi \in D(\mathbb{T}^d)$ and for some natural number N. We endow $D'(\mathbb{T}^d)$ with the weak topology. Precisely, $\{f_n\}_{n=1}^\infty \subset D'(\mathbb{T}^d)$ converges to $f \in D'(\mathbb{T}^d)$ if and only if $\lim_{n\to\infty} f_n(\varphi) = f(\varphi)$ holds for all $\varphi \in D(\mathbb{T}^d)$. The Fourier coefficients of a distribution $f \in D'(\mathbb{T}^d)$ are the complex numbers

$$\hat{f}(k) = f(e^{-i2\pi k\cdot x}) \quad , \quad k \in \mathbb{Z}^2.$$

In the sense of convergence in $D'(\mathbb{T}^2)$ we have $f = \sum_{k\in\mathbb{Z}^2} \hat{f}(k)e^{i2\pi k\cdot x}$. We call $T \in D'(\mathbb{T}^2)$ a regular distribution if a $\mathbb{T}^2$-integrable function $f : \mathbb{T}^2 \to \mathbb{C}$ exists with

$$T(\varphi) = \int_{\mathbb{T}^2} f(x)\cdot\varphi(x)\,dx \quad , \quad \varphi \in D(\mathbb{T}^2). \tag{2.1}$$

The computation of the Fourier coefficients is then performed by the well-known formula

$$\hat{f}(k) = \hat{T}(k) = \int_{\mathbb{T}^2} f(x)e^{-i2\pi k\cdot x}\,dx\,.$$

Let further denote $L_p(\mathbb{T}^2)$, $0 < p \le \infty$, the space of all measurable functions $f : \mathbb{T}^2 \to \mathbb{C}$ satisfying

$$\|f|L_p(\mathbb{T}^2)\| = \Big(\int_{\mathbb{T}^2} |f(x)|^p\,dx\Big)^{1/p} < \infty$$

with the usual modification in case $p = \infty$. The space $C(\mathbb{T}^2)$ is often used as a replacement for $L_\infty(\mathbb{T}^2)$. It denotes the collection of all continuous and bounded periodic functions equipped with the L_∞-topology.

The following inequality is commonly referred to as Bernstein-Nikolskij-inequality. The original (non-periodic) version is contained in the book [14]. We need a periodic version to bound the $L_q(\mathbb{T}^2)$(-quasi)-norm of a trigonometric polynomial from above by its L_p(-quasi)-norm whenever $0 < p \le q \le \infty$.

Lemma 2.1. *Let $0 < p \le q \le \infty$ and $\Lambda \subset \{k \in \mathbb{Z}^2 : |k_i| \le N_i, i = 1,2\}$ where N_1, N_2 are given natural numbers. Let further $(\alpha_1,\alpha_2) \in \mathbb{N}_0^2$ and $D^\alpha = \partial^\alpha/(\partial x_1^{\alpha_1}\partial x_2^{\alpha_2})$. Then there is a constant $c > 0$ independent of N_i and t such that*

$$\|D^\alpha t(x)|L_q(\mathbb{T}^2)\| \le cN_1^{\alpha_1+1/p-1/q}N_2^{\alpha_2+1/p-1/q}\|t(x)|L_p(\mathbb{T}^2)\|$$

for all trigonometric polynomials t with $\operatorname{supp}\hat{t} \subset \Lambda$.

Remark 2.2. *For vector norms $L_{\mathbf{q}}$, $L_{\mathbf{p}}$ with $\mathbf{q} = (q_1,q_2) \ge \mathbf{p} = (p_1,p_2) \ge (1,1)$ the result can be found in Temlyakov [19, Thms. 2.2.1, 2.2.2]. Here we use a version which is a direct consequence of Theorem 1.1 and Proposition 1.6 in [24] and extends to $0 < p \le q \le \infty$.*

Another main tool is a bivariate Marcinkiewicz-Zygmund type inequality.

Lemma 2.3. *Let N_1, N_2 be given natural numbers and $\Lambda \subset \{k \in \mathbb{Z}^2 : |k_i| \le N_i, i = 1,2\}$ be the same discrete set as in the previous lemma. Let further $1 \le p \le \infty$. Then there are two absolute constants $C > c > 0$ such that*

$$c\|t|L_p(\mathbb{T}^2)\| \le \left[\frac{1}{4N_14N_2}\sum_{\ell_1=0}^{4N_1-1}\sum_{\ell_2=0}^{4N_2-1}\Big|t\Big(\frac{\ell_1}{4N_1},\frac{\ell_2}{4N_2}\Big)\Big|^p\right]^{1/p} \le C\|t|L_p(\mathbb{T}^2)\|$$

for every trigonometric polynomial t with $\operatorname{supp}\hat{t} \subset \Lambda$.

Remark 2.4. (i) *We refer to the monograph [19, Thm. II.2.4] for a version in $L_{\mathbf{p}}(\mathbb{T}^d)$, $\mathbf{p} = (p_1,p_2), 1 \le p_i \le \infty, i = 1,2$. For our purpose the special case $d = 2$ and $\mathbf{p} = (p,p)$, where $1 \le p \le \infty$, is sufficient.*
(ii) *It turned out that there is also a version of Lemma 2.3 for $0 < p \le 1$, see [15], which makes it possible to extend the Faber basis characterization, Proposition 3.4, to $0 < p,q \le \infty$, $1/p < r < 2$. See Step 4 in the proof of Proposition 3.4 for the necessary modifications.*

2.2 Definition and basic properties

In this section we give the definition of Besov spaces with dominating mixed smoothness on $\mathbb{T}^2$. We closely follow [16, Chapt. 2] and [24, Chapt. 1]. To begin with, we recall the concept of a dyadic decomposition of unity.

Definition 2.5. *Let $\Phi(\mathbb{R})$ be the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \subset S(\mathbb{R})$ satisfying*

(i) $\operatorname{supp} \varphi_0 \subset \{x : |x| \leq 2\}$,

(ii) $\operatorname{supp} \varphi_j \subset \{x : 2^{j-1} \leq |x| \leq 2^{j+1}\}$, $j = 1, 2, ...,$

(iii) For all $\ell \in \mathbb{N}_0$ it holds $\sup\limits_{x,j} 2^{j\ell} \, |D^{\ell} \varphi_j(x)| \leq c_{\ell} < \infty$,

(iv) $\sum\limits_{j=0}^{\infty} \varphi_j(x) = 1$ *for all $x \in \mathbb{R}$.*

Remark 2.6. *The class $\Phi(\mathbb{R})$ is not empty. We consider the following standard example. Let $\varphi_0(x) \in S(\mathbb{R})$ be a smooth function with $\varphi_0(x) = 1$ on $[-1, 1]$ and $\varphi_0(x) = 0$ if $|x| > 2$. For $j > 0$ we define*

$$\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x).$$

It is easy to verify that the system $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty}$ satisfies (i) - (iv).

Now we fix a system $\{\varphi_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R})$. For $j = (j_1, j_2) \in \mathbb{Z}^2$ let the building blocks f_j be given by

$$f_j(x) = \sum_{k \in \mathbb{Z}^2} \varphi_{j_1}(k_1) \varphi_{j_2}(k_2) \hat{f}(k) e^{i2\pi k \cdot x} \,, \tag{2.2}$$

where we put $f_j = 0$ if $\min\{j_1, j_2\} < 0$.

Definition 2.7. *(Mixed periodic Besov space) Let $0 < p, q \leq \infty$ and $r \in \mathbb{R}$. Then $S^r_{p,q}B(\mathbb{T}^2)$ is the collection of all $f \in D'(\mathbb{T}^2)$ such that*

$$\|f|S^r_{p,q}B(\mathbb{T}^2)\|^{\varphi} := \Big(\sum_{j \in \mathbb{N}_0^2} 2^{|j|_1 rq} \|f_j|L_p\|^q \Big)^{1/q} \tag{2.3}$$

is finite (usual modification in case $q = \infty$).

Recall, that this definition is independent of the chosen system φ in the sense of equivalent (quasi-)norms. Moreover, in case $\min\{p, q\} \geq 1$ the defined spaces are Banach spaces, whereas they are quasi-Banach spaces in case $\min\{p, q\} < 1$. For details confer [16, 2.2.4] and [24, Sect. 1.4].

In this paper we are mainly concerned with spaces with positive smoothness parameter r in order to define a cubature formula in a reasonable way. In particular, the condition $r > 1/p$ ensures that the elements in $S^r_{p,q}B(\mathbb{T}^2)$ are regular distributions with a continuous representative (2.1). We have the following elementary embeddings, see [16, 2.2.3] and [24, Lem. 1.6].

Lemma 2.8. *Let $0 < p < \infty$, $r \in \mathbb{R}$, and $0 < q \leq \infty$.*

(i) If $\varepsilon > 0$ and $0 < v \le \infty$ then

$$S^{r+\varepsilon}_{p,q}B(\mathbb{T}^2) \hookrightarrow S^r_{p,v}B(\mathbb{T}^2)\,.$$

(ii) If $p < u \le \infty$ and $r - 1/p = t - 1/u$ then

$$S^r_{p,q}B(\mathbb{T}^2) \hookrightarrow S^t_{u,q}B(\mathbb{T}^2)\,.$$

(iii) If $r > 1/p$ then

$$S^r_{p,q}B(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)\,.$$

2.3 Characterization by mixed differences

There is also a direct characterization of the above defined function spaces. We will use mixed differences $\Delta^{m,m}_{h_1,h_1} f$ of a periodic function f instead of Fourier coefficients which represents the classical approach to these spaces [1]. We define differences of order $M \in \mathbb{N}$ as well as corresponding mixed differences. Essentially the same notation as in [16, 2.3.3] and [23] will be used. Fix $h \in \mathbb{R}$. Under a first order difference with step-length h of a function $f : \mathbb{R} \to \mathbb{C}$ we want to understand the function $\Delta_h f$ which is defined by

$$\Delta_h f(x) = f(x+h) - f(x) \quad , \quad x \in \mathbb{R}.$$

Iteration leads to M-th order differences, given by

$$\Delta^M_h f(x) = \Delta_h(\Delta^{M-1}_h f)(x) \quad , \quad M \in \mathbb{N} \quad , \quad \Delta^0_h = I. \tag{2.4}$$

Using mathematical induction one can show the explicit formula

$$\Delta^M_h f(x) = \sum_{j=0}^{M} (-1)^j \binom{M}{j} f(x + (M-j)h). \tag{2.5}$$

For our special purpose we need differences with respect to a certain component of f as well as mixed differences. Let us first define the operator $\Delta^m_{h,i} f$ applied to a function $f : \mathbb{R}^2 \to \mathbb{C}$. Having (2.5) in mind we put

$$\Delta^m_{h,1} f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x_1 + (m-j)h, x_2) \quad , \tag{2.6}$$

where $m \in \mathbb{N}_0$, $h \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$ (analogously for $i = 2$). The mixed difference $\Delta^{m,m}_{h_1,h_2} f$ is now given by the operator

$$\Delta^{m,m}_{h_1,h_2} f = (\Delta^m_{h_1} \circ \Delta^m_{h_2}) f$$

The following Lemma (in the case $\min\{p,q\} \ge 1$) is a well-known classical equivalent characterization of Besov spaces with dominating mixed smoothness, see for instance [1]. Some difficulties occur in the quasi-Banach case, i.e. $\min\{p,q\} < 1$. In this situation we mainly refer to [16, 2.3.4] where the non-periodic bivariate situation is treated and to the more recent paper [23, 3.7, 4.5]. For the sake of completeness we will recall the main steps in the proof.

Lemma 2.9. *Let $0 < p,q \le \infty$ and $m > r > 1/p$. Then the following quantity represents an equivalent (quasi-)norm in $S^r_{p,q}B(\mathbb{T}^2)$*

$$
\begin{aligned}
\|f|S^r_{p,q}B(\mathbb{T}^2)\|^M := & \|f|L_p(\mathbb{T}^2)\| \\
& + \Big(\sum_{j_1=0}^{\infty} 2^{rj_1q} \sup_{|h_1|\le 2^{-j_1}} \|\Delta^m_{h_1,1} f|L_p(\mathbb{T}^2)\|^q\Big)^{1/q} \\
& + \Big(\sum_{j_2=0}^{\infty} 2^{rj_2q} \sup_{|h_2|\le 2^{-j_2}} \|\Delta^m_{h_2,2} f|L_p(\mathbb{T}^2)\|^q\Big)^{1/q} \\
& + \Big(\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} 2^{r|j|_1q} \sup_{\substack{|h_1|\le 2^{-j_1}\\ |h_2|\le 2^{-j_2}}} \|\Delta^{m,m}_{h_1,h_2} f|L_p(\mathbb{T}^2)\|^q\Big)^{1/q}.
\end{aligned}
\tag{2.7}
$$

Proof. This assertion is a modified version of [23, Thm. 4.6.2] for the bivariate setting. Let us recall some basic steps in the proof. The relation

$$\|f|S^r_{p,q}B(\mathbb{T}^2)\|^M \le C_1\|f|S^r_{p,q}B(\mathbb{T}^2)\|^\varphi$$

is obtained by applying [23, Lem. 3.3.2] to the building blocks f_j in (2.1), which are indeed trigonometric polynomials, and using the proof technique in [23, Theorem 3.8.1].

To obtain the converse relation

$$\|f|S^r_{p,q}B(\mathbb{T}^2)\|^\varphi \le C_2\|f|S^r_{p,q}B(\mathbb{T}^2)\|^M$$

we take into account the characterization via rectangle means given in [23, Thm. 4.5.1]. We apply the techniques in Proposition 3.6.1 to switch from rectangle means to moduli of smoothness by following the arguments in the proof of Theorem 3.8.2. It remains to discretize the outer integral (with respect to the step length of the differences) in order to replace it by a sum. This is done by standard arguments. Thus, we almost arrived at (2.7). Indeed, the final step is to get rid of those summands where the summation index is negative. But this is trivially done by omitting the corresponding difference (translation invariance of L_p) such that the respective sum is just a converging geometric series (recall that $r > 0$). ■

Remark 2.10. *The condition $r > 1/p$ in Lemma 2.9 seems to be unnatural. We do not know whether it is necessary or not. So far, the condition is due to the proof technique. However, in case $1 \le p \le \infty$ this condition can be weakened to $r > 0$. In the sequel, we will deal with continuous functions $S^r_{p,q}B(\mathbb{T}^2)$ with $r > 1/p$. For this paper, Lemma 2.9 will be sufficient.*

3 Faber bases

3.1 The univariate Faber basis

Recently, Triebel [21, 22] and, independently, Dinh [4] observed the potential of the Faber basis for the approximation and integration of functions with dominating mixed smoothness. The latter reference is even more general and uses so-called B-spline representations of functions, where the Faber system is a special case. Let us briefly recall the basic facts about the Faber basis taken from [21, 3.2.1, 3.2.2]. Faber [5] observed that every continuous (non-periodic) function f on $[0,1]$ can be represented as

$$f(x) = f(0)\cdot(1-x) + f(1)\cdot x - \frac{1}{2}\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}(\Delta^2_{2^{-j-1}}f(2^{-j}k)v_{j,k} \tag{3.1}$$

with convergence at least point-wise. Consequently, every periodic function on $C(\mathbb{T})$ can be represented by

$$f(x) = f(0) - \frac{1}{2}\sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1}(\Delta^2_{2^{-j-1}}f(2^{-j}k)v_{j,k}\,. \tag{3.2}$$

Definition 3.1. *The univariate periodic Faber system is given by the system of functions on* $\mathbb{T} = [0,1]$

$$\{1, v_{j,k} : j \in \mathbb{N}_0, k = 0, ..., 2^j - 1\}\,,$$

where

$$v_{j,m}(x) = \begin{cases} 2^{j+1}(x - 2^{-j}m) & : \; 2^{-j}m \le x \le 2^{-j}m + 2^{-j-1}, \\ 2^{j+1}(2^{-j}(m+1) - x) & : \; 2^{-j}m + 2^{-j-1} \le x \le 2^{-j}(m+1), \\ 0 & : \; \textit{otherwise}\,. \end{cases} \tag{3.3}$$

For notional reasons we let $v_{-1,0} := 1$ *and obtain the Faber system*

$$F := \{v_{j,k} : j \in \mathbb{N}_{-1}, k \in D_j\}\,,$$

where $D_j := \{0, ..., \lceil 2^j \rceil - 1\}\,.$

3.2 The tensor Faber basis

Let now $f(\cdot,\cdot)$ be a bivariate function $f \in C(\mathbb{T}^2)$. By fixing one variable $y \in \mathbb{T}$ we obtain by $g(\cdot) = f(\cdot, y)$ a univariate periodic continuous function. By applying (3.2) in both components we obtain the representation

$$f(x) = \sum_{j\in\mathbb{N}^2_{-1}}\sum_{k\in D_j} d^2_{j,k}(f)v_{j,k}(x)\,, \tag{3.4}$$

where $v_{j,k}(x_1,x_2) := v_{j_1,k_1}(x_1)v_{j_2,k_2}(x_2)$, $j \in \mathbb{N}^2_{-1}, k \in D_j = D_{j_1}\times D_{j_2}$, and

$$d^2_{j,k}(f) = \begin{cases} f(0,0) & : \; j = (-1,-1), \\ -\frac{1}{2}\Delta^2_{2^{-j_1-1},1}f(2^{-j_1}k_1, 0) & : \; j = (j_1,-1), j_1 \in \mathbb{N}_0, \\ -\frac{1}{2}\Delta^2_{2^{-j_2-1},2}f(0, 2^{-j_2}k_2) & : \; j = (-1,j_2), j_2 \in \mathbb{N}_0, \\ \frac{1}{4}\Delta^{2,2}_{2^{-j_1-1},2^{-j_2-2}}f(2^{-j_1}k_1, 2^{-j_2}k_2) & : \; j = (j_1,j_2) \in \mathbb{N}^2_0\,. \end{cases}$$

Our goal is to discretize the spaces $S^r_{p,q}B(\mathbb{T}^2)$ using the Faber system $\{v_{j,k} : j \in \mathbb{N}^2_{-1}, k \in D_j\}$. We obtain a sequence space isomorphism performed by the coefficient mapping $d^2_{j,k}(f)$ above. In [21, 3.2.3, 3.2.4] and [4, Thm. 4.1] this was done for the non-periodic setting $S^r_{p,q}B(Q_2)$ and $S^r_pH(Q_2)$. Our proof is completely different and uses only classical tools. From my point of view this makes the proof a bit more transparent and self-contained. With these tools we show that one direction of the equivalence relation can be extended to $1/p < r < 2$.

Definition 3.2. *Let* $0 < p, q \leq \infty$ *and* $r \in \mathbb{R}$. *Then* $s^r_{p,q}b$ *is the collection of all sequences* $\{\lambda_{j,k}\}_{j\in N^2_{-1},k\in D_j}$ *such that*

$$\|\lambda_{j,k}|s^r_{p,q}b\| := \Big[\sum_{j\in\mathbb{N}^2_{-1}} 2^{|j|_1(r-1/p)q}\Big(\sum_{k\in D_j}|\lambda_{j,k}|^p\Big)^{q/p}\Big]^{1/q}$$

is finite.

Lemma 3.3. *Let Let* $0 < p, q \leq \infty$ *and* $r \in \mathbb{R}$. *The space* $s^r_{p,q}b$ *is a Banach space if* $\min\{p, q\} \geq 1$. *In case* $\min\{p, q\} < 1$ *the space* $s^r_{p,q}b$ *is a quasi-Banach space. Moreover, if* $u := \min\{p, q\}$ *it is a u-Banach space, i.e.,*

$$\|\lambda+\mu|s^r_{p,q}b\|^u \leq \|\lambda|s^r_{p,q}b\|^u + \|\mu|s^r_{p,q}b\|^u \quad , \quad \lambda,\mu \in s^r_{p,q}b\,.$$

Proposition 3.4. *Let* $0 < p, q \leq \infty$ *and* $1/p < r < 2$. *Then there exists a constant* $c > 0$ *such that*

$$\big\|d^2_{j,k}(f)|s^r_{p,q}b\big\| \leq c\|f|S^r_{p,q}B(\mathbb{T}^2)\|^{\varphi} \tag{3.5}$$

for all $f \in C(\mathbb{T}^2)$.

Proof. *Step 1.* The main idea is the same as in the proof of Lemma 2.9. We make use of the decomposition (2.2) in a slightly modified way. Let us first assume $1 \leq p, q \leq \infty$. We will point out the necessary modification in case $\min\{p, q\} < 1$ in Step 4 of the proof. For fixed $j \in \mathbb{N}^2_{-1}$ we write $f_j = \sum_{\ell\in\mathbb{Z}^2} f_{j+\ell}$. Putting this into (3.5) and using the triangle inequality yields

$$\big\|d^2_{j,k}(f)|s^r_{p,q}b\big\| \leq \sum_{\ell\in\mathbb{Z}^2}\Big[\sum_{j\in\mathbb{N}^2_{-1}} 2^{|j|_1(r-1/p)q}\Big(\sum_{k\in D_j}|d^2_{j,k}(f_{j+\ell})|^p\Big)^{q/p}\Big]^{1/q}. \tag{3.6}$$

Recall, that the numbers $\{d^2_{j,k}(f_{j+\ell})\}_k$ are samples of the trigonometric polynomial $t := \Delta^{2,2}_{2^{-j_1-1},2^{-j_2-1}}f_{j+\ell}$ (obvious modification if $j_1 = -1$ or $j_2 = -1$). We want to apply Lemma 2.3 in order to estimate the discrete ℓ_p-norm $(2^{-|j|_1}\sum_{k\in D_j}|d^2_{j,k}(f_{j+\ell})|^p)^{1/p}$ from above. Let $\tilde{\ell} = (\max\{0, \ell_1\}, \max\{0, \ell_2\})$. Since $\hat{t}$ is supported in the cube

$$Q_{j+\ell} = [-2^{j_1+\ell_1+1}, 2^{j_1+\ell_1+1}] \times [-2^{j_2+\ell_2+1}, 2^{j_2+\ell_2+1}]$$

we obtain by Lemma 2.3 the relation

$$\left[2^{-j_1-\tilde{\ell}_1-3}2^{-j_2-\tilde{\ell}_2-3}\sum_{k_1=0}^{2^{j_1+\tilde{\ell}_1+3}-1}\sum_{k_2=0}^{2^{j_2+\tilde{\ell}_2+3}-1}\Big|t\Big(\frac{k_1}{2^{j_1+\tilde{\ell}_1+3}},\frac{k_2}{2^{j_2+\tilde{\ell}_2+3}}\Big)\Big|^p\right]^{1/p} \leq C\|t|L_p(\mathbb{T}^2)\| \tag{3.7}$$

In the left-hand side of (3.7) we sample on a grid that includes all the grid points according to level $j = (j_1, j_2)$. Therefore, the left-hand side of (3.7) dominates the quantity

$$2^{-|\tilde{\ell}|_1/p}\Big(2^{-|j|_1}\sum_{k\in D_j}\Big|t\Big(\frac{k_1}{2^{j_1}},\frac{k_2}{2^{j_2}}\Big)\Big|^p\Big)^{1/p}.$$

This implies

$$\Big(2^{-|j|_1}\sum_{k\in D_j}|d^2_{j,k}(f_{j+\ell})|^p\Big)^{1/p} \lesssim 2^{|\tilde{\ell}|_1/p}\|\Delta^{2,2}_{2^{-j_1-1},2^{-j_2-1}}f_{j+\ell}|L_p(\mathbb{T}^2)\| \tag{3.8}$$

with an obvious modification in case $j_1 = -1$ or $j_2 = -1$.

Step 2. We continue estimating the right-hand side in (3.8). Let $h_i > 0$, $i = 1, 2$. Applying twice the classical mean-value theorem we obtain

$$\begin{aligned} |\Delta^2_{h_1,1} f_{j+\ell}(x_1, x_2)| &\lesssim |h_1|^2 \sup_{|y_1| \le 2|h_1|} |f^{(2,0)}_{j+\ell}(x_1 + y_1, x_2)| \\ &\lesssim |h_1|^2 \max\{1, 2^{j_1+\ell_1}|h_1|\}^a \sup_{y_1 \in \mathbb{R}} \frac{|f^{(2,0)}_{j+\ell}(x_1 + y_1, x_2)|}{(1 + 2^{j_1+\ell_1}|y_1|)^a} \\ &\lesssim |h_1|^2 \max\{1, 2^{j_1}|h_1|\}^a \sup_{y_1 \in \mathbb{R}} \frac{|f^{(2,0)}_{j+\ell}(x_1 + y_1, x_2)|}{(1 + 2^{j_1+\ell_1}|y_1|)^a} \end{aligned} \tag{3.9}$$

in case $\ell_1 \le 0$. In the same way we proceed with $|\Delta^2_{h_i,i} f_{j+\ell}(x_1, x_2)|$ if $\ell_i \le 0$. If $\ell_i > 0$, we simply use the triangle inequality and (2.6) to resolve the difference. Combining the scalar mixed Peetre maximal inequality (see Proposition 1.5 and Theorem 1.9 in [24]), Lemma 2.1, and the translation invariance in $L_p(\mathbb{T}^2)$ we obtain the estimate

$$\begin{aligned} &\|\Delta^{2,2}_{h_1,h_2} f_{j+\ell}|L_p(\mathbb{T}^2)\| \\ &\lesssim \min\{1, |h_1 2^{j_1+\ell_1}|\}^2 \min\{1, |h_2 2^{j_2+\ell_2}|\}^2 \max\{1, 2^{j_1}|h_1|\}^a \max\{1, 2^{j_2}|h_2|\}^a \|f_{j+\ell}|L_p(\mathbb{T}^2)\|, \end{aligned}$$

where we chose $a > 1/p$ in (3.9). Choosing $h_1 = 2^{-j_1-1}$ and $h_2 = 2^{-j_2-1}$ and putting $\bar{\ell} = (\min\{0, \ell_1\}, \min\{0, \ell_2\})$ yields

$$\|\Delta^{2,2}_{2^{-j_1-1},2^{-j_2-1}} f_{j+\ell}|L_p(\mathbb{T}^2)\| \lesssim 2^{2(\bar{\ell}_1+\bar{\ell}_2)} \|f_{j+\ell}|L_p(\mathbb{T}^2)\|. \tag{3.10}$$

Combining (3.8) and (3.10) gives

$$\Big(2^{-|j|_1} \sum_{k \in D_j} |d^2_{j,k}(f_{j+\ell})|^p\Big)^{1/p} \lesssim 2^{(\tilde{\ell}_1+\tilde{\ell}_2)/p} 2^{2(\bar{\ell}_1+\bar{\ell}_2)} \|f_{j+\ell}|L_p(\mathbb{T}^2)\|. \tag{3.11}$$

Step 3. Putting (3.11) into (3.6) yields

$$\begin{aligned} \|d^2_{j,k}|s^r_{p,q}b\| &\lesssim \sum_{\ell \in \mathbb{Z}^2} 2^{(\tilde{\ell}_1+\tilde{\ell}_2)/p} 2^{2(\bar{\ell}_1+\bar{\ell}_2)} 2^{-r(\ell_1+\ell_2)} \Big[\sum_{j \in \mathbb{N}^2_{-1}} 2^{|j+\ell|_1 rq} \|f_{j+\ell}|L_p(\mathbb{T}^2)\|^q\Big]^{1/q} \\ &\lesssim \|f|S^r_{p,q}B(\mathbb{T}^2)\| \cdot \sum_{\ell \in \mathbb{Z}^2} 2^{(\tilde{\ell}_1+\tilde{\ell}_2)/p} 2^{2(\bar{\ell}_1+\bar{\ell}_2)} 2^{-r(\ell_1+\ell_2)}. \end{aligned} \tag{3.12}$$

Finally, we split the sum over ℓ into four parts according to the indices $\ell_i \le 0$ and $\ell_i > 0, i = 1, 2$. In fact, the convergence of each sum is a consequence of the assumption $1/p < r < 2$ and the definition of $\tilde{\ell}$ and $\bar{\ell}$.

Step 4. Let us comment on the necessary modifications in case $\min\{p, q\} < 1$, $1/p < r < 2$. By taking Lemma 3.3 into account we replace (3.6) by

$$\|d^2_{j,k}(f)|s^r_{p,q}b\|^u \le \sum_{\ell \in \mathbb{Z}^2} \Big[\sum_{j \in \mathbb{N}^2_{-1}} 2^{|j|_1(r-1/p)q} \Big(\sum_{k \in D_j} |d^2_{j,k}(f_{j+\ell})|^p\Big)^{q/p}\Big]^{u/q},$$

where $u := \min\{p,q\}$. As already mentioned in Remark 2.4/(ii), Lemma 2.3 extends to $0<p<1$. Hence, we obtain (3.8) in the same way as above. The arguments in (3.9) to (3.11) apply for all $0<p\le\infty$. Therefore, instead of (3.12) we end up with

$$\big\|d^2_{j,k}(f)|s^r_{p,q}b\big\|^u \lesssim \|f|S^r_{p,q}B(\mathbb{T}^2)\|^u \cdot \sum_{\ell\in\mathbb{Z}^2}\Big[2^{(\tilde\ell_1+\tilde\ell_2)/p}2^{2(\bar\ell_1+\bar\ell_2)}2^{-r(\ell_1+\ell_2)}\Big]^u,$$

which proves the claim. ■

Remark 3.5. *The d-dimensional version of the statement in Proposition 3.4 for non-periodic functions on $[0,1]^d$ has been considered by Dinh [4, Thm. 4.1,(i)], see also Sections 5 and 6 in this paper. The techniques in the proof above heavily rely on the periodic setting. They essentially differ from the methods used in [4].*

Proposition 3.6. *Let $0<p,q\le\infty$ and $1/p<r<1+\min\{1/p,1\}$. Then there exists a constant $c>0$ such that*

$$\|f|S^r_{p,q}B(\mathbb{T}^2)\|^M \le c\|d^2_{j,k}(f)|s^r_{p,q}b\| \tag{3.13}$$

for all $f\in S^r_{p,q}B(\mathbb{T}^2)$.

Proof. *Step 1.* Since $f\in S^r_{p,q}B(\mathbb{T}^2)$ we obtain by the embedding result in Lemma 2.8/(ii),(iii) that $f\in S^\varepsilon_{\infty,1}B(\mathbb{T}^2)$ for an $\varepsilon>0$. As a consequence of Proposition 3.4 we obtain that (3.4) converges to f in $C(\mathbb{T}^2)$ and therefore in $L_p(\mathbb{T}^2)$. We will first prove the assertion in case $1\le p\le\infty$ and $0<q\le\infty$ and point out the necessary modifications in case $0<p<1$ afterwards. Let now $h_1,h_2\in\mathbb{R}$ such that $|h_i|\le 2^{-\ell_i}$ for a given $(\ell_1,\ell_2)\in\mathbb{N}_0^2$. With an eye on (2.7) we obtain the estimate (using (3.4) and the triangle inequality)

$$\begin{aligned}\Big(\int_{\mathbb{T}^2}|\Delta^{2,2}_{h_1,h_2}f(x)|^p\,dx\Big)^{1/p} &\le \sum_{j\in\mathbb{N}^2_{-1}}\Big(\int_{\mathbb{T}^2}\Big|\Delta^{2,2}_{h_1,h_2}\Big(\sum_{k\in D_j}d^2_{j,k}(f)v_{j,k}\Big)(x)\Big|^p\,dx\Big)^{1/p}\\ &= \sum_{j\in\mathbb{N}^2_{-1}}\Big(\int_{\mathbb{T}^2}\Big|\Delta^2_{h_1,1}\Big(\sum_{k\in D_j}d^2_{j,k}(f)\big(\Delta^2_{h_2,2}v_{j,k}\big)\Big)(x)\Big|^p\,dx\Big)^{1/p}\\ &= \sum_{j\in\mathbb{N}^2_{-1}}\Big(\int_{\mathbb{T}^2}\Big|\sum_{k\in D_j}d^2_{j,k}(f)\big(\Delta^{2,2}_{h_1,h_2}v_{j,k}\big)(x)\Big|^p\,dx\Big)^{1/p}.\end{aligned} \tag{3.14}$$

Which right-hand side we finally use depends of the relation between ℓ_i and j_i, $i=1,2$. The last one is used in case $(\ell_1,\ell_2)>(j_1,j_2)$. The second one is used in case $\ell_2>j_2$ and $\ell_1\le j_1$. And finally, the first one is used in case $(\ell_1,\ell_2)\le(j_1,j_2)$. By definition we have

$$\big(\Delta^{2,2}_{h_1,h_2}v_{j,k}\big)(x_1,x_2) = \big(\Delta^2_{h_1}v_{j_1,k_1}\big)(x_1)\cdot\big(\Delta^2_{h_2}v_{j_2,k_2}\big)(x_2).$$

Let us discuss the univariate function $\big(\Delta^2_{h_1}v_{j_1,k_1}\big)(x_1)$. Note first that v_{j_1,k_1} is a piecewise linear function. Therefore $\big(\Delta^2_{h_1}v_{j_1,k_1}\big)(x_1)$ vanishes unless x_1 belongs to one of the intervals I_L,I_M,I_R given by $I_L:=\{x\in\mathbb{T}:|x-2^{-j_1}k_1|<2^{-\ell_1+1}\}$, $I_M:=\{x\in\mathbb{T}:|x-2^{-j}k-2^{-j-1}|<2^{-\ell_1+1}\}$, and $I_R:=\{x\in\mathbb{T}:|x-2^{-j_1}(k_1+1)|<2^{-\ell_1+1}\}$. Further, if $\ell_1>j_1$ it is easy to verify that

$$|(\Delta^2_{h_1}v_{j_1,k_1})(x_1)|\lesssim 2^{j_1-\ell_1}\quad,\quad x\in I_L\cup I_M\cup I_R.$$

Indeed, this is a simple consequence of (2.5) and the definition of v_{j_1,k_1}, see (3.3). In particular, as a consequence of $|I_L \cup I_M \cup I_R| \lesssim 2^{-\ell_1}$ we obtain

$$\int_{\mathbb{T}} |(\Delta^2_{h_1} v_{j_1,k_1})(x_1)|^p \, dx_1 \leq 2^{p(j_1-\ell_1)} 2^{-\ell_1} \tag{3.15}$$

for $\ell_1 > j_1$. Let us assume $(\ell_1, \ell_2) > (j_1, j_2)$. We will use the last case of (3.14). Using (3.15) we can estimate

$$\begin{aligned}
&\int_{\mathbb{T}^2} \Big| \sum_{k \in D_j} d^2_{j,k}(f) \big(\Delta^{2,2}_{h_1,h_2} v_{j,k}\big)(x) \Big|^p \, dx \\
&= \int_{\mathbb{T}^2} \Big| \sum_{k_1} (\Delta^2_{h_1} v_{j_1,k_1})(x_1) \sum_{k_2} d^2_{j,k}(f) (\Delta^2_{h_2} v_{j_2,k_2})(x_2) \Big|^p \, dx \\
&\lesssim \sum_{k_1} \int_{\mathbb{T}} |(\Delta^2_{h_1} v_{j_1,k_1})(x_1)|^p \int_{\mathbb{T}} \Big| \sum_{k_2} |d^2_{j,k}(f)| (\Delta^2_{h_2} v_{j_2,k_2})(x_2) \Big|^p \, dx_2 \, dx_1 \\
&\lesssim 2^{p(j_1-\ell_1)} 2^{j_1-\ell_1} 2^{p(j_2-\ell_2)} 2^{j_2-\ell_2} 2^{-|j|_1} \sum_{k \in D_j} |d^2_{j,k}(f)|^p \, .
\end{aligned} \tag{3.16}$$

Note, that the last but one estimate can only be justified if $\ell_1 > j_1 + \lambda$, say $\lambda = 3$. Indeed, then just $\Delta^2_{h_1} v_{j_1,k_1}$ might have joint support with $\Delta^2_{h_1} v_{j_1,k_1+1}$, whereas all other functions are disjointly supported. Therefore $|\sum \cdots|^p \lesssim \sum |\cdots|^p$. However, if ℓ_1 is close to j_1 (or ℓ_2 close to j_2) we argue analogously to the case $\ell_1 \leq j_1$ and $\ell_2 > j_2$ below. This finishes the case $(\ell_1, \ell_2) > (j_1, j_2)$. In the case $\ell_1 \leq j_1$ and $\ell_2 > j_2$ we start with the second formula on the right-hand side of (3.14) and apply the translation invariance in $L_p(\mathbb{T}, x_1)$ first. Then we continue analogously as in the previous case and end up with an estimate similar as in (3.16) but where all factors involving ℓ_1 disappear. Therefore, we can collect all the cases in the following formula

$$\begin{aligned}
&\sup_{\substack{|h_i| \leq 2^{-\ell_i} \\ i=1,2}} \Big(\int_{\mathbb{T}^2} |\Delta^{2,2}_{h_1,h_2} f(x)|^p \, dx \Big)^{1/p} \\
&\lesssim \sum_{j \in \mathbb{N}^2_{-1}} M(j_1, \ell_1) M(j_2, \ell_2) \Big(2^{-|j|_1} \sum_{k \in D_j} |d^2_{j,k}(f)|^p \Big)^{1/p} .
\end{aligned} \tag{3.17}$$

where $M(j, \ell) = \min\{1, 2^{(j-\ell)(1+1/p)}\}$. With similar arguments we obtain corresponding estimates for $\sup_{|h_1| \leq 2^{-\ell}} \|\Delta^2_{h_1,1} f | L_p(\mathbb{T}^2)\|$ and $\sup_{|h_2| \leq 2^{-\ell}} \|\Delta^2_{h_2,2} f | L_p(\mathbb{T}^2)\|$ and $\|f | L_p(\mathbb{T}^2)\|$.

Step 2. In order to continue with (3.17) we need to introduce weighted sequence spaces of type $\ell^r_q(\mathbb{N}^2_0)$ with (quasi-)norm given by

$$\|\{\lambda_j\}_{j \in \mathbb{N}^2_0} | \ell^r_p(\mathbb{N}^2_0)\| := \Big(\sum_{j \in \mathbb{N}^2_0} 2^{r|j|_1 q} |\lambda_j|^q \Big)^{1/q}$$

with the usual modification in case $q = \infty$. The following Lemma gives information about mapping properties of a certain convolution type operator.

Lemma 3.7. *Let $0 < q \le \infty$ and $0 < r < s$. Let the operator A_s be given by*

$$\{A_s\lambda\}_\ell = \sum_{j\in\mathbb{N}_0^2} \min\{1, 2^{(j_1-\ell_1)s}\} \min\{1, 2^{(j_2-\ell_2)s}\}\lambda_j \quad , \quad \ell \in \mathbb{N}_0^2 .$$

Then A_s is a bounded operator $A_s : \ell_q^r(\mathbb{N}_0^2) \to \ell_q^r(\mathbb{N}_0^2)$.

Proof. Let us first consider the case $0 < q \le 1$. Let $\lambda \in \ell_q^r(\mathbb{N}_0^2)$ and $\mu = A\lambda$. Then we have

$$\begin{aligned}
\sum_{\ell\in\mathbb{N}_0^2} 2^{r|\ell|_1 q}|\mu_\ell|^q &\le \sum_{\ell\in\mathbb{N}_0^2}\sum_{j\in\mathbb{N}_0^2} 2^{r|\ell|_1 q} \min\{1, 2^{(j_1-\ell_1)s}\} \min\{1, 2^{(j_2-\ell_2)s}|\lambda_j|^q \\
&= \sum_{j\in\mathbb{N}_0^2} |\lambda_j|^q \Big[\sum_{\ell_1=0}^{j_1}\sum_{\ell_2=0}^{j_2} 2^{r|\ell|_1 q} + \sum_{\ell_1=0}^{j_1} 2^{r\ell_1 q}2^{j_2 sq} \sum_{\ell_2=j_2}^{\infty} 2^{-\ell_2(s-r)q} \\
&\qquad + 2^{j_1 sq} \sum_{\ell_1=j_1}^{\infty} 2^{-\ell_1(s-r)q} \sum_{\ell_2=0}^{j_2} 2^{\ell_2 r} \\
&\qquad + 2^{j_1 sq} \sum_{\ell_1=j_1}^{\infty} 2^{-\ell_1(s-r)q} 2^{j_2 sq} \sum_{\ell_2=j_2}^{\infty} 2^{-\ell_2(s-r)q}\Big] \\
&\lesssim \sum_{j\in\mathbb{N}_0^2} 2^{r|j|_1 q}|\lambda_j|^q .
\end{aligned}$$

In case $q = \infty$ we interchange the supremum over ℓ with the sum over j and argue in a similar way. It remains the case $1 < q < \infty$. This is a simple consequence of the well-known complex interpolation formula $[\ell_{q_0}^{r_0}(\mathbb{N}_0^2), \ell_{q_1}^{r_1}(\mathbb{N}_0^2)]_\theta = \ell_q^r(\mathbb{N}_0^2)$ with $(\theta - 1)(1/q_0, r_0) + \theta(1/q_1, r_1) = (1/q, r)$ where $r_0, r_1 \in \mathbb{R}$ and $1 \le q_0, q_1 \le \infty$, see for instance [20, 1.18.4]. Indeed, this formula applied to $q_0 = 1$, $q_1 = \infty$, $r_0 = r_1 = r$ and $\theta = 1 - 1/q$ together with the results above give the boundedness of $A_s : \ell_q^r(\mathbb{N}_0^2) \to \ell_q^r(\mathbb{N}_0^2)$. ■

Let us continue with the proof of Proposition 3.6. Applying Lemma 3.7 with $s = 1 + 1/p$ to the relation (3.17) and its modifications we are now able to bound every summand on the right-hand side of (2.7) from above by $\|d^2_{j,k}(f)|s^r_{p,q}b\|$ which finishes the proof in case $1 \le p \le \infty$ and $0 < q \le \infty$.

Step 3. Let us comment on the case $0 < p < 1$. First of all, the additional restriction $r < 2$ comes from the definition of the left-hand side in (3.13) in connection with Lemma 2.9. By using the p-triangle inequality in $L_p(\mathbb{T}^2)$ we start with replacing (3.14) by the similar estimate without the powers $1/p$. The subsequent considerations (3.14) to (3.16) apply as well. We have to replace (3.17) and its modifications by

$$\begin{aligned}
&\sup_{\substack{|h_i|\le 2^{-\ell_i}\\ i=1,2}} \int_{\mathbb{T}^2} |\Delta^{2,2}_{h_1,h_2} f(x)|^p\, dx \\
&\lesssim \sum_{j\in\mathbb{N}_{-1}^2} \tilde{M}(j_1,\ell_1)\tilde{M}(j_2,\ell_2)\Big(2^{-|j|_1} \sum_{k\in D_j} |d^2_{j,k}(f)|^p\Big) ,
\end{aligned}$$

where this time $\tilde{M}(j,\ell) = \min\{1, 2^{(j-\ell)(p+1)}\}$. Now we apply Lemma 3.7 with q', r' and s', where $q' = q/p$, $r' = rp$ and $s' = p + 1$. Thus, the claim follows and the proof is complete. ■

Remark 3.8. *The d-dimensional version of the statement in Proposition 3.6 for non-periodic spaces on* $[0,1]^d$ *has been proved by Dinh [4, Thm. 4.1,(ii)]. Some of the arguments in the proof above are already used in [4].*

4 Optimal QMC integration on Hammersley points

In the sequel we consider cubature formulas for continuous periodic functions $f \in C(\mathbb{T}^2)$ of type

$$\Lambda_N(X,f) := \sum_{x_i \in X} \lambda_i f(x_i)\,,$$

where $X = \{x_1, .., x_N\} \subset \mathbb{T}^2$ represents the fixed set of integration knots and $\Lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{R}^N$ the fixed vector of weights. A QMC method has equal weights which sum up to 1, i.e., $\Lambda = (1/N, ..., 1/N)$. In this case we denote $I_N(X,f) := \Lambda_N(X,f)$. Furthermore,

$$I(f) := \int_{\mathbb{T}^2} f(x)\,dx$$

denotes the exact value of the integral of the function $f \in C(\mathbb{T}^2)$ over the 2-torus $\mathbb{T}^2$. Once, we have fixed a cubature formula $\Lambda_N(X,f)$ we will consider the error

$$R_N(f) := I(f) - \Lambda_N(X,f) \quad , \quad f \in C(\mathbb{T}^2)\,. \tag{4.1}$$

4.1 Hammersley type point sets

In this paper we mainly consider cubature formulas on Hammersley type point sets

$$\mathcal{H}_n = \Big\{ \Big(\frac{t_n}{2} + \frac{t_{n-1}}{2^2} + ... + \frac{t_1}{2^n}, \frac{s_1}{2} + \frac{s_2}{2^2} + ... + \frac{s_n}{2^n} \Big) : t_i \in \{0,1\}, i = 1, ..., n \Big\} \subset Q_2\,. \tag{4.2}$$

Here, $s_i = t_i$ or $s_i = 1 - t_i$ depending on i. in particular, every set $\mathcal{H}_n$ contains $N = 2^n$ points. The original van der Corput point set [25] is given by putting $s_i = t_i$ for all $i = 1, ..., n$. The above setting admits certain modifications. For instance, the symmetrized Hammersley point set, considered by Halton and Zaremba [6], is obtained by choosing $s_i = t_i$ if i is even and $s_i = 1 - t_i$ if i is odd. In literature the name Hammersley seems to be commonly associated with the above point sets, although Hammersley rather proposed a multidimensional generalization of the van der Corput point set $\mathcal{H}_n$.

4.2 Error estimates

In the sequel, we will investigate the quality of the approximation $I_N(\mathcal{H}_n, f)$ of $I(f)$ with any (fixed) Hammersley type point set $\mathcal{H}_n$ for functions from $S^r_{p,q}B(\mathbb{T}^2)$. Let us now fix a cubature formula

$$I_N(\mathcal{H}_n, f) := \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} f(x_i)$$

and a space $S^r_{p,q}B(\mathbb{T}^2)$ with $1 \le p, q \le \infty$ and $1/p < r < 2$. Applying the argument in Step 1 of the proof of Proposition 3.6 together with Proposition 3.4 we obtain that the representation in

(3.4) converges in $C(\mathbb{T}^2)$ and therefore in any $L_p(\mathbb{T}^2)$. Therefore, the integration error, defined in (4.1), can be written as follows

$$
\begin{aligned}
|R_N(f)| &= \Big| \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} f(x_i) - \int_{\mathbb{T}^2} f(x)\, dx \Big| \\
&= \Big| \sum_{j \in \mathbb{N}^2_{-1}} \sum_{m \in D_j} d^2_{j,m}(f) \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} v_{j,m}(x_i) - \sum_{j \in \mathbb{N}^2_{-1}} \sum_{m \in D_j} d^2_{j,m}(f) \int_{\mathbb{T}^2} v_{j,m}(x)\, dx \Big| \\
&= \Big| \sum_{j \in \mathbb{N}^2_{-1}} \sum_{m \in D_j} d^2_{j,m}(f) c_{j,m} \Big| ,
\end{aligned}
\tag{4.3}
$$

where

$$
c_{j,m} := \frac{1}{N} \sum_{x_i \in \mathcal{H}_n} v_{j,m}(x_i) - \int_{\mathbb{T}^2} v_{j,m}(x)\, dx \quad , \quad j \in \mathbb{N}^2_{-1}, m \in D_j\,. \tag{4.4}
$$

Lemma 4.1. *Let $j \in \mathbb{N}^2_{-1}$ and $m \in D_j$ then*

$$
\int_{\mathbb{T}^2} v_{j,m}(x) dx = \begin{cases} 1 & : \quad j = (-1,-1), \\ 2^{-(j_1+1)} & : \quad j = (j_1, -1), j_1 \in \mathbb{N}_0, , \\ 2^{-(j_2+1)} & : \quad j = (-1, j_2), j_2 \in \mathbb{N}_0, \\ 2^{-(j_1+j_2+2)} & : \quad j = (j_1, j_2) \in \mathbb{N}^2_0\,. \end{cases} \tag{4.5}
$$

Proof. By definition (see the line after (3.4)) the functions $v_{j,m}(x_1,x_2) = v_{j_1,m_1}(x_1) \cdot v_{j_2,m_2}(x_2)$ are tensor products of univariate hat functions supported in

$$
I_{j,m} = I_{j_1,m_1} \times I_{j_2,m_2}\,, \text{ where } I_{j_i,m_i} = [m_i 2^{-j_i}, (m_i+1)2^{-j_i}]\,, i = 1,2\,. \tag{4.6}
$$

Note, that $I_{-1,0} = I_{0,0}$ but $v_{-1,0} = 1_{\mathbb{T}}$. Thus, performing the integration coordinate-wise gives immediately (4.5). ■

In the sequel we will need a series of technical lemmas in order to compute the first summand in (4.4), namely the value of $\frac{1}{N} \sum_{x_i \in \mathcal{H}_n} v_{j,m}(x_i)$ for fixed $j \in \mathbb{N}^2_{-1}$ and $m \in D_j$.

Lemma 4.2. *Let $j \in \mathbb{N}^2_{-1}$, $m \in D_j$ and let further $z \in I_{j,m}$. Then*

$$
v_{j,m}(z) = \begin{cases} 1 & : \quad j = (-1,-1) \\ 1 - |2m_1 + 1 - 2^{j_1+1} z_1| & : \quad j = (j_1,-1), j_1 \in \mathbb{N}_0, \\ 1 - |2m_2 + 1 - 2^{j_2+1} z_2| & : \quad j = (-1, j_2), j_2 \in \mathbb{N}_0, \\ (1 - |2m_1 + 1 - 2^{j_1+1} z_1|)(1 - |2m_2 + 1 - 2^{j_2+1} z_2|) & : \quad j = (j_1, j_2) \in \mathbb{N}^2_0\,. \end{cases}
$$

Proof. The result is a direct consequence of the definition of the univariate hat functions in (3.3) and their tensorization. ■

Lemma 4.3. *Let $\mathcal{H}_n$ be a Hammersley type point set with $N = 2^n$ points, $j \in \mathbb{N}^2_0$ with $j_1 + j_2 < n$ and $m \in D_j$. Then we have*

$$
\sum_{z \in \mathcal{H}_n \cap I_{j,m}} (1 - |2m_1 + 1 - 2^{j_1+1} z_1|) = \sum_{z \in \mathcal{H}_n \cap I_{j,m}} (1 - |2m_2 + 1 - 2^{j_2+1} z_2|) = 2^{n-j_1-j_2-1}\,.
$$

Proof. This Lemma is Lemma 3.4 in [7]. A detailed proof can be found there. ■

Lemma 4.4. *Let $\mathcal{H}_n$ be a Hammersley type point set with $N = 2^n$ points, $j \in \mathbb{N}_0^2$ with $j_1 + j_2 < n-1$ and $m \in D_j$. Then we have*

$$\sum_{z \in \mathcal{H}_n \cap I_{j,m}} |2m_1 + 1 - 2^{j_1+1} z_1| \cdot |2m_2 + 1 - 2^{j_2+1} z_2| = 2^{n-j_1-j_2-2} + 2^{j_1+j_2-n}\,.$$

Proof. This Lemma is Lemma 3.5 in [7]. A detailed proof can be found there. ■

Lemma 4.5. *Let $\mathcal{H}_n$ be a Hammersley type point set with $N = 2^n$ points. Let $j \in \mathbb{N}_0^2$ such that $j_1 + j_2 < n-1$ and $m \in D_j$. Then*

$$\sum_{z \in \mathcal{H}_n \cap I_{j,m}} (1 - |2m_1 + 1 - 2^{j_1+1} z_1|) \cdot (1 - |2m_2 + 1 - 2^{j_2+1} z_2|) = 2^{n-j_1-j_2-2} + 2^{j_1+j_2-n}\,.$$

Proof. This statement is a direct consequence of Lemmas 4.3 and 4.4 and the fact that $\sharp(\mathcal{H}_n \cap I_{j,m}) = 2^{n-j_1-j_2}$. See also Lemma 3.6 in [7]. ■

The following proposition states an estimate for the numbers $c_{j,m}$ from (4.4) for all possible indices $j \in \mathbb{N}_{-1}^2$ and $m \in D_j$.

Proposition 4.6. *Let $j \in \mathbb{N}_{-1}^2$ and $m \in D_j$. Then the numbers $c_{j,m}$ in (4.4) satisfy the relations*

$$|c_{j,m}| \le \begin{cases} 0 & : \; j = (-1,-1), \\ 0 & : \; j = (k,-1) \vee j = (-1,k), 0 \le k < n, \\ 2^{-(k+1)} & : \; j = (-1,k) \vee j = (k,-1), k \ge n, \\ 2^{j_1+j_2-2n} & : \; j_1 + j_2 < n-1, \\ 2^{-(j_1+j_2+2)} & : \; j_1 \ge n \vee j_2 \ge n, \\ c2^{-n} & : \; j_1 + j_2 \ge n-1, m \in A_j, \\ 2^{-(j_1+j_2+2)} & : \; j_1 + j_2 \ge n-1, m \in D_j \setminus A_j, \end{cases} \tag{4.7}$$

where A_j denotes the set of indices m (depending on j) such that $I_{j,m} \cap \mathcal{H}_n$ is non-empty. Moreover, in all, except the sixth case in (4.7)*, we even have equality.*

Proof. *(i).* The first relation is trivial since $v_{(-1,-1),(0,0)} = 1_{\mathbb{T}^2}$. Thus, (4.4) vanishes.
(ii). The second case follows from the second relation in Lemma 4.2 together with Lemma 4.3. Indeed, if $j = (-1,k)$, then by definition

$$\begin{aligned} \sum_{z \in \mathcal{H}_n} v_{j,m}(z) &= \sum_{z \in \mathcal{H}_n \cap I_{j,m}} (1 - |2m_2 + 1 - 2^{k+1} z_1|) \\ &= \sum_{z \in \mathcal{H}_n \cap I_{(0,k),m}} (1 - |2m_2 + 1 - 2^{k+1} z_1|) \\ &= 2^{n-(k+1)}\,, \end{aligned} \tag{4.8}$$

where the last identity follows from Lemma 4.3. Taking the second relation in (4.5) into account, we see that (4.4) again vanishes in this case.
(iii), (v). Clearly, none of the points of $\mathcal{H}_n$ is contained in the interior of boxes $I_{j,m}$ where $j_1 \ge n$ or $j_2 \ge n$. Thus, the first summand in (4.4) vanishes. What remains is the second

summand, which is given by (4.5).
(iv). This relation follows by combining Lemma 4.2 (fourth relation) with Lemma 4.5 to get

$$\begin{aligned}\sum_{z\in\mathcal{H}_n} v_{j,m}(z) &= \sum_{z\in\mathcal{H}_n\cap I_{j,m}} (1-|2m_1+1-2^{j_1+1}z_1|)\cdot(1-|2m_2+1-2^{j_2+1}z_2|)\\ &= 2^{n-j_1-j_2-2}+2^{j_1+j_2-n}\,.\end{aligned}$$

Putting this into into (4.4) and taking (4.5) into account yields the statement.
(vi),(vii). On a fixed level $j\in\mathbb{N}_0^2$ we have $2^{j_1+j_2}$ (interior) disjoint boxes $I_{j,m}$. Clearly, at most 2^n of these boxes contain a point of $\mathcal{H}_n$ in their interior. The corresponding indices $m\in D_j$ are collected in the set A_j. In particular, we have $\sharp A_j\le 2^n$. Since, $j_1+j_2\ge n-1$ every such box $I_{j,m}$ with $m\in A_j$ can contain at most two points from $\mathcal{H}_n$. Therefore, the absolute value of the first summand in (4.4) can be estimated from above by 2^{-n}, whereas the absolute value of the second summand is of upper order $2^{-(j_1+j_2)}$. Thus, by $j_1+j_2\ge n-1$ the sum is of order 2^{-n}. On the other hand, if $m\in D_j\setminus A_j$, the first summand in (4.4) vanishes. Hence, $|c_{j,m}|$ equals $2^{-(j_1+j_2+2)}$ by (4.5). ■

We are ready to state and prove the main result of this section.

Theorem 4.7. *Let $\mathcal{H}_n$ be a Hammersley type point set with $N=2^n$ points. Let further $1\le p,q\le\infty$ and $1/p<r<2$. Then we have*

$$\sup_{\|f|S^r_{p,q}B(\mathbb{T}^2)\|\le 1}|R_N(f)| = \sup_{\|f|S^r_{p,q}B(\mathbb{T}^2)\|\le 1}|I(f)-I_N(\mathcal{H}_n,f)|\lesssim N^{-r}(\log N)^{1-1/q}\,. \qquad (4.9)$$

Proof. *Step 1.* Our starting point will be the representation of the error $R_N(f)$ in (4.3). We have

$$|R_N(f)|\le\sum_{j\in\mathbb{N}^2_{-1}}\sum_{m\in D_j}|d^2_{j,m}(f)|\cdot|c_{j,m}|\,.$$

By applying Hölder's inequality twice, to the inner sum with $1/p+1/p'=1$ and afterwards to the outer sum with $1/q+1/q'=1$, we obtain

$$\begin{aligned}|R_N(f)|\le&\Big[\sum_{j\in\mathbb{N}^2_{-1}}2^{(r-1/p)(j_1+j_2)q}\Big(\sum_{m\in D_j}|d^2_{j,m}(f)|^p\Big)^{q/p}\Big]^{1/q}\\ &\times\Big[\sum_{j\in\mathbb{N}^2_{-1}}2^{-(r-1/p)(j_1+j_2)q'}\Big(\sum_{m\in D_j}|c_{j,m}|^{p'}\Big)^{q'/p'}\Big]^{1/q'}\\ \lesssim&\|f|S^r_{p,q}B(\mathbb{T}^2)\|\cdot\Big[\sum_{j\in\mathbb{N}^2_{-1}}2^{-(r-1/p)(j_1+j_2)q'}\Big(\sum_{m\in D_j}|c_{j,m}|^{p'}\Big)^{q'/p'}\Big]^{1/q'}.\end{aligned} \qquad (4.10)$$

The last relation is a consequence of Proposition 3.4 and Definition 3.2 . Note, that here the condition $r<2$ is relevant. It remains to estimate the quantity involving the numbers $c_{j,m}$ in (4.10) with the help of Proposition 4.6.

Step 2. Let us split the sum over j in (4.10) into several parts.

(i) To begin with we deal with the case $j_1 + j_2 < n - 1$, $j \in \mathbb{N}_0^2$. With the fourth case in Proposition 4.6 we obtain

$$\begin{aligned}
&\sum_{\substack{j_1+j_2<n-1\\ j_1,j_2\geq 0}} 2^{-(r-1/p)(j_1+j_2)q'}\Big(\sum_{m\in D_j}|c_{j,m}|^{p'}\Big)^{q'/p'}\\
&= \sum_{\substack{j_1+j_2<n-1\\ j_1,j_2\geq 0}} 2^{-(j_1+j_2)(r-1/p)q'}2^{-2nq'}2^{(j_1+j_2)(1/p'+1)q'}\\
&= 2^{-2nq'}\sum_{\substack{j_1+j_2<n-1\\ j_1,j_2\geq 0}} 2^{-(j_1+j_2)(r-1/p-1/p'-1)q'}\\
&= 2^{-2nq'}\sum_{\substack{j_1+j_2<n-1\\ j_1,j_2\geq 0}} 2^{-(j_1+j_2)(r-2)q'}\\
&\asymp 2^{-2nq'}2^{-n(r-2)q'}n\\
&\asymp 2^{-rnq'}n\,.
\end{aligned}$$

In the last but one step again the condition $r < 2$ is required.
(ii) At next we will deal with the sum over $j = (k, -1)$ with $k \geq n$. With the third case in Proposition 4.6 we obtain

$$\sum_{k=n}^{\infty} 2^{-(r-1/p)kq'}\Big(\sum_{m\in D_{(k,-1)}} 2^{-(k+1)p'}\Big)^{q'/p'} \asymp \sum_{k=n}^{\infty} 2^{-k(r-1/p+1-1/p')q'} \asymp 2^{-nrq'}\,.$$

The same estimate holds true for the sum over $j = (-1, k)$.
(iii) Now we consider the sum over all $j \in \mathbb{N}_0^2$ with $\max\{j_1, j_2\} \geq n$. The fifth case in Proposition 4.6 yields

$$\begin{aligned}
&\sum_{\substack{\max\{j_1,j_2\}\geq n\\ j_1,j_2\geq 0}} 2^{-(r-1/p)(j_1+j_2)q'}\Big(\sum_{m\in D_j} 2^{-(j_1+j_2)p'}\Big)^{q'/p'}\\
&= \Big(\sum_{j_1=n}^{\infty}\sum_{j_2=0}^{n-1} + \sum_{j_1=n}^{\infty}\sum_{j_2=n}^{\infty} + \sum_{j_1=0}^{n-1}\sum_{j_2=n}^{\infty}\Big)2^{-(j_1+j_2)(r-1/p+1-1/p')q'}\\
&\asymp 2^{-rnq'}\,.
\end{aligned}$$

(iv) Let us deal with the sum over $j \in \mathbb{N}_0^2$ with $j_1 + j_2 \geq n - 1$ and $\max\{j_1, j_2\} \leq n$. We split

the sum over m according to the last two cases in Proposition 4.6. This yields

$$
\begin{aligned}
&\sum_{\substack{j_1+j_2\geq n-1\\ 0\leq j_1,j_2\leq n}} 2^{-(r-1/p)(j_1+j_2)q'}\Big(\sum_{m\in D_j}|c_{j,m}|^{p'}\Big)^{q'/p'}\\
&\lesssim \sum_{\substack{j_1+j_2\geq n-1\\ 0\leq j_1,j_2\leq n}} 2^{-(r-1/p)(j_1+j_2)q'}\Big(\sum_{m\in A_j}2^{-np'}\Big)^{q'/p'}\\
&\qquad + \sum_{\substack{j_1+j_2\geq n-1\\ 0\leq j_1,j_2\leq n}} 2^{-(r-1/p)(j_1+j_2)q'}\Big(\sum_{m\in D_j\setminus A_j}2^{-(j_1+j_2)p'}\Big)^{q'/p'}\\
&\lesssim 2^{-nq'}2^{nq'/p'}\sum_{\substack{j_1+j_2\geq n-1\\ 0\leq j_1,j_2\leq n}} 2^{-(r-1/p)(j_1+j_2)q'} + \sum_{\substack{j_1+j_2\geq n-1\\ 0\leq j_1,j_2\leq n}} 2^{-(j_1+j_2)(r-1/p+1-1/p')q'}\\
&\lesssim 2^{(1/p'-1)nq'}2^{-(r-1/p)nq'}n + 2^{-rnq'}n\\
&\asymp 2^{-rnq'}n\,,
\end{aligned}
$$

where in the last but one step the assumption $r > 1/p$ is required. It remains to add up the bounds in (i),(ii),(iii),(iv) and take the power $1/q'$. Finally, (4.10) implies the required upper estimate (4.9). The proof is complete. ■

Remark 4.8. (i) *Note, that we do not have to specify the Hammersley point set in Theorem 4.7. There is no need for restricting the number $a = \#\{i = 1, ..., n : s_i = t_i\}$ in contrast to [7, Thm. 3.1,(vi)]. There the Haar coefficient $\mu_{(-1,-1),(0,0)}$ depends on the number a and is small if $a = \lfloor n/2 \rfloor$, see also [6, p. 318]. However, the index $j = (-1,-1)$ causes less problems in our situation due to the periodicity of the functions from the space $S^r_{p,q}B(\mathbb{T}^2)$. In fact, the basis function according to the lowest level is $v_{-1,0} \equiv 1$ rather than $v_{-1,0}(t) = 1-t$ and $v_{-1,1}(t) = t$.*
(ii) *For the lower bounds we refer to Step 3 and Step 4 in the proof of [21, Thm. 4.15] and particularly to the functions defined in (4.149) and (4.153). Literally the same method works in the periodic setting by taking Proposition 3.6 into account. This yields*

$$\mathrm{Int}_N(S^r_{p,q}B(\mathbb{T}^2)) \gtrsim N^{-r}(\log N)^{1-1/q} \tag{4.11}$$

in case $1 \leq p, q \leq \infty$ and $1/p < r < 1 + 1/p$. For the extension of (4.11) *to arbitrary $r > 1/p$ let us refer to a forthcoming paper of Dinh and the author.*

5 Non-periodic spaces on the unit square

It is already mentioned in the introduction that the classes $S^r_{p,q}B(Q_2)$ are formally defined as restrictions to the unit square $Q_2 = [0,1]^2$ of functions from the classes $S^r_{p,q}B(\mathbb{R}^2)$, see, e.g., [21, Def. 1.38]. In fact, the resulting spaces do not consist of periodic functions. Consequently, the spaces $S^r_{p,q}B(Q_2)$ and $S^r_{p,q}B(\mathbb{T}^2)$ differ essentially. However, we will see below that every function in $S^r_{p,q}B(Q_2)$ can be decomposed into a periodic function and some "boundary terms". The main tool for this insight is again provided by the (non-periodic) Faber basis decomposition. Let us return to (3.1) and re-consider the univariate Faber system

$$\{v_0(x), v_1(x), v_{j,m}(x) \,:\, j \in \mathbb{N}_0, m \in D_j\}\,,$$

where $v_0(x) = 1-x$, $v_1(x) = x$, and $v_{j,m}(x)$ is given by (3.3). For $j \in \mathbb{N}_{-1}$ we put $D_{-1} = \{0,1\}$ and $D_j := \{0,...,2^j-1\}$ if $j \geq 0$. Let now $j = (j_1, j_2) = \mathbb{N}_{-1}^2$, $D_j = D_{j_1} \times D_{j_2}$ and $m = (m_1, m_2) \in D_j$. The bivariate (non-periodic) Faber basis functions result from a tensorization of the univariate ones, i.e.,

$$v_{(j_1,j_2),(m_1,m_2)}(x_1,x_2) = \begin{cases} v_{m_1}(x_1)v_{m_2}(x_2) & : \ j_1 = j_2 = -1, \\ v_{m_1}(x_1)v_{j_2,m_2}(x_2) & : \ j_1 = -1, j_2 \in \mathbb{N}_0, \\ v_{j_1,m_1}(x_1)v_{m_2}(x_2) & : \ j_1 \in \mathbb{N}_0, j_2 = -1, \\ v_{j_1,m_1}(x_1)v_{j_2,m_2}(x_2) & : \ j_1, j_2 \in \mathbb{N}_0\,, \end{cases} \tag{5.1}$$

see also [21, 3.2]. In contrast to the periodic decomposition in (3.4) we obtain for every continuous bivariate function $f \in C(Q_2)$ the representation

$$f(x) = \sum_{j\in\mathbb{N}_{-1}^2} \sum_{m\in D_j} D_{j,m}^2(f) v_{j,m}(x)\,, \tag{5.2}$$

where now

$$D_{j,k}^2(f) = \begin{cases} f(m_1,m_2) & : \ j = (-1,-1), \\ -\frac{1}{2}\Delta^2_{2^{-j_1-1},1} f(2^{-j_1}m_1, 0) & : \ j = (j_1,-1), \\ -\frac{1}{2}\Delta^2_{2^{-j_2-1},2} f(0, 2^{-j_2}m_2) & : \ j = (-1,j_2), \\ \frac{1}{4}\Delta^{2,2}_{2^{-j_1-1},2^{-j_2-2}} f(2^{-j_1}m_1, 2^{-j_2}m_2) & : \ j = (j_1,j_2)\,. \end{cases}$$

Consequently, a function $f \in S^r_{p,q}B(Q_2)$ with $r > 1/p$ admits a decomposition into three parts

$$f = f^Q + f^{\partial Q} + f^{\partial^2 Q}, \tag{5.3}$$

where f^Q involves the summands with $(j_1, j_2) \in \mathbb{N}_0^2$ in (5.2), $f^{\partial^2 Q}$ the summands with $j_1 = j_2 = -1$ and $f^{\partial Q}$ the rest. The function f^Q is periodic in each direction and even belongs to $S^r_{p,q}B(\mathbb{T}^2)$. Indeed, it is shown in [4, Thm. 4.1] that the statements from Propositions 3.4 and 3.6 transfer almost literally to non-periodic functions from $S^r_{p,q}B([0,1]^2)$.

Proposition 5.1. *Let $0 < p, q \leq \infty$ and $1/p < r < 2$. Then there exists a constant $c > 0$ such that*

$$\big\|D_{j,m}^2(f)|s^r_{p,q}b\big\| \leq c\|f|S^r_{p,q}B([0,1]^2)\| \tag{5.4}$$

for all $f \in C([0,1]^2)$.

Proof. See [4, Thm. 4.1] or [21, Thm. 3.16]. Note, that in the latter reference the additional restriction $1/p < r < \min\{2, 1 + 1/p\}$ is used. However, it is not needed for this direction. ■

6 Optimal cubature of non-periodic functions

In this section we will present optimal cubature formulas for the numerical integration of non-periodic functions from the class $S^r_{p,q}B(Q_2)$ where $Q_2 = [0,1]^2$. Based on the observation (5.3) these formulas are adaptions of the Hammersley QMC methods (Sections 4) for integrating periodic functions. However, the presented cubature formulas are not longer QMC rules since the integration weights $\Lambda = (\lambda_1, ..., \lambda_N)$ (computed out of the chosen integration knots) are non-equal in general. We were not able to construct optimal QMC rules in this context, so we pose it as an open problem here.

6.1 The cubature formula

Let $\mathcal{H}_n$ be a fixed Hammersley type points set with $N = 2^n$ points. We define the functional

$$\begin{aligned} Q_N(\mathcal{H}_n, f) :=& \frac{1}{2^n} \sum_{(x_i,y_i)\in\mathcal{H}_n} f(x_i, y_i) \\ &+ \frac{1}{2^n} \sum_{(x_i,y_i)\in\mathcal{H}_n} \Big[\Big(y_i - \frac{1}{2}\Big)\big(f(x_i, 0) - f(x_i, 1)\big) + \Big(x_i - \frac{1}{2}\Big)\big(f(0, y_i) - f(1, y_i)\big)\Big] \\ &+ \Big(\frac{1}{2^{n+1}} - \frac{1}{4} + \frac{1}{2^n} \sum_{(x_i,y_i)\in\mathcal{H}_n} x_i y_i\Big)\big(f(0, 0) - f(1, 0) + f(1, 1) - f(0, 1)\big) . \end{aligned} \tag{6.1}$$

Note the analogy to the decomposition in (5.3). The first summand in (6.1) coincides with the QMC method $I_N(\mathcal{H}_n, f)$ considered in Section 4. The second and third summand represent certain correction terms in order to deal with the boundary. Finally, it is obvious that all the integration weights in (6.1) sum up to 1.

6.2 Error estimates

To estimate the error $Q_N(\mathcal{H}_n, f) - I(f)$ we proceed as done in (4.3) by using (5.2). Doing so we end up with

$$|R_N(f)| \le \Big| \sum_{j\in\mathbb{N}^2_{-1}} \sum_{m\in D_j} D^2_{j,m}(f) C_{j,m} \Big| \tag{6.2}$$

where

$$C_{j,m} := Q_N(\mathcal{H}_n, v_{j,m}) - \int_{[0,1]^2} v_{j,m}(x)\, dx \quad , \quad j \in \mathbb{N}^2_{-1} , m \in D_j . \tag{6.3}$$

Hence, the error analysis reduces to the error made by integrating the basis functions. The counterparts of Lemma 4.1 and Proposition 4.6 read as follows.

Lemma 6.1. *Let $j \in \mathbb{N}^2_{-1}$ and $m \in D_j$ then*

$$\int_{[0,1]^2} v_{j,m}(x)dx = \begin{cases} 2^{-2} & : \ j = (-1, -1), \\ 2^{-(j_1+2)} & : \ j = (j_1, -1), j_1 \in \mathbb{N}_0, \\ 2^{-(j_2+2)} & : \ j = (-1, j_2), j_2 \in \mathbb{N}_0, \\ 2^{-(j_1+j_2+2)} & : \ j = (j_1, j_2) \in \mathbb{N}^2_0 . \end{cases} \tag{6.4}$$

Note, that Lemmas 4.1 and 6.1 differ in the second line of (4.5) and (6.4), respectively.

Proposition 6.2. *Let $j \in \mathbb{N}^2_{-1}$ and $m \in D_j$. Then we have*

$$|C_{j,m}| \le \begin{cases} 0 & : \ j = (-1, -1), \\ 0 & : \ j = (k, -1) \vee j = (-1, k), 0 \le k < n, \\ 2^{-(k+2)} & : \ j = (-1, k) \vee j = (k, -1), k \ge n, \\ 2^{j_1+j_2-2n} & : \ j_1 + j_2 < n - 1, \\ 2^{-(j_1+j_2+2)} & : \ j_1 \ge n \vee j_2 \ge n, \\ c2^{-n} & : \ j_1 + j_2 \ge n - 1, m \in A_j, \\ 2^{-(j_1+j_2+2)} & : \ j_1 + j_2 \ge n - 1, m \in D_j \setminus A_j. \end{cases} \tag{6.5}$$

Proof. The construction of $Q_N(\mathcal{H}_n, \cdot)$ immediately implies that

$$Q_N(\mathcal{H}_n, v_{j,m}) = I_N(\mathcal{H}_n, v_{j,m})$$

whenever $j \in \mathbb{N}_0^2$. Hence, the cases (iv) to (vii) in (6.5) coincide with the corresponding cases in (4.7). By taking the modified Lemma 6.1 into account, case (iii) follows by the same argument as used for (4.7). Finally, for (i) and (ii) the particular selection of the integration weights in (6.1) plays the crucial role. Let us first deal with (ii) and suppose that $j = (-1, k)$ with $k \in \mathbb{N}_0$, $n > k$, and $m = (0, m_2)$. By definition we have

$$Q_N(\mathcal{H}_n, v_{j,m}) = \frac{1}{2^n} \sum_{(x_i,y_i)\in\mathcal{H}_n} \Big[v_{j,m}(x_i, y_i) + \Big(x_i - \frac{1}{2}\Big)v_{j,m}(0, y_i)\Big]$$

since $v_{j,m}$ has non-vanishing boundary values only on the line strictly between $(0,0)$ and $(0,1)$. Moreover, $v_{j,m}(0,0) = v_{j,m}(0,1) = v_{j,m}(1,0) = v_{j,m}(1,1) = 0$ unless $j_1 = j_2 = -1$. The tensor structure of $v_{j,m}$ implies

$$\begin{aligned}
Q_N(\mathcal{H}_n, v_{j,m}) &= \frac{1}{2^n} \sum_{(x_i,y_i)\in\mathcal{H}_n} \Big[(1 - x_i)v_{k,m_2}(y_i) + \Big(x_i - \frac{1}{2}\Big)v_{k,m_2}(y_i)\Big] \\
&= \frac{1}{2^{n+1}} \sum_{(x_i,y_i)\in\mathcal{H}_n} v_{k,m_2}(y_i) \\
&= 2^{-(k+2)},
\end{aligned}$$

where we used (4.8) in the last step. This together with (6.4) gives the second case in (6.5) for $j = (-1, k)$ and $m = (0, m_2)$. The remaining cases for j and m follow in a similar fashion. It remains to deal with case (i). Let for instance $j = (-1, -1)$ and $m = (1, 1)$ or, equivalently, $v_{j,m}(x, y) = xy$. Then

$$\begin{aligned}
Q_N(\mathcal{H}_n, v_{j,m}) =& \frac{1}{2^n} \sum_{(x_i,y_i)\in\mathcal{H}_n} v_{j,m}(x_i, y_i) \\
&+ \frac{1}{2^n} \sum_{(x_i,y_i)\in\mathcal{H}_n} \Big[\Big(\frac{1}{2} - y_i\Big)v_{j,m}(x_i, 1) + \Big(\frac{1}{2} - x_i\Big)v_{j,m}(1, y_i)\Big] \\
&+ \Big(\frac{1}{2^{n+1}} - \frac{1}{4} + \frac{1}{2^n} \sum_{(x_i,y_i)\in\mathcal{H}_n} x_i y_i\Big)v_{j,m}(1, 1) \\
=& \frac{1}{2^{n+1}} - \frac{1}{4} + \frac{1}{2^{n+1}} \sum_{(x_i,y_i)\in\mathcal{H}_n} x_i + \frac{1}{2^{n+1}} \sum_{(x_i,y_i)\in\mathcal{H}_n} y_i \\
=& \frac{1}{4}.
\end{aligned}$$

Together with (6.4) we obtain the first case in (6.5). The remaining cases for m follow in a similar fashion. The proof is complete. ∎

We are ready to state our main result in this section. Surprisingly, the following construction works for arbitrary Hammersley type point sets. We do not have to specify the number a, see Remark 4.8,(i). In fact, we start as we did in the periodic setting and add the points which we get by projecting the Hammersley points on the respective four boundary lines. However, to these new points we attach weights which depend on the Hammersley set from the start, see (6.1). Hence, we did not construct a QMC rule here.

Theorem 6.3. *Let $\mathcal{H}_n$ be a Hammersley type point set with $N = 2^n$ points. Let further $1 \le p, q \le \infty$ and $1/p < r < 2$. Then we have*

$$\begin{aligned} \sup_{\|f|S^r_{p,q}B(Q_2)\|\le 1} |R_N(f)| &= \sup_{\|f|S^r_{p,q}B(Q_2)\|\le 1} |I(f) - Q_N(\mathcal{H}_n, f)| \\ &\lesssim N^{-r}(\log N)^{1-1/q} \lesssim M^{-r}(\log M)^{1-1/q}\,, \end{aligned} \tag{6.6}$$

where $M = 5N - 2$ or $M = 5N$, respectively, denotes the number of points used by Q_N.

Proof. By combining (6.2) with (6.5) the proof is almost literally the same as the proof of Theorem 4.7. It is clear, that Q_N uses $M = 5 \cdot 2^n = 5N$ points if $(0,0) \in \mathcal{H}_n$. Otherwise, we have $M = 5 \cdot 2^n - 2 = 5N - 2$ points since we have 2 distinct points in $\mathcal{H}_n$ lying on $x-$ and $y-$axis, respectively. ■

Corollary 6.4. *Let $1 \le p, q \le \infty$.*
(i) *If $1/p < r < 2$ it holds*

$$\mathrm{Int}_N(S^r_{p,q}B(Q_2)) \lesssim N^{-r}(\log N)^{1-1/q}\,, \quad N \in \mathbb{N}\,.$$

(ii) *If $1/p < r < 1 + 1/p$ it holds*

$$\mathrm{Int}_N(S^r_{p,q}B(Q_2)) \gtrsim N^{-r}(\log N)^{1-1/q}\,, \quad N \in \mathbb{N}\,.$$

Proof. Statement (i) is a direct consequence of Theorem 6.3. For the lower bound we refer to Step 3 and Step 4 in the proof of [21, Thm. 4.15] and particularly to the functions defined in (4.149) and (4.153). ■

Remark 6.5. *In a forthcoming paper by Dinh and the author the lower bound in (ii) will be extended to all $r > 1/p$.*

6.3 Consequences for optimal discrepancy

Let us finally take a look to consequences for discrepancy numbers in $S^r_{p,q}B(\mathbb{T}^2)$ with negative smoothness r. We start with the definition of the discrepancy function on Q_2 associated to a discrete set of points $\Gamma = \{x^j\}_{j=1}^N \subset Q_2$ and weights $A = \{a^j\}_{j=1}^N \subset \mathbb{C}$. The discrepancy function $\mathrm{disc}_{\Gamma,A}(x)$ on Q_2 is given by

$$\mathrm{disc}_{\Gamma,A}(x_1, x_2) = x_1 x_2 - \sum_{j=1}^N a^j \chi_{[x^j,1]}(x)\,, \quad x = (x_1, x_2) \in Q_2\,,$$

where $[x^j, 1] := [x^j_1, 1] \times [x^j_2, 1]$. In fact, this function computes the error between the exact integral of the characteristic function $\chi_{[0,x]}$ over the cube Q_2 and the result of the cubature formula with knots Γ and weights A. In the case $a^j = N^{-1}$, $j = 1, ..., N$, we reduce the notation to $\mathrm{disc}_\Gamma := \mathrm{disc}_{\Gamma,A}$. The respective function disc_Γ measures the deviation between the uniform distribution and the discrete distribution of given points Γ. A generalization [21, Thm. 6.11] of the well-known Hlawka-Zaremba identity connects the optimal cubature error $\mathrm{Int}_N(F')$ with the discrepancy number $\mathrm{disc}_N(F)$. The latter quantity is the smallest norm of $\mathrm{disc}_{\Gamma,A}$ in a fixed function space F on Q_2 for all choices of Γ and A, i.e.,

$$\mathrm{disc}_N(F) := \inf_{\Gamma,A} \|\mathrm{disc}_{\Gamma,A}|F\|\,.$$

For the scale of spaces $F = S^r_{p,q}B(Q_2)$ we have $F' = S^{1-r}_{p',q'}(Q_2)^{\urcorner}$ which is $S^{1-r}_{p',q'}(Q_2)$ with zero boundary on the upper and right boundary line. For the following result and its proof we refer to [21, Thm. 6.11].

Proposition 6.6. *Let $(p,q) \in (1,\infty] \times (1,\infty] \cup \{(1,1)\}$ and $1/p - 1 < r < 1/p$. Then*

$$\mathrm{disc}_N(S^r_{p,q}B(Q_2)) \asymp \mathrm{Int}_N(S^{1-r}_{p',q'}B(Q_2)^{\urcorner}), \quad N \in \mathbb{N}.$$

By an application of Corollary 6.4 in connection with Proposition 6.6 we obtain the main result of this subsection.

Theorem 6.7. *Let $(p,q) \in (1,\infty] \times (1,\infty] \cup \{(1,1)\}$ and $1/p - 1 < r < 1/p$. Then*

$$\mathrm{disc}_N(S^r_{p,q}B(Q_2)) \asymp N^{r-1}(\log N)^{1/q}, \quad N \in \mathbb{N}.$$

Remark 6.8. (i) *The restriction $r < 1/p$ is necessary. We need that characteristic functions belong to the respective space. However, the restriction $r > 1/p - 1$ and the remaining restrictions on p and q come from Proposition 6.6 and [21, 3.215]. We do not know if they are necessary.*
(ii) *The result in Theorem 6.7 improves on [21, Thm. 6.13] in the exponent of the logarithm ($d = 2$). We have an explicit point set and associated weights which are optimal, see* (6.1). *Note, that we do not need the samples on the right and upper boundary line of Q_2 due to zero boundary values of the respective function. Hence, the optimal point set consists of Hammersley points together with their projection on the lower and left boundary of Q_2. The weights can be computed explicitly, see* (6.1). *Surprisingly, the same procedure works for Fibonacci lattices. This will be shown in a forthcoming paper by Dinh and the author.*
(iii) *The result in Theorem 6.7 shows that a slight modification of the Hammersley points together with properly chosen (non-equal) weights yield optimality even for negative smoothness r. However, the interesting question remains what happens in the case of equal weights $a^j = 1/N$ and negative smoothness r, see the question posed at the end of [7]. In other words, do we still have*

$$\inf_{\Gamma} \|\mathrm{disc}_\Gamma | S^r_{p,q}B(Q_2)\| \lesssim N^{r-1}(\log N)^{1/q}, \quad N \in \mathbb{N}\,?$$

Note, that Hammersley points alone do not even provide the optimal order in the main polynomial term in N, as shown in [7].

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