Preasymptotics for Multivariate Approximation

Organizers: Thomas Kühn, University of Leipzig and Tino Ullrich, Hausdorff-Center for Mathematics, Bonn

In the investigation of high-dimensional function approximation or recovery problems gaining control over associated worst-case errors is central in many respects. Such estimates are needed, for instance, in the search for optimal algorithms, for complexity-theoretical considerations (tractability analysis), or practically, to obtain reliable a priori error estimates. Deriving worst-case error bounds requires model assumptions. These narrow down the set of admissible functions, which potentially fit the given data, to a function class F_d. The function class could be the unit ball of a smoothness space (e.g., Sobolev, Gevrey, or a class of analytic functions), which is a naturally appearing model assumption in the context of Galerkin methods for PDEs. Another type of model assumption are structured dependencies. The idea to allow only specific functional dependencies is very prominent in statistics and machine learning to mitigate the burden of high-dimensionality. Classically, people were interested in the correct asymptotical behavior of worst-case errors. Those investigations are, of course, important for determining optimal algorithms. However, the obtained asymptotical error bounds are often useless for practical implementations since they are only relevant for very large numbers n which is, related to the amount of data an algorithm uses, beyond the range a computer is able to process. Moreover, asymptotical error bounds often hide dependencies on the ambient dimension d in the constants, which makes a rigorous tractability analysis impossible. Controlling the dependence of the underlying constants is a first step to improve on the classical error estimates. But this does not automatically provide preasymptotic error guarantees, that is, error guarantees for algorithms using smaller amounts of data.

The goal of this minisymposium is to present recent progress in the direction of finding reliable preasymptotic error estimates and corresponding algorithms. It turned out, that this problem is mathematically rather challenging and connected to deep problems in Banach space geometry. Some recent findings highlight and elaborate a connection to metric entropy and present a generic tool how to obtain preasymptotics for several classes of functions F_d when dealing with general linear information. This rather new viewpoint opens several research directions such that we are already faced with a huge list of open problems.

Monday August 15. 3^{50}–5^{50}, Berg C

Preasymptotic Estimates for Approximation in Periodic Function Spaces

Thomas Kühn
Universität Leipzig, Germany
kuehn@math.uni-leipzig.de

In this talk I will give a survey on some recent results on preasymptotic estimates for approximation in multivariate periodic function spaces, including a motivation for this type of problems.

For embeddings of classical Sobolev spaces the exact rate of approximation numbers has been known for many years. However, almost all asymptotic estimates in the literature involve unspecified constants which depend on the parameters of the spaces, in particular on the dimension d of the underlying domain, and on the chosen norm. Often the asymptotic rate can only be seen after "waiting exponentially long".

In high-dimensional numerical problems, as they appear e.g. in financial mathematics, this might be beyond the capacity of computational methods. Therefore one needs additional information on the d-dependence of these constants, and also on the behaviour in the so-called preasymptotic range, i.e. before the asymptotic rate becomes visible.

I will discuss these problems for the classical isotropic Sobolev spaces, Sobolev spaces of dominating mixed smoothness, and for periodic Gevrey spaces. The latter have a long history, they consist of C^∞-functions and have been used in numerous problems related to partial differential equations.

Preasymptotics via Entropy

Sebastian Mayer
University of Bonn
mayer@ins.uni-bonn.de

I will present an abstract characterization result which allows to readily derive preasymptotic error bounds for approximation numbers of certain Sobolev type spaces from the well-known entropy numbers of the embedding $\mathcal{I} : \ell_p \to \ell_\infty$.

Point Distributions on the Sphere, Almost Isometric Embeddings, One-Bit Sensing, and Energy Optimization

Dmitriy Bilyk
University of Minnesota
dbilyk@math.umn.edu

We shall consider various versions and applications of questions of equidistribution of points on the sphere. First we address the question of uniform tessellation of the sphere $\mathbb{S}^d$ (or its subsets) by hyperplanes, or equivalently, almost isometric embeddings of subsets of the sphere into the Hamming cube. We find that in the general case this problem is equivalent to the study of discrepancy with respect to certain spherical "wedges", which behaves very similarly to the well-studied situation of the spherical cap discrepancy. In particular, asymptotically the so-called "jittered sampling" yields better results than random selection of points.

However, in one-bit compressed sensing (which is closely related to this problem), one is interested in the preasymptotic case, where the number of points (measurements) is small relative to the dimension. We address this issue and obtain a series of results, which may be roughly summarized as: Theoretically, one-bit compressed sensing is just as good as the standard compressed sensing. In particular, we prove direct one-bit analogs of several classical compressed sensing results (including the famous Johnson-Lindenstrauss Lemma).

Finally, we obtain several different analogs of the Stolarsky principle, which states that minimizing the L^2 spherical cap discrepancy is equivalent to maximizing the sum of pairwise Euclidean distances between the points. These results bring up some interesting connections with frame theory and energy optimization. In particular, we show that geodesic energy integrals yield surprisingly different behavior from their Euclidean counterparts.

Parts of this work have been done in collaboration with Michael Lacey and Feng Dai.

Non-Optimality of Rank-1 Lattice Sampling in Spaces of Mixed Smoothness

Glenn Byrenheid
Hausdorff Center for Mathematics, University of Bonn
byrenheid.glenn@gmail.com

We consider the approximation of functions with mixed smoothness by algorithms using m nodes on a rank-1 lattice

$$\Lambda(z,m) := \Big\{ \frac{k}{m} z : \ k = 0, \ldots, m-1 \Big\}, \quad z \in \mathbb{Z}^d.$$

We prove upper and lower bounds for the sampling rates with respect to the number of lattice points in various situations and achieve improvements in the main error rate compared to earlier contributions to the subject. This main rate (without logarithmic factors) is half the optimal main rate coming for instance from sparse grid sampling and suprisingly turns out to be best possible among all algorithms taking samples on rank-1 lattices.

This is joint work with Lutz Kämmerer (TU Chemnitz, Germany), Tino Ullrich (HCM Bonn, Germany) and Toni Volkmer (TU Chemnitz, Germany).